C-clause calculi and refutation search in first-order classical logic

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Abstract. The paper describes an approach to the construction of a resolution-type technique basing on a certain generalization of the resolution notion of a clause. This generalization called a conjunctive clause (c-clause) leads to a possibility to introduce two different inference rules and determine two c-clause calculi oriented to refutation search in first-order classical logic both with and without equality. Using the connection of these calculi with Robinson's clash-resolution method, a simple way for the proving of their soundness and completeness is given. Analogs of some of the well-known resolution strategies for the calculi are suggested. Besides, the treatment of Maslov's inverse method in the resolution terms is given. This research can be used in (e-)learning systems for the intelligent testing of knowledge of trainees learning a mathematical subject.

Keywords: first-order classical logic, refutation search, calculus, soundness, completeness, clash-resolution method, paramodulation, strategy

Key Terms: MachineIntelligence

1 Introduction

This paper is devoted to the description of special calculi intended for the establishing of the unsatisfiability of a formula F of a certain form or a set S of such formulas in first-order classical logic maybe with equality. The calculi relate to the class of refutation-search methods based on the ideas firstly presented in Robinsin's paper [1] on the well-known resolution method.

After the appearance of the resolution method, the main efforts of automated theorem-proving community were concentrated on its development in the direction of the construction of its different modifications and strategies oriented to increasing the efficiency of deduction search. All such attempts based on the use of a clause being a well-formed expression of the resolution method leaving aside the possibility of building efficient methods using modifications of the notion of a clause, to which this paper is devoted. Besides, the problem of the interpretation of the Maslov inverse method [2] in resolution terms is solved in it.

Our calculi completely are determined by their (resolution-type) inference rules. The deducibility of a special expression Λ in such a calculus Π is equivalent to the unsatisfiability of F or S. At that, Π is called a *sound* calculus, if the deducibility of Λ implies the unsatisfiability of F or S; Π is called a *complete* calculus, if the unsatisfiability of F or S implies the deducibility of Λ.

If we put certain restrictions on inferences of Λ in a calculus Π, these restrictions are said to determine a *strategy* for proof search in Π.

All the above-said takes place for the clash-resolution method [3] being called the clause calculus below. It deals with clauses and contains the unique inference rule – the latent class-resolution rule. The empty clause plays the role of Λ.

Let us stop on the way of the construction of an initial set of clauses for a formula F (or a set S of such formulas) being investigated on unsatisfiability. First of all, we can consider that F is a closed formula. Further, let us suppose that F already is presented in Skolem functional form (for satisfiability), all the quantifiers of which are omitted. Then, under the condition that all its variables implicitly are bound by the universal quantifier, the following question is reasonable: Can we refrain from the obligatory presentation F (or S) as a set of clauses and develop a technique similar to the resolution one? Research in this direction is described in what follows. At that, note that the papers [4] and [5] are a starting point for the development of the approach presented here.

We usually give references to the original papers, which laid the foundations for the research in a particular direction, although for the modern description of most of them, one can turn to [6] or [7]. QED indicates the end of any proof.

2 Preliminaries

First-order classical logic with functional symbols and equality is considered.

The notions of terms, atomic formula, and formulas are assumed to be known. A formula being the result of renaming of variables in a formula F is called a *variant* of F. A *literal* is an atomic formula or its negation. For a literal L of the form $\neg A$, its *complementary* $\tilde{L}$ is A. If L is an atomic formula A, then its *complementary* $\tilde{L}$ is $\neg A$.

As it was said above, we restrict ourselves by the consideration of only closed formulas F presented in Skolem functional form for satisfiability by means of the elimination of positive quantifiers. That is, F may be considered as a formula of the form $\forall x_1 \ldots \forall x_m\ G(x_1, , x_m)$, where $x_1, , x_m$ all the variables of F, and $G(x_1, , x_m)$ a quantifier-free formula. I. e. it can be assumed that in the case of reasoning on satisfiability, one has to deal with only quantifier-free formulas, all variables of which implicitly are universally bound.

We can reduce $G(x_1, , x_m)$ to a formula $D_1 \wedge \ldots \wedge D_n$, where D_i is a formula presented in disjunctive normal form (DNF). As a result, we can make investigation of the set $\{D_1, \ldots, D_n\}$ on unsatisfiability instead of making the appropriate investigation of $G(x_1, , x_m)$. This leads to the following notions.

If $L_1, \ldots, L_m$ are literals, then the expression $L_1 \wedge \ldots \wedge L_m$ ($m \geq 1$) is called a *conjunct*. An expression of the form $C_1 \vee \ldots \vee C_n$, where $C_1, \ldots, C_n$ are conjuncts, is called a *conjunctive clause*, or a *c-clause* ($n \geq 0$).

A c-clause not containing any conjunct (that is, if $n = 0$) is called an *empty clause* (or *empty c-clause*) and denoted by $\Box$.

In what follows, any conjunct is considered to be the set of its literals and any c-clause – the set of its conjuncts. Thus, in the case when any conjunct of a c-clause contains exactly one literal, this c-clause can be considered as a usual *clause* (see, for example, [1] or [6]).

The introduced definitions allow us to use all the semantic notions of first-order classical logic for c-clauses and sets of c-clauses under the assumption that every variable in any c-clause is universally bound. The empty clause is considered to be an unsatisfiable formula.

Our main purpose is to prove that the inferring of $\Box$ in our calculi is equivalent to the unsatisfiability of an initial set of c-clauses.

An *inference* from an initial set S of c-clauses in a calculi under consideration is a sequence $D_1, \ldots, D_n$, where every D_i $(i = 1, \ldots, n)$ is either a variant of an c-clause from S or a variant of a conclusion of a rule applied to some of the c-clauses preceding D_i. Therefore, our calculi *uniquely are identified* by their inference rules. That is why the names of rules will serve as unique names of the calculi under consideration. The *deducibility* of a c-clause C from a set S of c-clauses in a calculus Π is denoted by $S \vdash_{\Pi} C$.

The resolution method first was published in [1] in 1965. It contained the only resolution rule of the arity 2. In [8], J.A.Robinson proposed its modification of this rule under the name of the hyper-resolution. Its further generalization led to the clash-resolution method [3]. The peculiarity of this generalization is that it contains the only latent clash-resolution rule (denoted by RR below) that can be applied to any finite number of clauses. The corresponding clash-resolution method (being the clause calculus with the RR-rule) is sound and complete [3].

Let us give some necessary notations.

A *substitution*, σ, is a finite mapping from variables to terms that has the form $\sigma = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$, where variables $x_1, \ldots, x_n$ are pairwise different and for any i $(1 \leq i \leq n)$, the term t_i is distinct from x_i.

A substitution σ is called a *variant substitution* if t_1, …, t_n from σ are only variables that are pairwise different. In this case, the inverse (one-one) correspondence σ^{-1} exists and presents itself a (variant) substitution.

For an expression Ex and a substitution σ, the result of the application of σ to the expression of Ex is understood in the usual sense; it is denoted by $Ex \cdot \sigma$.

The *composition* of substitutions (as mappings) σ and λ is denoted by $\sigma \cdot \lambda$. It has the property that for any expression Ex, $Ex \cdot (\sigma \cdot \lambda) = (Ex \cdot \sigma) \cdot \lambda$.

For any set Ξ of expressions, $\Xi \cdot \sigma$ denotes the set obtained by the application of σ to each expression in Ξ. If Ξ is a set of (at least two) expressions and $\Xi \cdot \sigma$ a singleton, then σ is called a *unifier* of Ξ. If $\Xi_1, \ldots, \Xi_n$ $(n \geq 1)$ are sets of expressions and for a substitution σ, the set $\Xi_i \cdot \sigma$ is a singleton $(i = 1, \ldots, n)$, then σ is called a *simultaneous unifier* of $\Xi_1, \ldots, \Xi_n$.

