

Service Centers Finding by Fuzzy Antibases of Fuzzy Graph

Leonid Bershtein^{1,1}, Alexander Bozhenyuk², Igor Rozenberg³,

¹ Taganrog Institute of Technology of Southern Federal University,
Nekrasovskiy 44, 347928, Taganrog, Russia

² Scientific and Technical Center “Intech” of Southern Federal University,
Oktyabrskaya Square 4, 347922, Taganrog, Russia

³ Public Corporation “Research and Development Institute of Railway Engineers”,
Nizhegorodskaya Street, 27/1, 109029, Moscow, Russia
Avb@itt.net.ru, Avb002@yandex.ru, I.kudreyko@gismaps.ru

Abstract. In this paper the questions of definition optimum allocation of the service centres of some territory are observed. It is supposed that territory is described by fuzzy graph. In this case a task of definition optimum allocation of the service centres may be transformed into the task of definition of fuzzy antibases of fuzzy graph. The method of definition of fuzzy antibases is considered in this paper. The example of founding optimum allocation of the service centres as definition of fuzzy antibases is considered too.

Keywords: fuzzy directed way, accessible degree, fuzzy transitive closure, fuzzy reciprocal transitive closure, fuzzy set of antibases.

1 Introduction

There are many tasks of optimum allocation of the service centres [1]. They are an allocation of radio and TV station in some region; an allocation of military bases, which control some territory; an allocation of shops, which serve some region and so on.

However, the information about the allocation of the service centres is inaccurate or not reliable very frequently [2]. The calculation of a service degree (or quality) can be carried out by several, including to contradicting each other, criteria. For example, the definition of number and allocation of shops can be made by taking into account quality of roads, cost of ground in the given area, distance from other areas, and other criteria. Ranking of such criteria is frequently made subjectively, on the basis of the human factor.

We consider that some territory is divided into n areas. There are k service centres, which may be placed into these areas. It is supposed that each centre may be placed

¹ This work has been supported by the Russian Foundation for Basic Research project № 10-01-00029a.

into some stationary place of each area. From this place the centre serves all area, and also some neighbor areas with the given degree of service. The service centres can fail during the exploitation (for example, for planned or extraordinary repair). It is necessary for the given number of the service centres to define the places of their best allocation. In other words, it is necessary to define the places of k service centres into n areas such that the control of all territory is carried out with the greatest possible degree of service.

2 Main concepts and definitions

In this paper we suppose that the service degree of region is defined as the minimal meaning of service degrees of each area. Taking into account, that the service degree can not always have symmetry property (for example, by specific character and relief of the region) the model of such task is a fuzzy directed graph $\tilde{G}=(X, \tilde{U})$ [3]. Here, set $X=\{x_i\}$, $i \in I=\{1,2,\dots,n\}$ is a set of vertices and $\tilde{U}=\{\langle \mu_U \langle x_i, x_j \rangle / \langle x_i, x_j \rangle \rangle\}$, $\langle x_i, x_j \rangle \in X^2$ is a fuzzy set of directed edges with a membership function $\mu_U: X^2 \rightarrow [0, 1]$. The membership function $\mu_U \langle x_i, x_j \rangle$ of graph $\tilde{G}=(X, \tilde{U})$ defines a service degree of area j in the case when a service center is placed into area i . We assume, that the service degree has property of transitivity, i.e. if the service centre is in the area x_i and serves area x_j with a degree $\mu_U \langle x_i, x_j \rangle$, and if the service centre is in area x_j and serves area x_k with a degree $\mu_U \langle x_j, x_k \rangle$ then a degree of service of area x_k from area x_i not less than $\mu_U \langle x_i, x_j \rangle \& \mu_U \langle x_j, x_k \rangle$.

For consideration of questions of optimum allocation of the service centres we shall consider concepts of a fuzzy directed way and fuzzy antibase of the fuzzy graph [4].

Definition 1. Fuzzy directed way $\tilde{L}(x_i, x_m)$ of fuzzy directed graph $\tilde{G}=(X, \tilde{U})$ is called the sequence of fuzzy directed edges from vertex x_i to vertex x_m :

$$\tilde{L}(x_i, x_m) = \langle \mu_U \langle x_i, x_j \rangle / \langle x_i, x_j \rangle \rangle, \langle \mu_U \langle x_j, x_k \rangle / \langle x_j, x_k \rangle \rangle, \dots, \langle \mu_U \langle x_1, x_m \rangle / \langle x_1, x_m \rangle \rangle.$$

Conjunctive durability of way $\mu(\tilde{L}(x_i, x_m))$ is defined as:

$$\mu(\tilde{L}(x_i, x_m)) = \bigwedge_{\langle x_\alpha, x_\beta \rangle \in \tilde{L}(x_i, x_m)} \mu_U \langle x_\alpha, x_\beta \rangle.$$

Fuzzy directed way $\tilde{L}(x_i, x_m)$ is called simple way between vertices x_i and x_m if its part is not a way between the same vertices.

