A Tractable Paraconsistent Fuzzy Description Logic

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Abstract. In this paper, we introduce the tractable pf-$\mathcal{EL}^{++}$ logic, a paraconsistent version of the fuzzy logic f-$\mathcal{EL}^{++}$. Within pf-$\mathcal{EL}^{++}$, it is possible to tolerate contradictions under incomplete and vague knowledge. pf-$\mathcal{EL}^{++}$ extends the f-$\mathcal{EL}^{++}$ language with a paraconsistent negation in order to represent contradictions. This paraconsistent negation is defined under Belnap's bilattices. It is important to observe that pf-$\mathcal{EL}^{++}$ is a conservative extension of f-$\mathcal{EL}^{++}$, thus assuring that the polynomial-time reasoning algorithm used in f-$\mathcal{EL}^{++}$ can also be used in pf-$\mathcal{EL}^{++}$.

1 Introduction

A difficult task in a knowledge base that aims to formalise a real world application is to deal with incomplete, imprecise and contradictory information. Hence, it is unreasonable to expect that a knowledge base which allows realistic reasoning based on partial knowledge must always be kept logically consistent. In this sense, in the last century, the paraconsistent logics were designed to handle inconsistencies without deriving anything from a contradiction. Here, we are particularly interested in the paraconsistent logic introduced by Belnap [3]. In addition, there are some logical approaches that attempt to formalise reasoning under incomplete and imprecise knowledge as the fuzzy logic introduced by Zadeh [10].

Although expressive enough to deal with incomplete, imprecise and contradictory information, the satisfiability problem for paraconsistent and fuzzy logics is undecidable. Since real world applications demand efficient inference systems, a family of logics, the Description Logics (DLs) [1], have been proposed. DLs are decidable fragments of classical first-order logic, and they have been customarily used in the definition of ontologies and applications for the Semantic Web.

In [7], a fuzzy logic f-$\mathcal{EL}^{++}$ with a polynomial-time subsumption algorithm was specially defined to deal with imprecise and vague knowledge. Unfortunately, this logic cannot express negative information. In fact, it was proved that the introduction of the classical negation in DLs leads to undecidability [2].

In this paper, we introduce the tractable pf-$\mathcal{EL}^{++}$ logic, a paraconsistent version of f-$\mathcal{EL}^{++}$ that is able to tolerate contradiction under incomplete and vague knowledge. It extends the f-$\mathcal{EL}^{++}$ language with a paraconsistent negation in order to represent contradictions.

2 Bilattices

In [3] Belnap introduced a logic intended to deal with inconsistent and incomplete information. This logic is capable of representing four truth values: t (true), f (false), $\ddot{\top}$ (overdefined) and $\ddot{\bot}$ (underdefined). The underdefined value represents the total lack of knowledge, while the overdefined one represents the excess of knowledge (conflicts between information). Belnap's logic was generalized by Ginsberg [4], who introduced the notion of bilattices, which are algebraic structures containing an arbitrary number of truth values simultaneously arranged in two partial orders. In the sequel, we will show the definition of bilattices and introduce the particular billatice employed in the representation of fuzzy truth-values in our proposal:

Definition 1 (Complete Bilattice) *Given two complete lattices*[1] $\langle C, \leq_1 \rangle$ *and* $\langle D, \leq_2 \rangle$, *the structure* $\mathcal{B}(C, D)$=$\langle C \times D, \leq_k, \leq_t, \neg \rangle$ *is a complete bilattice, in which:* $\langle c_1, d_1 \rangle \leq_k \langle c_2, d_2 \rangle$ if $c_1 \leq_1 c_2$ and $d_1 \leq_2 d_2$, $\langle c_1, d_1 \rangle \leq_t \langle c_2, d_2 \rangle$ if $c_1 \leq_1 c_2$ and $d_2 \leq_2 d_1$. *Furthermore,* $\neg : C \times D \to D \times C$ *is a negation operation such that: (1)* $a \leq_k b \Rightarrow \neg a \leq_k \neg b$, *(2)* $a \leq_t b \Rightarrow \neg b \leq_t \neg a$, *(3)* $\neg\neg a = a$.

$\mathcal{B}^2$ = $\langle [0,1] \times [0,1], \leq_t, \leq_k, \neg \rangle$ *is a complete bilattice where* $\neg \langle x_1, x_2 \rangle = \langle x_2, x_1 \rangle$.

In an element $x = \langle x_1, x_2 \rangle$ in $[0,1] \times [0,1]$, x_1 and x_2 represent, respectively, the membership and non-membership degrees of x in $[0,1]$. This means that x_2 can be any value in $[0,1]$ and not necessarily $1 - x_1$ as one would expect in the classical case. It is a very important distinction because it will allow us to identify contradictory truth-values. A truth-value $x = \langle x_1, x_2 \rangle$ is *contradictory* whenever $x_1 + x_2 > 1$.