It is known (see, for example, [6] or [3]) that in the case the existence of a unifier σ of sets $\Xi_1, \ldots, \Xi_n$, there exist such substitutions λ and σ' that $\Xi_1 \cdot \lambda, \ldots, \Xi_n \cdot \lambda$ are singletons and $\Xi_1 \cdot \sigma = (\Xi_1 \cdot \lambda) \cdot \sigma', \ldots, \Xi_n \cdot \sigma = (\Xi_n \cdot \lambda) \cdot \sigma'$. The substitution λ is unique up to renaming of its variables. It is called the *most general simultaneous unifier (mgsu)* of $\Xi_1, \ldots, \Xi_n$.

Obviously, we can consider that any mgsu σ has the *idempotence property* that means that $\sigma \cdot \sigma = \sigma$. This fact will often be used in what follows implicitly.

Robinson's latent clash-resolution rule (RR). Let clauses $C_0, C_1, \ldots, C_q$ ($q \geq 1$) with mutually distinct variables be of the forms $C'_0 \vee L_{1,1} \ldots \vee L_{1,r_1} \ldots \vee L_{q,1} \vee \ldots \vee L_{q,r_q}$, $C'_1 \vee E_{1,1} \vee \ldots \vee E_{1,p_1}$, ..., $C'_q \vee E_{q,1} \vee \ldots \vee E_{q,p_q}$ respectively, where $C'_0, C'_1, \ldots, C'_q$ are clauses and $L_1, \ldots, L_q, E_{1,1}, \ldots E_{q,p_q}$ literals. Suppose that there exists the mgsu σ of the sets $\{\tilde{L}_{1,1},\ldots,\tilde{L}_{1,r_1},\ E_{1,1},\ldots,\ E_{1,p_1}\}$, ..., $\{\tilde{L}_{q,1},\ldots,\ \tilde{L}_{q,r_q},\ E_{q,1},\ldots,\ E_{q,p_q}\}$. Then the clause $C'_0 \cdot \sigma \vee C'_1 \cdot \sigma \vee \ldots \vee C'_q \cdot \sigma$ is said to be deducible from $C_0, C_1, \ldots, C_q$ by the rule *RR*.

The *RR*-rule with two clauses as its premises will be denoted by RR_2.

The paper [3] contains the following result (see, also, [6]).

Robinson's Proposition. An initial set S of clauses is unsatisfiable if and only if the empty clause $\Box$ is inferred in the RR-calculus.

3 C-clause calculi for logic without equality

Below, we introduce two resolution-type rules in order to define two specific c-clause calculi. These calculi have a number of similar properties. That is why their proofs are detailed only for one of them. As to the other calculus, the corresponding proofs for it can be obtained in the same way.

3.1 CR calculus

Let us start with the consideration of the calculus that is based on the analog of Robinson's rule *RR*.

Clash-resolution (CR). Let c-clauses $D_0, D_1, \ldots, D_q$ ($q \geq 1$) pairwise without common variables be of the forms $D'_0 \vee K_{1,1} \vee \ldots \vee K_{1,r_1} \vee \ldots \vee K_{q,1} \vee \ldots \vee K_{q,r_q}$, $D'_1 \vee M_{1,1} \vee \ldots \vee M_{1,p_1}$, ..., $D'_q \vee M_{q,1} \vee \ldots \vee M_{q,p_q}$ respectively, where $D'_0, \ldots, D'_q$ are c-clauses and $K_{1,1}, \ldots, K_{q,r_g}, M_{1,1}, \ldots M_{q,p_q}$ conjuncts. Suppose that $K_{1,1}, \ldots, K_{q,r_q}$ contain literals $L_{1,1}, \ldots, L_{q,r_q}$ respectively and for every $j = 1, \ldots, q$, $M_{j,1},\ldots M_{j,p_j}$ contain literals $E_{j,1},\ldots E_{j,p_j}$ respectively such that there exists the mgsu σ of the sets $\{\tilde{L}_{1,1},\ldots,\tilde{L}_{1,r_1},\ E_{1,1},\ldots,\ E_{1,p_1}\}$, ..., $\{\tilde{L}_{q,1},\ldots,\ \tilde{L}_{q,r_q},\ E_{q,1},\ldots,\ E_{q,p_q}\}$. Then the c-clause $D'_0 \cdot \sigma \vee D'_1 \cdot \sigma \vee \ldots \vee D'_q \cdot \sigma$ is said to *be inferred* from the *nucleus* D_0 and *electrons* $D_1, \ldots, D_q$ by the *CR-rule*. Besides, the q-tuple $\langle D_0,\ D_1, \ldots,\ D_q\rangle$ is called a *CR-clash* and $D'_0 \cdot \sigma \vee D'_1 \cdot \sigma \vee \ldots \vee D'_q \cdot \sigma$ its *CR-resolvent*.

Remark. If $D_0, D_1, \ldots, D_q$ are only clauses, the definitions of CR and RR are coincides, which gives a simple way for proving some results relating to CR.

Proposition 1. *The CR-rule is sound.*

Proof. Since we implicitly consider every variable in any c-clause to be bound by the universal quantifier, obviously it is enough to prove that a CR-resolvent is the logical conclusion of its premises only in the propositional case. For this, it is enough to check the validity of the propositional formula:

$((D'_0 \vee (\tilde{L}_{1,1} \wedge K'_{1,1}) \vee \ldots \vee (\tilde{L}_{1,1} \wedge K'_{1,r_1}) \vee \ldots \vee (\tilde{L}_{q,1} \wedge K'_{q,1}) \vee \ldots \vee (\tilde{L}_{q,1} \wedge K'_{q,r_q})) \wedge (D'_1 \vee (L_{1,1} \wedge M'_{1,1}) \vee \ldots \vee (L_{1,1} \wedge M'_{1,p_1})) \wedge \ldots \wedge (D'_q \vee (L_{q,1} \wedge M'_{q,1}) \vee \ldots \vee (L_{q,1} \wedge M'_{q,p_q}))) \supset (D'_0 \vee D'_1 \ldots \vee D'_q)$, where $\supset$ is the implication symbol, which can be made by applying induction on q. QED.

Let a c-clause D distinguished from $\Box$ be of the form $K_1 \vee \ldots \vee K_n$. Then $\rho(D)$ is the set $\{L_1 \vee \ldots \vee L_n : L_1$ occurs in $K_1, \ldots, L_n$ occurs in $K_n\}$.

For $\Box$, we suppose that $\rho(\Box)$ contains $\Box$ and only it.

If S is a set of c-clauses, then $\rho(S)$ denotes the set $\bigcup_{D \in S} \rho(D)$.

It is obvious that for any non-empty set S of c-clauses, $\rho(S)$ is a finite non-empty set and contains only clauses. Moreover, considering D as a formula, we can produce $\rho(D)$ my means of applying the following propositional tautology: $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$, where $\equiv$ is the logical equivalence symbol. Therefore, *a set S is unsatisfiable if and only if $\rho(S)$ is an unsatisfiable set.*

Remark. According to the previous remark, we conclude that Robinson's clash-resolution technique is used when we are interested in the establishing of the deducibility of $\Box$ from $\rho(S)$ in the CR-calculus. Thus, for any finite set S of c-clauses, it is true that S is unsatisfiable if and only if $\rho(S) \vdash_{CR} \Box$.