Obviously, that this definition coincides with the same definition for nonfuzzy graphs.

Definition 2. Vertex y is called fuzzy accessible of vertex x in the graph $\tilde{G}=(X, \tilde{U})$ if exists a fuzzy directed way from vertex x to vertex y .

The accessible degree of vertex y from vertex x , ($x \neq y$) is defined by expression:

$$\gamma(x, y) = \max_{\alpha} (\mu(\tilde{L}_{\alpha}(x, y))), \quad \alpha=1, 2, \dots, p,$$

where p - number of various simple directed ways from vertex x to vertex y . Let's consider, that each vertex $x \in X$ in the graph $\tilde{G}=(X, \tilde{U})$ is accessible from itself with an accessible degree $\gamma(x,x)=1$.

Example 1. For the fuzzy graph 1 presented on Fig.1, vertex x_5 is fuzzy accessible vertex from x_1 with an accessible degree:

$$\gamma(x_1, x_5) = \max \{ (0,7 \& 0,3); (0,6 \& 0,8) \} = \max \{ 0,3; 0,6 \} = 0,6.$$

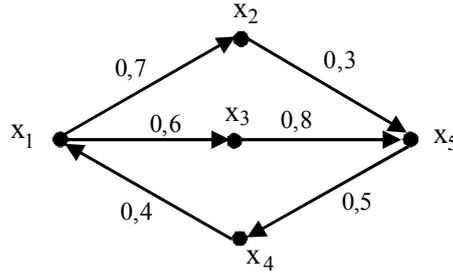


Fig. 1. Fuzzy graph 1.

Let a fuzzy graph $\tilde{G}=(X, \tilde{U})$ is given. Let's define fuzzy multiple-valued reflections $\tilde{\Gamma}^1, \tilde{\Gamma}^2, \tilde{\Gamma}^3, \dots, \tilde{\Gamma}^k$ as:

$$\tilde{\Gamma}^1(x_i) = \{ \langle \mu_{\tilde{\Gamma}^1(x_i)}(x_j) / (x_j) \rangle \}, \text{ here } (\forall x_j \in X) [\mu_{\tilde{\Gamma}^1(x_i)}(x_j) = \mu_U \langle x_i, x_j \rangle],$$

$$\tilde{\Gamma}^2(x_i) = \tilde{\Gamma} \{ \tilde{\Gamma}(x_i) \}, \tilde{\Gamma}^3(x_i) = \tilde{\Gamma} \{ \tilde{\Gamma}^2(x_i) \}, \dots,$$

$$\tilde{\Gamma}^k(x_i) = \tilde{\Gamma} \{ \tilde{\Gamma}^{k-1}(x_i) \} = \{ \langle \mu_{\tilde{\Gamma}^k(x_i)}(x_j) / x_j \rangle \}, \text{ here}$$

$$(\forall x_j \in X) [\mu_{\tilde{\Gamma}^k(x_i)}(x_j) = \bigvee_{x_l \in X} \mu_{\tilde{\Gamma}^{k-1}(x_l)}(x_j) \& \mu_U \langle x_i, x_l \rangle].$$

It is obvious, that $\tilde{\Gamma}^k(x_i)$ is a fuzzy subset of vertices, which it is accessible to reach from x_i , using fuzzy ways of length k .

Example 2. For the fuzzy graph presented on Fig.1, we have:

$$\tilde{\Gamma}^1(x_1) = \{ \langle 0,7 / (x_2) \rangle, \langle 0,6 / x_3 \rangle \}, \tilde{\Gamma}^2(x_1) = \{ \langle 0,6 / x_5 \rangle \}.$$

Definition 3. Fuzzy transitive closure $\tilde{\Gamma}(x_i)$ is fuzzy multiple-valued reflection:

$$\tilde{\Gamma}(x_i) = \tilde{\Gamma}^0(x_i) \cup \tilde{\Gamma}^1(x_i) \cup \tilde{\Gamma}^2(x_i) \cup \dots = \bigcup_{j=0}^{\infty} \tilde{\Gamma}^j(x_i).$$

Here, by definition: $\tilde{\Gamma}^0(x_i) = \{ \langle 1 / x_i \rangle \}$.

In other words, $\tilde{\Gamma}(x_i)$ is fuzzy subset of vertices, which it is accessible to reach from x_i by some fuzzy way with the greatest possible conjunctive durability. As we consider final graphs, it is possible to put, that:

$$\tilde{\Gamma}(x_i) = \bigcup_{j=0}^{n-1} \tilde{\Gamma}^j(x_i).$$

Example 3. For the fuzzy graph presented on Fig.1, fuzzy transitive closure of vertex x_i is defined as $\tilde{\Gamma}(x_i) = \{<1/x_1>, <0,7/x_2>, <0,6/x_3>, <0,5/x_4>, <0,6/x_5>\}$.