3 The pf-$\mathcal{EL}^{++}$ Logic

Here we propose a new Description Logic, pf-$\mathcal{EL}^{++}$, by extending f-$\mathcal{EL}^{++}$ [7] with the negation operator $\neg$. Motivated by [6,5], we will employ a bilattice of truth-values to represent the degree of inclusion and non-inclusion of an individual to a concept. The differences between the syntax of pf-$\mathcal{EL}^{++}$ and f-$\mathcal{EL}^{++}$ concepts is that in our proposal we introduce the negation in the alphabet and $\sqcup$ and $\exists$ are replaced respectively by $\otimes_k$ and $\exists_k$. Now it is also possible to use negation to concepts and atomic roles:

Definition 2 (Concept Semantics) *The semantics of* pf-$\mathcal{EL}^{++}$ *individuals and atomic concepts/roles is given by* I = $(\Delta^I, .^I)$, *where the domain* Δ^I *is a nonempty set of elements and* $.^I$ *is a mapping function defined by: each individual* $a \in N_I$ *is mapped to* $a^I \in \Delta^I$; *each atomic concept name* $A \in N_C$ *is mapped to* $A^I : \Delta^I \to [0,1] \times [0,1]$; *each atomic role name* $R \in N_R$ *is mapped to* $R^I : \Delta^I \times \Delta^I \to [0,1] \times [0,1]$.

Each atomic concept/role C is mapped to a pair $\langle P, N \rangle$, where $P, N \in [0,1]$. Intuitively, P denotes the degree in which an element belongs to C, while N denotes the degree in which it does not belong to C. Note that $P + N$ is not necessarily equal to 1 as in the classical case. We define the functions $proj^+ \langle P, N \rangle = P$ and $proj^- \langle P, N \rangle = N$. Concepts can be interpreted inductively as follows, where for all $x \in \Delta^I$:

[1] Let L be a nonempty set and $\leq$ a partial order on L. The pair $\langle L, \leq \rangle$ is a complete lattice if every subset of L has both a least upper bound and a greatest lower bound according to $\leq$.

Syntax	Semantics
$\top$	$\top^I(x) = \langle 1, 0\rangle$
$\bot$	$\bot^I(x) = \langle 0, 1\rangle$
$\neg C$	$(\neg C)^I(x) = \langle N, P\rangle$, if $C^I(x) = \langle P, N\rangle$
$\{a\}$	$\{a\}^I(x) = \begin{cases} \langle 1, 0\rangle & \text{if } x = a^I \\ \langle 0, 1\rangle & \text{otherwise} \end{cases}$
$C \otimes_k D$	$(C \otimes_k D)^I(x) = \langle min(P_1, P_2), min(N_1, N_2)\rangle$, if $C^I(x) = \langle P_1, N_1\rangle$ and $D^I(x) = \langle P_2, N_2\rangle$
$\exists_k R.C$	$(\exists_k R.C)^I(x) = \langle \sup_{y\in\Delta^I} (min(proj^+(R^I(x,y)), proj^+(C^I(y)))), \sup_{y\in\Delta^I} (min(proj^+(R^I(x,y)), proj^-(C^I(y)))) \rangle$

The controversial part refers to $\otimes_k$ and $\exists_k$, which were designed in a way that $\neg(C \otimes_k D)^I(x) = (\neg C \otimes_k \neg D)^I(x)$ and $\neg(\exists_k R.C)^I(x) = (\exists_k R.\neg C)^I(x)$. Roughly speaking we can understand them as the counterpart in $\leq_k$ of conjunction ($\sqcap$) and role restriction ($\exists$) respectively. In fact, we can simulate $\sqcap$ and $\exists$ presented in f-$\mathcal{EL}^{++}$ respectively as $(C \sqcap D)^I(x) \equiv (C \otimes_k D \otimes_k \top)^I(x)$ and $(\exists R.C)^I(x) \equiv (\exists_k R.C \otimes_k \top)^I(x)$. The problem is that we cannot introduce them in pf-$\mathcal{EL}^{++}$ language because $\neg(C \sqcap D)^I(x) = (\neg C \sqcup \neg D)^I(x)$ and $\neg(\exists R.C)^I(x) = (\forall R.\neg C)^I(x)$. Then, since our aim is to present a tractable paraconsistent fuzzy extension for $\mathcal{EL}^{++}$, the inclusions of disjunction ($\sqcup$) and universal restriction ($\forall$) in $\mathcal{EL}^{++}$ are not allowed. Otherwise, as proved in [2], the algorithm of decidibility will grow exponentially!

We define the notions of Terminological Box (TBox), Assertional Box (ABox) and ontology in pf-$\mathcal{EL}^{++}$. For now on, consider $T_1, \ldots, T_k, T$ refer to atomic roles or the negation of them. The semantics of negation of roles is similar to negation of concepts.