Lemma 1. *Let $D_0, D_1, \ldots, D_q$ be c-clauses and $A_0, A_1, \ldots, A_q$ clauses such that $A_0 \in \rho(D_0)$, $A_1 \in \rho(D_1)$, $\ldots$, $A_q \in \rho(D_q)$. If for $A_0, A_1, \ldots, A_q$, there is the CR-clash $\langle A_0, A_1, \ldots, A_q \rangle$ with A_0 as a nucleus and $A_1, \ldots, A_q$ as electrons and A is its CR-resolvent, then there exists the CR-clash $\langle D_0, D_1, \ldots, D_q \rangle$ with D_0 as a nucleus, $D_1, \ldots, D_q$ as electrons, and D as its CR-resolvent such that $A \in \rho(D)$.*

Proof. Let us consider $A_0, A_1, \ldots, A_q$ from the lemma conditions. Let they be of the form: $B_0 \vee L_{1,1} \vee \ldots \vee L_{1,r_1} \vee \ldots \vee L_{q,1} \vee \ldots \vee L_{q,r_q}$, $B_1 \vee E_{1,1} \vee \ldots \vee E_{1,p_1}$, $\ldots$, $B_q \vee L_{q,1} \vee \ldots \vee L_{q,p_q}$ respectively, where B_0, $\ldots$, B_q are clauses and $L_{1,1}$, $\ldots$, L_{q,r_q}, $E_{1,1}$, $\ldots$ E_{q,p_q} literals such that there exists the mgsu σ of the sets $\{\tilde{L}_{1,1}, \ldots, \tilde{L}_{1,r_1}, E_{1,1}, \ldots, E_{1,p_1}\}$, $\ldots$, $\{\tilde{L}_{q,1}, \ldots, \tilde{L}_{q,r_q}, E_{q,1}, \ldots, E_{q,p_q}\}$. Then the clause $B_0 \cdot \sigma \vee B_1 \cdot \sigma \vee \ldots \vee B_n \cdot \sigma$ is a CR-resolvent of the above-given CR-clash.

Since $A_0 \in \rho(D_0)$, D_0 can be presented in the form $D'_0 \vee K_{1,1} \vee \ldots \vee K_{1,r_1} \vee \ldots \vee K_{q,1} \vee \ldots \vee K_{q,r_q}$, where D'_0 is a c-clause and $K_{1,1}$, $\ldots$, K_{q,r_g} conjuncts such that $B_0 \in \rho(D'_0)$ and $K_{1,1}$, $\ldots$, K_{q,r_g} contain literals $L_{1,1}$, $\ldots$, L_{q,r_q} respectively.

Making reasoning in the similar way, we obtain that D_1, $\ldots$, D_q can be presented in the form $D'_1 \vee M_{1,1} \vee \ldots \vee M_{1,p_1}$, $\ldots$, $D'_q \vee M_{q,1} \vee \ldots \vee M_{q,p_q}$ respectively, where D'_1, $\ldots$, D'_q are c-clauses and $M_{1,1}$, $\ldots$ M_{q,p_q} conjuncts such that $B_1 \in \rho(D'_1)$, $\ldots$, $B_q \in \rho(D'_q)$ and $M_{1,1}$, $\ldots$, M_{q,p_q} contain $E_{1,1}$, $\ldots$ E_{q,p_q}.

In accordance with the definition of CR, this means that $D_0, D_1, \ldots, D_q$ form a clash with D_0 as a nucleus and $D_1, \ldots, D_q$ as electrons. For this CR-clash, $D'_0 \cdot \sigma \vee D'_1 \cdot \sigma \vee \ldots \vee D'_q \cdot \sigma$ is its CR-resolvent. Obviously, $B_0 \cdot \sigma \vee B_1 \cdot \sigma \vee \ldots \vee B_n \cdot \sigma \in \rho(D'_0 \cdot \sigma \vee D'_1 \cdot \sigma \vee \ldots \vee D'_q \cdot \sigma)$. QED.

Proposition 2. *Let S be a set of c-clauses and B'_1, $\ldots$, B'_n an inference of $\Box$ from $\rho(S)$ in the RR-calculus. Then there exists an inference B_1, $\ldots$, B_n of $\Box$*

from S in the CR-calculus such that for every j ($j = 1, \ldots, n$) $B'_j \in \rho(B_j)$ and if B'_j is a variant of a CR-resolvent of the CR-clash $\langle B'_{i_r}, \ldots, B'_{i_1} \rangle$ with B'_{i_r} as its nucleus, then B_j is a variant of a CR-resolvent of the CR-clash $\langle B_{i_r}, \ldots, B_{i_1} \rangle$ with B_{i_r} as its nucleus ($i_1 < \ldots < i_r < j$).

Proof. Let $B'_1, \ldots, B'_n$ be an inference of $\Box$ from $\rho(S)$ in the RR-calculus. It is an inference of $\Box$ from $\rho(S)$ in the CR-calculus

For each $i = 1, \ldots, n$, assign a c-clause B_i to a clause B'_i in the following way.

$j = 1$. The definition of an inference implies that B'_1 is a variant of a clause $C \in \rho(S)$. That is there exists a variant substitution λ such that B'_1 is $C \cdot \lambda$. Hence, we can select such a c-clause D in S that $C \in \rho(D)$. Take $D \cdot \lambda$ as B_1. Obviously, $B'_1 \in \rho(B_1)$.

Suppose that $j > 1$ and we have c-clauses $B_1 \ldots, B_{j-1}$ that pairwise have no common variables and satisfy the conditions: $B'_1 \in \rho(B_1), \ldots, B'_{j-1} \in \rho(B_{j-1})$. Two cases are possible.

(1) B'_j is a variant of a clause $C \in \rho(S)$. Proceeding in the same manner as in the case of $j = 1$, we easily achieve the necessary renaming some of the variables of $D \cdot \lambda$ in order the result B_j has no common variables with $B_1, \ldots, B_{j-1}$.

(2) B'_i is a variant of a CR-resolvent C of a CR-clash $\langle B'_{i_r}, \ldots, B'_{i_1} \rangle$ with B'_{i_r} as its nucleus ($i_1 < \ldots < i_r$). Accordantly to Lemma 1, we can construct the CR-clash $\langle B_{i_r}, \ldots, B_{i_1} \rangle$ with B_{i_r} as its nucleus and D as its CR-resolvent, for which $C \in \rho(D)$.

Let λ be a variant substitution such that B'_{i_r} is $C \cdot \lambda$. Obviously, we can select a variant B of $D \cdot \lambda$ not having common variables with $B_1, \ldots, B_{j-1}$ and satisfying the condition $B'_j \in B$. Denote this B by B_j.

Let us consider $B_1, \ldots, B_n$. Since B'_n is $\Box$ and $\rho(\Box)$ contains only $\Box$, B_n is the empty clause $\Box$. Thus, accordingly to the construction of $B_1, \ldots, B_n$, this sequence is an inference of $\Box$ satisfying the conclusion of the proposition. QED.

Now, it is easy to obtain the soundness and completeness of the CR-calculus.

Theorem 1 *(Soundness and completeness of CR-calculus). A set S of c-clauses is unsatisfiable if and only if $S \vdash_{CR} \Box$.*

Proof. The *soundness* of CR is provided by Prop. 1.

Completeness. If S is an unsatisfiable set of c-clauses, then $\rho(S)$ is an unsatisfiable set of clauses. The calculus RR is complete (Robinson's proposition). Hence, $\rho(S) \vdash_{RR} \Box$. Thus, $S \vdash_{CR} \Box$ on the basis of Prop. 2. QED.

Let us consider an example of a deduction in the CR-calculus. Note that all the examples in the paper are given only for propositional case since the resolution-type technique under consideration uses the usual unification.

Example 1. Let U denote the following set of c-clauses: $\{(A \wedge \neg A) \vee (B \wedge C) \vee (E \wedge L), \neg B \vee \neg C, \neg E \vee \neg L\}$, where A, B, C, E, and L are atomic formulas. The (minimal) inference of $\Box$ from U in CR is as follows:

1. $(A \wedge \neg A) \vee (B \wedge C) \vee (E \wedge L)$ $(\in U)$,
2. $(A \wedge \neg A) \vee (B \wedge C) \vee (E \wedge L)$ $(\in U)$,
3. $(B \wedge C) \vee (E \wedge L)$ (by CR from (1) as a nucleus and (2) as an electron),

4. $(B \wedge C) \vee (E \wedge L)$ (a variant of (3)),
5. $\neg B \vee \neg C$ ($\in U$),
6. $E \wedge L$ (by CR from (5) as a nucleus and (3) and (4) as electrons),
7. $E \wedge L$ (a variant of (6)),
8. $\neg E \vee \neg L$ ($\in U$),
9. $\Box$ (by CR from (8) as a nucleus and (6) and (7) as electrons).

Therefore, the set U is unsatisfiable.

3.2 IR calculus

Maslov's inverse method (denoted by MIM here) and Robinson's resolution method (the calculus of clauses in our terminology) appeared approximately at the same time: MIM – in 1964 [2] and RR – in 1965 [1].

After their appearance, the problem of the interpretation of MIM in the resolution terms has arisen. This problem has attracted the attention of a number of researchers in inference search (see, for example, [11] and [12]) also because MIM was defined as a special calculus of so-called favorable assortments and its description was made in the terms that did not correspond to traditional logical terminology and resolution one applied at that time.

In [11], S. Maslov gave himself some MIM explanation in the resolution notions for a restricted case. Later, after an attentive analysis of MIM, the author of this paper "discovered" that MIM interpretation was preferable to do in the terms of a special c-clause[1] calculus [5], the enough description detailed of which is given below. Also it was found that this calculus has an independent significance. It echoes the CR-calculus and, at the same time, it differs from CR.

Inverse resolution (IR). Let c-clauses $D_0, D_1, \ldots, D_q$ $(q \geq 1)$ pairwise without common variables be of the forms $D'_0 \vee K_1 \vee \ldots \vee K_q$, $D'_1 \vee N^1_{1,1} \vee \ldots \vee N^1_{1,p_{1,1}} \vee \ldots \vee N^{r_1}_{1,1} \vee \ldots \vee N^{r_1}_{1,p_{1,r_1}}$, …, $D'_q \vee N^1_{q,1} \vee \ldots \vee N^1_{q,p_{q,1}} \vee \ldots \vee N^{r_q}_{q,1} \vee \ldots \vee N^{r_q}_{q,p_{n,r_n}}$ respectively, where $D'_0, \ldots, D'_q$ are c-clauses and $K_1, \ldots, K_q$, $N^1_{1,1}, \ldots, N^{r_q}_{q,p_{n,r_n}}$ conjuncts. Suppose that for every j $(1 \leq j \leq q)$, K_j contains literals $L_{j,1}, \ldots, L_{j,r_j}$ and $N^1_{j,1}, \ldots, N^1_{j,p_{j,1}}, \ldots, N^{r_j}_{j,1}, \ldots, N^{r_j}_{j,p_{j,r_j}}$ contain literals $E^1_{j,1}, \ldots, E^1_{j,p_{j,1}}, \ldots, E^{r_j}_{j,1}, \ldots, E^{r_j}_{j,p_{j,r_j}}$ respectively such that there exists the mgsu σ of the sets $\{\tilde{L}_{1,1}, E^1_{1,1}, \ldots, E^1_{1,p_{1,1}}\}, \ldots, \{\tilde{L}_{1,r_1}, E^{r_1}_{1,1}, \ldots, E^{r_1}_{1,p_{1,r_1}}\}, \ldots, \{\tilde{L}_{q,1}, E^1_{q,1}, \ldots, E^1_{q,p_{q,1}}\}, \ldots, \{\tilde{L}_{q,r_q}, E^{r_q}_{q,1}, \ldots, E^{r_q}_{q,p_{q,r_q}}\}$. Then the c-clause $D'_0 \cdot \sigma \vee D'_1 \cdot \sigma \vee \ldots \vee D'_q \cdot \sigma$ is said to *be inferred* from the *nucleus* D_0 and *electrons* $D_1, \ldots, D_q$ by the IR-*rule*. Besides, the q-tuple $\langle D_0, D_1, \ldots, D_q \rangle$ is called its *IR-clash* and $D'_0 \cdot \sigma \vee D'_1 \cdot \sigma \vee \ldots \vee D'_q \cdot \sigma$ its *IR-resolvent*.

Having the *IR*-rule, we can speak about the IR-calculus.

[1] In 1989, V. Lifschitz independently introducing the notion of a c-clause under the name of a super-clause improved such interpretation [13]. In [14], T. Bollinger extended Loveland's model elimination method [15] to the case of c-clauses using the name of a generalized clause for a c-clause.

The comparative analysis of IR and CR shows that the only difference between them is in the ways of the selection of cutting literals for their applications. The following statement contains a more detailed explanation of this observation.

Lemma 2. *If $\langle D_0, D_1, \ldots, D_q\rangle$ is a CR-clash with D_0 an its nucleus and $D_1, \ldots, D_q$ as its electrons, then for any its CR-resolvent D, it is possible to construct an IR-clash with D_0 as its nucleus and certain variants of $D_1, \ldots, D_q$ as its electrons such that for its some IR-resolvent D' and a substitution τ, $D = D' \cdot \tau$.*

Proof. If $\langle D_0, D_1, \ldots, D_q\rangle$ is the CR-clash from the definition of CR-rule, then the c-clauses $D_0, D_1, \ldots, D_q$ can be presented as $D'_0 \vee K_{1,1} \vee \ldots \vee K_{1,r_1} \vee \ldots \vee K_{q,1} \vee \ldots \vee K_{q,r_q}$, $D'_1 \vee M_{1,1} \vee \ldots \vee M_{1,p_1}$, $\ldots$, $D'_q \vee M_{q,1} \vee \ldots \vee M_{q,p_q}$ respectively, where $D'_0, \ldots, D'_q$ are c-clauses and $K_{1,1}, \ldots, K_{q,r_g}, M_{1,1}, \ldots M_{q,p_q}$ conjuncts and moreover for literals $L_{1,1}, \ldots, L_{q,r_q}, E_{j,1}, \ldots E_{j,p_j}$ from $K_{1,1}, \ldots, K_{q,r_g}, M_{1,1}, \ldots M_{q,p_q}$ respectively, there exists the mgsu σ of the sets $\Theta_1 = \{\tilde{L}_{1,1}, \ldots, \tilde{L}_{1,r_1}, E_{1,1}, \ldots, E_{1,p_1}\}, \ldots, \Theta_q = \{\tilde{L}_{q,1}, \ldots, \tilde{L}_{q,r_q}, E_{q,1}, \ldots, E_{q,p_q}\}$ such that $D = D'_0 \cdot \sigma \vee D'_1 \cdot \sigma \vee \ldots \vee D'_q \cdot \sigma$.

Let us take such variant substitutions $\lambda_{1,1}, \ldots, \lambda_{1,r_1}, \ldots, \lambda_{q,1}, \ldots, \lambda_{q,r_q}$ that $D_1 \cdot \lambda_{1,1}, \ldots, D_1 \cdot \lambda_{1,r_1}, \ldots, D_q \cdot \lambda_{q,1} \ldots D_q \cdot \lambda_{q,r_q}$ have no common variables with D_0 and each other. Considering $\lambda^{-1}_{1,1}, \ldots, \lambda^{-1}_{q,r_q}$ as mapping graphs, construct the set $\lambda^{-1}_{1,1} \cup \ldots \cup \lambda^{-1}_{q,r_q}$. Obviously, it is a (variant) substitution. Let us denote it by μ and the c-clause $D'_j \cdot \lambda_{j,k} \vee M_{j,1} \cdot \lambda_{j,k} \vee \ldots \vee M_{j,p_j} \cdot \lambda_{j,k}$ by D^k_j.