Definition 3 (TBox/ABox) *A paraconsistent fuzzy TBox in pf-$\mathcal{EL}^{++}$ is a finite set of internal fuzzy inclusion axioms ($C \sqsubset_n D$), strong fuzzy inclusion axioms ($C \rightarrow_n D$), internal role inclusion axioms ($T_1 \circ \ldots \circ T_k \sqsubset T$) and strong role inclusion axioms ($T_1 \circ \ldots \circ T_k \rightarrow T$). A paraconsistent fuzzy ABox in pf-$\mathcal{EL}^{++}$ consists of a finite set of assertion axioms of the form $C(a) \geq n$ and $T(a, b) \geq n$, where $n \in [0, 1]$.*

Definition 4 (Ontology) *An ontology or knowledge base in pf-$\mathcal{EL}^{++}$ is a set composed by a paraconsistent fuzzy TBox and a paraconsistent fuzzy ABox.*

The semantics of both paraconsistent fuzzy general concept inclusions, role inclusions, concept assertion and role assertion is given as follows, where for all $x, y \in \Delta^I$:

Axiom Name	Syntax	Semantics
Internal f-GCI	$C_1 \sqsubset_n C_2$	$min(proj^+(C_1^I(x)), n) \leq proj^+(C_2^I(x))$
Strong f-GCI	$C_1 \rightarrow_n C_2$	$min(proj^+(C_1^I(x)), n) \leq proj^+(C_2^I(x))$, $min(proj^-(C_2^I(x)), n) \leq proj^-(C_1^I(x))$
Internal RIA	$T_1 \circ \ldots \circ T_k \sqsubset T$	$proj^+([T_1^I \circ^t \ldots \circ^t T_k^I](x, y)) \leq proj^+(T^I(x, y))$
Strong RIA	$T_1 \circ \ldots \circ T_k \rightarrow T$	$proj^+([T_1^I \circ^t \ldots \circ^t T_k^I](x, y)) \leq proj^+(T^I(x, y))$, $proj^-(T^I(x, y)) \leq proj^-([T_1^I \circ^t \ldots \circ^t T_k^I](x, y))$
Concept assertion	$C(a) \geq n$	$proj^+(C^I(a^I)) \geq n$
Role assertion	$T(a, b) \geq n$	$proj^+(T^I(a^I, b^I)) \geq n$

Finally, we show the notions of satisfiability and logical consequence in pf-$\mathcal{EL}^{++}$:

Definition 5 (Satisfiability) *The satisfiability of an axiom α by a fuzzy interpretation I, denoted $I \models \alpha$, is defined as $I \models C_1 \sqsubseteq_n C_2$ iff $\forall x \in \Delta^I$, $min(proj^+(C_1^I(x)), n) \leq proj^+(C_2^I(x))$. The notion is similarly applied to the other axioms shown in the table above. I is a model of an ontology O iff I satisfies each axiom of O.*

Definition 6 (Logical Consequence) *An axiom α is a logical consequence of an ontology O, denoted by $O \models \alpha$, iff every model of O satisfies α.*

Paraconsistency comes to deal with the principle that $\alpha, \neg\alpha \not\models \bot$, where α is an axiom. Note that in pf-$\mathcal{EL}^{++}$, $\bot$ is not logical consequence of α and $\neg\alpha$. For example, consider the axioms $(C(a) \geq 0)$, $(\neg C(a) \geq 0)$ and $(\bot(a) \geq 1)$. We have that $(C(a) \geq 0), (\neg C(a) \geq 0) \not\models (\bot(a) \geq 1)$, because there is an interpretation I (say $C^I(a^I) = \langle 0, 0\rangle$) such that $(C(a) \geq 0)^I$ and $(\neg C(a) \geq 0)^I$ are true and $(\bot(a) \geq 1)^I$ is false.

4 Conclusions and Future Works

In this paper, we introduced pf-$\mathcal{EL}^{++}$, a paraconsistent extension of the fuzzy description logic f-$\mathcal{EL}^{++}$, that deals with negation on concepts and roles. Inspired in [6], we can show how to translate pf-$\mathcal{EL}^{++}$ into f-$\mathcal{EL}^{++}$, preserving logical consequence, and under linear time and space in the size of the ontology. Since there is an algorithm for deciding fuzzy concept subsumptions operating in polynomial time [8], we know that paraconsistency can be simulated by f-$\mathcal{EL}^{++}$ without the loss of tractability.

Regarding future works, we plan to investigate and extend another approach to fuzzy $\mathcal{EL}$, presented by Vojtás [9], where conjunction is interpreted as a fuzzy aggregation function rather than fuzzy intersection. Another line of research is to extend tractable DLs to deal with probabilistic and possibilistic knowledge.

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