Let us consider $D^1_1, \ldots, D^{r_1}_1, \ldots, D^1_q, \ldots, D^{r_q}_q$. Accordantly to their definition and the definition of μ, we have that $D^k_j \cdot \mu$ is the same as $D^k_j \cdot \lambda^{-1}_{j,k}$ and, therefore, it is the same as D_j ($j = 1, \ldots, q$; $k = 1, \ldots, r_j$). Thus, we can select literals $E^1_{1,1}, \ldots, E^1_{1,p_1}, \ldots, E^{r_1}_{1,1}, \ldots, E^{r_1}_{1,p_1}, \ldots, E^1_{q,1}, \ldots, E^1_{q,p_q}, \ldots, E^{r_q}_{q,1}, \ldots, E^{r_q}_{q,p_q}$ in $M_{1,1} \cdot \lambda_{1,1}, \ldots, M_{1,p_1} \cdot \lambda_{1,1}, \ldots, M_{1,1} \cdot \lambda_{1,r_1}, \ldots, M_{1,p_1} \cdot \lambda_{1,r_1}, \ldots, M_{q,1} \cdot \lambda_{q,1}, \ldots, M_{q,p_q} \cdot \lambda_{q,1}, \ldots, M_{q,1} \cdot \lambda_{q,r_q}, \ldots, M_{q,p_q} \cdot \lambda_{q,r_q}$ respectively, such that $E^k_{i,j} \cdot \lambda^{-1}_{i,k} = E^k_{i,j} \cdot \mu = E_{i,j}$ ($i = 1, \ldots, q$; $j = 1, \ldots, p_q$; $k = 1, \ldots, r_q$).

Considering σ and μ as mapping graphs, we conclude that $\zeta = \mu \cdot \sigma \cup \sigma$ is a substitution. Because σ is the mgsu of the sets $\Theta_1, \ldots, \Theta_q$, the definition of ζ and the idempotence of σ imply that ζ is a simultaneous unifier of the sets of literals $\{\tilde{L}_{1,1}\ E^1_{1,1}, \ldots, E^1_{1,p_1}\}, \ldots, \{\tilde{L}_{1,r_1}\ E^{r_1}_{1,1}, \ldots, E^{r_1}_{1,p_1}\}, \ldots, \{\tilde{L}_{q,1}\ E^1_{q,1}, \ldots, E^1_{q,p_q}\}, \ldots, \{\tilde{L}_{q,r_q}\ E^{r_q}_{q,1}, \ldots, E^{r_q}_{j,p_q}\}$. Therefore, there exists the mgsu θ of these sets, for which $\zeta = \theta \cdot \tau$, where τ is a substitution.

As a result, we have that $D_0, D^1_1, \ldots, D^{r_1}_1, \ldots, D^q_1, \ldots, D^{r_q}_q$ can form the IR-clash with D_0 as its nucleus and $D^1_1, \ldots, D^{r_1}_1, \ldots, D^q_1, \ldots, D^{r_q}_q$ as its electrons that produces the IR-resolvent $D' = D'_0 \cdot \theta \vee D'_1 \cdot (\lambda_{1,1} \cdot \theta) \vee \ldots \vee D'_1 \cdot (\lambda_{1,r_1} \cdot \theta) \vee \ldots \vee D'_q \cdot (\lambda_{q,1} \cdot \theta) \vee \ldots \vee D'_q \cdot (\lambda_{q,r_q} \cdot \theta)$.

Since $\theta \cdot \tau = \zeta$ and $\zeta = \mu \cdot \sigma \cup \sigma$, it is obvious that $D' \cdot \tau = D$. QED.

This result permits to "simulate" any inference in CR by an inference in IR.

Proposition 3. *Let S be a set of c-clauses and $B_1, \ldots, B_n$ an inference of $\Box$ from S in the CR-calculus. Then there exists an inference $B'_1, \ldots, B'_m$ of $\Box$ from S in the IR-calculus ($m \geq n$) such that if B_j is a variant of a CR-resolvent of an CR-clash with B_r as its nucleus, then for some j' and r' ($j' \geq j, r' \geq r$),*

$B'_{j'}$ *is a variant of an* IR*-resolvent of the corresponding* IR*-clash with* $B'_{r'}$ *as its nucleus; moreover,* $B_j = B'_{j'} \cdot \tau$ *for some substitution* τ.

Proposition 4. *The* IR*-rule is sound.*

Proof. As in the case of the CR-rule, it is enough to establish the validity of the following formula, "extracted" from the definition of IR-rule:

$(D'_0 \vee (\tilde{L}_{1,1} \wedge \ldots \wedge \tilde{L}_{1,r_1} \wedge K'_1) \vee \ldots \vee (\tilde{L}_{q,1} \wedge \ldots \wedge \tilde{L}_{q,r_q} \wedge K'_q)) \wedge (D'_1 \vee (L_{1,1} \wedge M^1_{1,1}) \vee \ldots \vee (L_{1,1} \wedge M^1_{1,p_{1,1}}) \vee \ldots \vee (L_{1,r_1} \wedge M^{r_1}_{1,1}) \vee \ldots \vee (L_{1,r_1} \wedge M^{r_1}_{1,p_{1,r_1}})) \wedge \ldots \wedge (D'_q \vee (L_{q,1} \wedge M^1_{q,1}) \vee \ldots \vee (L_{q,1} \wedge M^1_{q,p_{q,1}}) \vee \ldots \vee (L_{q,r_q} \wedge M^{r_q}_{q,1}) \vee \ldots \vee (L_{q,r_q} \wedge M^{r_q}_{q,p_{q,r_q}}))) \supset (D'_0 \vee D'_1 \ldots \vee D'_q)$. QED.

Theorem 2 *(Soundness and completeness of IR-calculus). A set* S *of c-clauses is unsatisfiable if and only if* $S \vdash_{IR} \Box$.

Proof. The *soundness* is provided by Prop. 4.

Completeness. If S is unsatisfiable set, then $S \vdash_{IR} \Box$ by Theorem 1. By Prop. 3, any inference of $\Box$ from S in CR can be transformed into an inference of $\Box$ from S but already in the IR-calculus, that is $S \vdash_{IR} \Box$. QED.

Example 2. Let us consider the set U from Example 1 and construct the (minimal) inference of $\Box$ from U in IR is as follows:

1. $(A \wedge \neg A) \vee (B \wedge C) \vee (E \wedge L)$ $(\in U)$,
2. $(A \wedge \neg A) \vee (B \wedge C) \vee (E \wedge L)$ $(\in U)$,
3. $(B \wedge C) \vee (E \wedge L)$ (by IR from (1) ac a nucleus and (2) as an electron),
4. $\neg B \vee \neg C$ $(\in U)$,
5. $\neg E \vee \neg L$ $(\in U)$,
6. $\Box$ (by IR from (3) as a nucleus and (4) and (5) as electrons).

We have again proved the unsatisfiability of U.

Draw your attention to the fact that this inference in IR is shorter than the inference in CR from Example 1. This situation is more or less standard for these calculi (see the section containing a comparison of CR and IR).

4 C-clause calculi for logic with equality

The CR- and IR-calculi admit equality handling based on a modification of the paramodulation rule that was proposed in [9] for inference search in first-order theories with equality (denoted by $\simeq$).

We are needed in the following notions that provide us with a possibility to reduce the establishing of the validity of the first-order statement with equality to the search of the refutation of a certain set of c-clauses.

Let S be a set of c-clauses. Then $S^{\simeq}$ denotes the *set of equality axioms* for S in the form of clauses, in which $x, y, z, x_0, \ldots, x_p$ are variables (see, for example, [6]): consists of the following (1) $x \simeq x$, (2) $x \not\simeq y \vee y \simeq x$, (3) $x \not\simeq y \ \vee \ y \not\simeq z \ \vee \ x \simeq z$, (4) $x_i \not\simeq x_o \vee \tilde{R}(x_1, \ldots, x_i, \ldots, x_p) \vee R(x_1, \ldots, x_0, \ldots, x_p)$ for each p-arity predicate symbol R occurring in S and for each $i = 1, 2, \ldots, p$, (5) $x_i \not\simeq x_o$

$\vee\ f(x_1,\ldots,x_i,\ldots,x_p) \simeq f(x_1,\ldots,x_0,\ldots,x_p)$ for each p-arity function symbol f occurring in S and for each $i = 1, 2, \ldots, p$.

A set S of c-clauses is called *equationally unsatisfiable* if and only if the set $S \cup S^{\simeq}$ is unsatisfiable.

Thus, in the case when we have deals with S requiring equality handling, we must establish the equationally unsatisfiability of the set S, which can be achieved by deducing the empty clause $\Box$ from $S \cup S^{\simeq}$. But such approach leads to the extreme large growth of the searching space. For the optimization of such growth, we use a modification of the paramodulation rule [9].

Paramodulation rule PP. Let we have two c-clauses D and $D' \vee (K \wedge s \simeq t)$, where D' is a c-clause and K conjunct (possibly, empty). If there exists mgsu σ of the set of terms $\{s, u\}$, where u is a term occurring in D at a selected position, then the c-clause $D' \cdot \sigma \vee (D \cdot \sigma)[t \cdot \sigma]$ is said to *be inferred* from these c-clauses *by the rule PP*, where $(D \cdot \sigma)[t \cdot \sigma]$ denotes the result of replacing in $D \cdot \sigma$ the term $u \cdot \sigma$ being at the selected position by $t \cdot \sigma$. At that, the ordered pair $\langle\ D,\ D' \vee (K \wedge s \simeq t)\ \rangle$ is called a *PP-clash* (w.r.t. $s \simeq t$) with the *PP-paramodulant* $D' \cdot \sigma \vee (D \cdot \sigma)[t \cdot \sigma]$, *nucleus* D, and *electron* $D' \vee (K \wedge s \simeq t)$.

The *set* S^f *of functionally reflexive axioms* for a set S of c-clauses consists of all the clauses of the form $f(x_1,\ldots,x_p) \simeq f(x_1,\ldots,x_p)$, where f is a p-arity function symbol occurring in S.

Adding PP to the CR- and IR-calculi, we get the calculi CR+PP and IR+PP intended for inference search in first-order classical logic with equality.

Remark. If in the above-given definition, PP is applied to only clauses, we have the usual paramodulation rule from [9] being denoted by P here.

Because of the completeness of the inference system "negative hyper-resolution + paramodulation" (see, for example, [6]), the following result takes place on the basis that a set S of c-clauses is equationally unsatisfiable if and only if $\rho(S)$ is equationally unsatisfiable.

Robinson-Wos's Proposition. *A set S of c-clauses is equationally unsatisfiable if and only if* $\rho(S) \cup \{x = x\} \cup S^f \vdash_{RR+P} \Box$.

Taking into account the well-known result [10] about the completeness of the system "resolution + paramodulation" without using functionally reflexive axioms, we obtain the further reinforcement of Robinson-Wos's Proposition.

Corollary. *A set S of c-clauses is equationally unsatisfiable if and only if* $\rho(S) \cup \{x = x\} \vdash_{RR+P} \Box$. *Moreover, RR can denote the only binary rule.*

Now, we have all the necessary for obtaining the results about the completeness of the calculi CR+PP and IR+PP.

First of all, the following analog of Lemma 1 for the PP-rule is obvious.

Lemma 3. *Let D and $D' \vee (K \wedge s \simeq t)$ are c-clauses from the definition of PP. If for $C \in \rho(D)$ and $C' \in \rho(D')$, there exists the PP-clash $\langle C,\ C' \vee s \simeq t\rangle$ w.r.t. $s \simeq t$ with a PP-paramodulant A, then there exists the PP-clash $\langle D,$ $D' \vee (K \wedge s \simeq t)\rangle$ w.r.t. $s \simeq t$ with such a PP-paramodulant B that $A \in \rho(B)$.*

Using this lemma and Prop. 2 and 3, it is easy to obtain the following result.

Proposition 5. *Let S be a set of c-clauses and $B'_1, \ldots, B'_n$ an inference of $\Box$ from $\rho(S) \cup \{x = x\} \cup S^f$ in the calculus RR+P. Then there exists an inference $B_1, \ldots, B_n$ of $\Box$ from $S \cup \{x = x\} \cup S^f$ in the calculus CR+PP (IR+PP) such that: (1) if B'_j is a variant of a resolvent of an RR-clash with B'_r as its nucleus, then for some j' and r', $B_{j'}$ is a variant of a resolvent of the corresponding CR-clash (IR-clash) with $B_{r'}$ as its nucleus and, additionally, $B'_r \in \rho(B_{r'} \cdot \tau)$ for some substitution τ; (2) if B'_j is a variant of a paramodulant of a PP-clash with B'_r as its nucleus, then for some j' and r', $B_{j'}$ is a variant of a paramodulant of the PP-clash with $B_{r'}$ as its nucleus and $B'_r \in \rho(B_{r'} \cdot \tau)$ for some substitution τ.*

This proposition, in fact, guarantees the completeness of the *paramodulation extensions* of the CR- and IR-calculi as well as their methods and strategies, some of which are given in the next section. Note that the *soundness* of such extensions is provided by Prop. 1 and 4 and the obvious fact that PP-paramodulant is a logical conclusion of the conjunction of all the c-clauses from $\{N, E\} \cup \{N, E\}^{\simeq}$, where N is a nucleus and E an electron of a PP-rule application.

Theorem 3 *(Soundness and completeness of CR+PP and IR+PP). A set S of c-clauses is equationally unsatisfiable if and only if $S \cup \{x = x\} \vdash_{IR+PP} \Box$ ($S \cup \{x = x\} \vdash_{CR+PP} \Box$). Moreover, CR (IR) can be the only binary rule.*

Proof. The *soundness* of CR+PP and IR+PP is provided by the remark in the preceding paragraph. *Completeness* takes place for CR+PP and IR+PP due to Corollary and Prop. 5. The completeness of CR+PP with the binary CR-rule is obvious. For proving the completeness of IR+PP with the binary IR-rule, it is enough to note that any binary application of CR can be "decomposed" into r_1 binary applications of IR (see the proof of Lemma 2 for the binary case). QED.

5 Methods and strategies for CR and IR

Prop. 5 gives a simple way for transferring most part of the methods and strategies taking place for the usual clash-resolution (RR) to the ones for the CR- and IR-calculi for classical logic both with and without equality. For the demonstration of how it is possible to do, let us consider the usual liner resolution and positive and negative hyper-resolutions in their wording from [6].

Note that they are given for logic with equality. To obtain them for the case without equality, it is enough to delete all parts concerning the PP-rule in the definitions and wordings of the theorems given below. Also note that their *soundness* is provided by the soundness of the rules CR, IR, and PP. That is why a soundness proof is absent in corresponding theorems.

Linear strategy for CR_2+PP and IR_2+PP. It permits to apply CR_2 (IR_2) or PP to the pair of c-clauses when beginning with the second rule application in an inference, any its c-clause is either a CR_2-resolvent (IR_2-resolvent) or PP-paramodulant of the previous application of the rule CR_2 (IR_2) or PP, and the other c-clause is a variant of either a c-clause from an initial set S of c-clauses or a c-clause that was deduced earlier.

Theorem 4 *(Soundness and completeness of linear strategy for CR_2+PP and IR_2+PP). A set S of c-clauses is equationally unsatisfiable if and only if there exists an inference of $\Box$ from $S \cup \{x = x\} \cup S^f$ satisfying to the linear strategy for CR_2+PP (IR_2+PP).*

Proof. Completeness takes place due to the completeness of the usual linear resolution with paramodulation [6], Robinson-Wos's Proposition, and Prop. 5. QED.

Positive and negative hyper-resolution for CR_2+PP (IR_2+PP). An atomic formula is called a *positive* literal. A literal of the form $\neg A$, where A is an atomic formula, is a *negative* one.

A c-clause is called a *positive* (*negative*) if each its conjunct contains at least one positive (negative) literal. Note that there are c-clauses being positive and negative at the same time, for example, $\neg A \wedge A$.

A CR- or IR-clash $\langle D_0, D_1, \ldots, D_q \rangle$ with D_0 as a nucleus and $D_1, \ldots, D_q$ as electrons is called *positive* (*negative*), if $D_1, \ldots, D_q$ are positive (negative) c-clauses and the cut literals $L_{j,k}$ in the definitions of CR or IR respectively are negative (positive).

For logic without equality, *positive (negative) hyper-resolution strategy* for CR and IR permits constructing inferences containing only the positive (negative) hyper-resolution clashes with positive (negative) CR- or IR-resolvents.

In the case of logic with equality, we additionally permit to apply the PP-rule only to positive nucleus and electron; moreover, a literal containing the selected occurrence of the term u (see the definition of PP-rule) must be positive.

Theorem 5 *(Soundness and completeness of positive and negative hyper-resolutions with PP-rule). A set S of c-clauses is equationally unsatisfiable if and only if there exists an inference of $\Box$ from $S \cup \{x = x\} \cup S^f$ satisfying to the positive and negative hyper-resolution with CR- (IR-) and PP-rules.*

Proof. Completeness. Since there exists an inference of $\Box$ from $\rho(S) \cup \{x = x\} \cup S^f$ satisfying to the usual positive (negative) hyper-resolution and paramodulation (see [6]), this inference can be transformed into an inference of $\Box$ from $S \cup \{x = x\} \cup S^f$ satisfying to the positive and negative hyper-resolution with CR (IR) and PP on the basis of Robinson-Wos's Proposition and Prop. 5. QED.

Remark. In Theorems 9 and 10, the adding of functionally reflexive axioms to the set S is the necessary condition for completeness. Examples demonstrating this for clauses (when CR, IR, and RR are coincided) can be found in [6].

6 IR calculus and Maslov's inverse method

Below, we give the description of MIM in the form of a special strategy for IR.

Maslov's inverse method deals with so-called favorable assortments. In this connection, we consider MIM as a calculus of favorable assortments that has two inference rule: A and B. The A rule determines an initial set of favorable assortments, while the B rule produces new favorable assortments from the already deduced ones. That is why we treat assortments as clauses and favorable assortments as favorable clauses being produced by the α and β rules (see below).

If C is a conjunct $L_1 \wedge \ldots \wedge L_r$, where $L_1, \ldots, L_r$ are literals, then $\tilde{C}$ denotes the clause $\tilde{L}_1 \vee \ldots \vee \tilde{L}_r$.

Rule α. Let S be a set of c-clauses and $S^d = \{\tilde{C} : C$ is a conjunct from a c-clause belonging to $S\}$. If $S^\alpha = \{C : C = C' \cdot \sigma \vee C'' \cdot \sigma$, where C', $C'' \in S^d$ and C' and C'' contain literals L and L' respectively such that there exists the mgsu σ of $\{\tilde{L}, L'\}\}$, then any clause from S^α is called a *favorable* one deduced from S by the α-*rule*.

Obviously, S^α is a finite set if S is the same. Besides, each its (favorable) clause contains both a literal and its complementary. That is why S is a unsatisfiable set of c-clauses if and only if the set $S \cup S^\alpha$ is unsatisfiable.

Rule β. Let S be a set of c-clauses, $D \in S$, D consists of q conjuncts, and C_1, …, C_q be favorable clauses. If the IR-rule can be applied to D as a nucleus and C_1, …, C_q as electrons, than the IR-resolvent of this application is called a *favorable* clause that is deducible from D, C_1, …, C_q by the β-*rule*.

Note that the requirement that the number of conjuncts in D ids equal to q leads to the fact that any IR-resolvent of β-rule is a clause.

In these terms, MIM presents itself the following strategy for IR-calculus called a MIM-*strategy*: First of all, we produce all the possible favorable clauses applying the α-rule; then, we apply only the β-rule attempting to deduce $\Box$.

The soundness of the MIM-strategy provides the soundness of IR-rule and the above-given remark about $S \cup S^\alpha$. As to completeness, the proof of it is omitted here; we simply give the rewording of the main result for MIM from [2].

Theorem 6 *(Soundness and completeness of MIM-strategy). A set S of c-clauses that pairwise have no common variables is unsatisfiable if and only if there exists an inference of $\Box$ from S satisfying to the MIM-strategy.*

This result seams unexpected because of the requirement that D from the definition of the β-rule must consist of exact q conjuncts. This apparent contradiction is explained by the fact that when using the MIM-strategy, we construct S^α containing clauses, the usage of which in an application of the β-rule can be considered as a "latent" way for reducing the number of electrons.

The below-given example demonstrates some of the features of inferences satisfying to the MIM-strategy.

Example 3. It is easy to see that for U from Example 1, $U^d = \{\neg A \vee A, \neg B \vee \neg C, \neg E \vee \neg L, B, C, E, L\}$. As a result, $U^\alpha = \{\neg A \vee A \vee \neg A \vee A, \neg B \vee \neg C \vee B, \neg B \vee \neg C \vee C, \neg E \vee \neg L \vee E, \neg E \vee \neg L \vee L\}$. We have the following MIM-inference:

1. $(A \wedge \neg A) \vee (B \wedge C) \vee (E \wedge L)$ $(\in U)$,
2. $\neg B \vee \neg C$ $(\in U)$,
3. $\neg E \vee \neg L$ $(\in U)$,
4. $\neg A \vee A \vee \neg A \vee A$ (by α-rule),

5. $\neg B \vee \neg C \vee B$ (by α-rule),
6. $\neg B \vee \neg C \vee C$ (by α-rule),
7. $\neg E \vee \neg L \vee E$ (by α-rule),
8. $\neg E \vee \neg L \vee L$ (by α-rule),
9. $B \vee E$ (by β-rule from (1) as a nucleus and (4), (5), and (7) as electrons),
10 . $C \vee E$ (by β-rule from (1) as a nucleus and (4), (6), and (7) as electrons),
11. E (by β-rule from (2) as a nucleus and (9) and (10) as electrons),
12. $B \vee L$ (by β-rule from (1) as a nucleus and (4), (5), and (8) as electrons),
13 . $C \vee L$ (by β-rule from (1) as a nucleus and (4), (6), and (8) as electrons),
14. L (by β-rule from (2) as a nucleus and (12) and (13) as electrons),
15. $\Box$ (by β-rule from (3) as a nucleus and (11) and (14) as electrons).

We have proved the unsatisfiability of U at the 3rd time.

7 Comparison of CR- and IR-calculi

One can see that the obtained results on the CR- and IR-calculi "echo" each other. In this connection, it is interesting to know is there any advantages of one of them over the other? Moreover that Prop. 3 states that any inference of $\Box$ in CR can be simulated by an inference of $\Box$ in IR with the same number of rule applications. This section contains an answer on this question when comparison is made w.r.t. inferences being minimal on the number of rule applications.

By $\psi(\Pi, \Delta, S)$, denote the number all the c-clauses in an inference Δ of a c-clause C from a set S in a calculus Π that are deduced by different rule applications. The inference Δ is *minimal on the number of rule applications* if for any other inference Δ' of a variant of C from S in Π, the inequality $\psi(\Pi, \Delta, S) \leq \psi(\Pi, \Delta', S)$ holds.

Let Δ denote an inference of $\Box$ from S in CR. Using Prop. 3, it is easy to construct an inference Γ of $\Box$ from S in IR such that $\psi(IR, \Gamma, S) \leq \psi(CR, \Delta, S)$. Thus, in the case when Δ_{min} and Γ_{min} denotes the minimal inferences on the introduced characteristic, we have that $\psi(CR, \Delta_{min}, S) - \psi(IR, \Gamma_{min}, S) \geq 0$.

Let us make an attempt to find an upper bound for this difference restricting us by the case when an initial set S contains only c-clauses without variables.

Let us consider an application of IR-rule to a nucleus c-clause D_0 and electron clauses $D_1, \ldots, D_n$ $(n \geq n)$ with an IR-resolvent D. Its attentive analysis demonstrates that this $(n+1)$-arity application can be slitted into n binary applications of CR-rule in the following way: first we make a binary application of CR to D_0 and D_1, then to an obtained CR-resolvent and D_2, and so on. That is we can split any $n+1$-arity IR-application into n binary CR-applications in such a way that for the result C of such CR-rule applications, C will contain all or some of conjuncts belonging to D.

This observation leads to the following upper bound for the difference given above: $\psi(CR,\, \Delta_{min},\, S) - \psi(IR,\, \Gamma_{min},\, S) \leq \sum(m_i - 2)$, where m_i is the arity of the ith CR-rule application in Δ_{min} and the sum is taken over all of m_i.

To demonstrate that this upper bound is achieved, let us take the sets $S_n = \{(L_1 \wedge E_1) \vee \ldots \vee (L_n \wedge E_n),\, (A_{1,1} \wedge B_{1,1}) \vee \ldots \vee (A_{1,m_1} \wedge B_{1,m_1}) \vee \tilde{L}_1 \vee$

$\tilde{E}_1$, $\tilde{A}_{1,1} \vee \tilde{B}_{1,1} \vee \tilde{L}_1 \vee \tilde{E}_1$, …, $\tilde{A}_{1,m_1} \vee \tilde{B}_{1,m_1} \vee \tilde{L}_1 \vee \tilde{E}_1$, …, $(A_{n,1} \wedge B_{n,1}) \vee \ldots \vee (A_{n,m_n} \wedge B_{n,m_n}) \vee \tilde{L}_n \vee \tilde{E}_n$, $\tilde{A}_{n,1} \vee \tilde{B}_{n,1} \vee \tilde{L}_n \vee \tilde{E}_n$, …, $\tilde{A}_{n,m_n} \vee \tilde{B}_{n,m_n} \vee \tilde{L}_n \vee \tilde{E}_n\}$, where $L_1, \ldots, L_n, E_1, \ldots, E_n, A_{1,1}, \ldots, A_{n,m_n}, B_{1,1}, \ldots, B_{n,m_n}$ are literals.

Below we give an inference $\overline{\Delta}$ of $\Box$ from S_n in the IR-calculus. (Thus, S_n is an unsatisfiable set.)

$\lceil$ $(A_{1,1} \wedge B_{1,1}) \vee \ldots \vee (A_{1,m_1} \wedge B_{1,m_1}) \vee \tilde{L}_1 \vee \tilde{E}_1$ $(\in \overline{S})$,
$|$ $\tilde{A}_{1,1} \vee \tilde{B}_{1,1} \vee \tilde{L}_1 \vee \tilde{E}_1$ $(\in \overline{S})$,
$|$ …
$\lfloor$ $\tilde{A}_{1,m_1} \vee \tilde{B}_{1,m_1} \vee \tilde{L}_1 \vee \tilde{E}_1$ $(\in \overline{S})$,
…
$\lceil$ $(A_{n,1} \wedge B_{n,1}) \vee \ldots \vee (A_{n,m_n} \wedge B_{n,m_n}) \vee \tilde{L}_n \vee \tilde{E}_n$ $(\in \overline{S})$,
$|$ $\tilde{A}_{n,1} \vee \tilde{B}_{n,1} \vee \tilde{L}_n \vee \tilde{E}_n$ $(\in \overline{S})$,
$|$ …
$\lfloor$ $\tilde{A}_{n,m_n} \vee \tilde{B}_{n,m_n} \vee \tilde{L}_n \vee \tilde{E}_n$ $(\in \overline{S})$,
$\lceil$ $(L_1 \wedge E_1) \vee \ldots \vee (L_n \wedge E_n)$ $(\in \overline{S})$,
$|$ $\tilde{L}_1 \vee \tilde{E}_1$ (by IR from the 1st-block c-clauses with the 1st c-clause as a nucleus),
$|$ …
$\lfloor$ $\tilde{L}_n \vee \tilde{E}_n$ (by IR from the nst-block c-clauses with the 1st c-clause as a nucleus),
$\Box$ (by IR from the $(n+1)$st block c-clauses with the 1st c-clause as a nucleus).

Using the ideas from [16], we can prove that $\overline{\Delta}$ is a minimal inference in IR containing $n+1$ rule applications with the arities $m_1+1, \ldots, m_n+1$, and $n+1$.

Now, let us convert $\overline{\Delta}$ into an inference $\overline{\Gamma}$ of $\Box$ from S_n, but already in the CR-calculus in the following way:

For each i ($i = 1, \ldots, n$), let us replace the c-clause $\tilde{L}_i \vee \tilde{E}_i$ by the sequence of c-clauses $(A_{i,2} \wedge B_{i,2}) \vee \ldots \vee (A_{i,m_i} \wedge B_{i,m_i}) \vee \tilde{L}_i \vee \tilde{E}_i$, …, $(A_{i,m_i} \wedge B_{i,m_i}) \vee \tilde{L}_i \vee \tilde{E}_i$, $\tilde{L}_i \vee \tilde{E}_i$ that along with the all c-clauses form the i th block is an inference of $\tilde{L}_i \vee \tilde{E}_i$ in CR. Replace the empty clause $\Box$ by the sequence $(L_2 \wedge E_2) \vee \ldots \vee (L_n \wedge E_n)$, …, … $(L_n \wedge E_n)$, $\Box$, being an inference of $\Box$ in CR since $(L_2 \wedge E_2) \vee \ldots \vee (L_n \wedge E_n)$ is deduced from $(L_1 \wedge E_1) \vee (L_2 \wedge E_2) \vee \ldots \vee (L_n \wedge E_n)$ and $(L_1 \wedge E_1)$ by the CR-rule, …, $(L_n \wedge E_n)$ is deduced from $(L_{n-1} \wedge E_{n-1}) \vee \ldots \vee (L_n \wedge E_n)$ and $(L_{n-1} \wedge E_{n-1})$ by the CR-rule, $\Box$ is deduced from $(L_n \wedge E_n)$ and $(L_n \wedge E_n)$ by CR.

We have that $\overline{\Gamma}$ is an inference of $\Box$ from S_n in CR, for which $\psi(CR, \overline{\Gamma}, S_n) = (\sum_{i=1}^{n} m_i) + (n+1)$. Again using the ideas from [16], we can conclude that $\overline{\Gamma}$ is a minimal inference in CR.

Finally, we get $\psi(CR, \overline{\Gamma}, S_n) - \psi(IR, \overline{\Delta}, S_n) = (n-1) + \sum_{i=1}^{n}(m_i - 1)$, that is the upper bound is reachable.

8 Conclusion

The paper does not touch any practical aspects and is purely theoretical. Nevertheless, the author considers that it may be useful for researchers involved in the implementation of intelligent systems, in particular, e-learning systems requiring tools for proof search in classical logic at least for the following reasons.

The research demonstrates that the transition to c-clauses being the generalization of the widely-used resolution notion as a clause gave the possibility to construct the calculi possessing different properties in general and not worsening such an important characteristic as the minimum number of rule applications in comparison with the usual resolution methods. Although now it is difficult to say that the “behavior” of provers based on these calculi will be better than the “behavior” of the well-know resolution provers such as Vampire or Prover 9, we may expect that more detailed analysis of the proposed approach will lead to the further improvement of the traditional resolution technique. From this point of view, MIM seems to be a more attractive method, possessing a number of positive features not mentioned in the paper and requiring a separate study.

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