Let's define fuzzy reciprocal multiple-valued reflections $\tilde{\Gamma}^{-1}, \tilde{\Gamma}^{-2}, \tilde{\Gamma}^{-3}, \dots, \tilde{\Gamma}^{-k}$ as:

$$\tilde{\Gamma}^{-1}(x_i) = \{<\mu_{\Gamma^{-1}(x_i)}(x_j)/x_j>\}, \text{ here } (\forall x_j \in X)[\mu_{\Gamma^{-1}(x_i)}(x_j) = \mu_U <x_j, x_i>],$$

$$\tilde{\Gamma}^{-1}(x_i) = \tilde{\Gamma}^{-1}\{\tilde{\Gamma}^{-1}(x_i)\}, \tilde{\Gamma}^{-3}(x_i) = \tilde{\Gamma}^{-1}\{\tilde{\Gamma}^{-2}(x_i)\}, \dots,$$

$$\tilde{\Gamma}^{-k}(x_i) = \tilde{\Gamma}^{-1}\{\tilde{\Gamma}^{-(k-1)}(x_i)\} = \{<\mu_{\Gamma^{-k}(x_i)}(x_j)/x_j>\}, \text{ here}$$

$$(\forall x_j \in X)[\mu_{\Gamma^{-k}(x_i)}(x_j) = \bigvee_{\forall x_l \in X} \mu_{\Gamma^{k-1}(x_i)}(x_l) \& \mu_U <x_j, x_i>].$$

It is obvious, that $\tilde{\Gamma}^{-k}(x_i)$ is a fuzzy subset of vertices, from which it is accessible to reach vertex x_i , using fuzzy ways of length k .

Example 4. For the fuzzy graph presented on Fig.1, we have $\tilde{\Gamma}^{-1}(x_1) = \{<0,4/x_4>\}$, $\tilde{\Gamma}^{-2}(x_1) = \{<0,4/x_5>\}$.

Definition 4. Fuzzy reciprocal transitive closure $\tilde{\Gamma}^{-}(x_i)$ is fuzzy reciprocal multiple-valued reflection:

$$\tilde{\Gamma}^{-}(x_i) = \tilde{\Gamma}^0(x_i) \cup \tilde{\Gamma}^{-1}(x_i) \cup \tilde{\Gamma}^{-2}(x_i) \cup \dots = \bigcup_{j=0}^{n-1} \tilde{\Gamma}^{-j}(x_i).$$

In other words, $\tilde{\Gamma}^{-}(x_i)$ is fuzzy subset of vertices, from which it is accessible to reach vertex x_i by some fuzzy way with the greatest possible conjunctive durability.

Example 5. For the fuzzy graph presented on Fig.1, fuzzy reciprocal transitive closure of vertex x_1 is $\tilde{\Gamma}^{-}(x_1) = \{<1/x_1>, <0,3/x_2>, <0,4/x_3>, <0,4/x_4>, <0,4/x_5>\}$.

Definition 5. Graph $\tilde{G}=(X, \tilde{U})$ is named fuzzy strongly connected graph if the condition is satisfied:

$$(\forall x_i \in X)(S_{\tilde{\Gamma}^{-}(x_i)} = X). \quad (1)$$

Here $S_{\tilde{\Gamma}^{-}(x_i)}$ is the carrier of fuzzy transitive closure $\tilde{\Gamma}^{-}(x_i)$.

Differently, graph $\tilde{G}=(X, \tilde{U})$ is fuzzy strongly connected graph if between any two vertices there is a fuzzy directed way with the conjunctive durability which is distinct from 0.

It is easy to show, that expression (1) is equivalent to expression (2):

$$(\forall x_i \in X)(S_{\tilde{\Gamma}^{-}(x_i)} = X). \quad (2)$$

Here $S_{\tilde{\Gamma}^-(x_i)}$ - is the carrier of fuzzy reciprocal transitive closure $\tilde{\Gamma}^-(x_i)$.

Definition 6. Let fuzzy transitive closure for vertex x_i looks like $\tilde{\Gamma}^-(x_i) = \{ \langle \mu_{i1}(x_1)/x_1 \rangle, \langle \mu_{i2}(x_2)/x_2 \rangle, \dots, \langle \mu_{in}(x_n)/x_n \rangle \}$, then the volume $\rho(\tilde{G}) = \bigwedge_{j=1, n} \bigwedge_{i=1, n} \mu_{ij}(x_j)$ we name a degree of strong connectivity of fuzzy graph \tilde{G} .

Let $\tilde{G} = (X, \tilde{U})$ is fuzzy graph with degree of strong connectivity $\rho(\tilde{G})$, and $\tilde{G}' = (X', \tilde{U}')$ is fuzzy subgraph with degree of strong connectivity $\rho(\tilde{G}')$.

Definition 7. Fuzzy subgraph $\tilde{G}' = (X', \tilde{U}')$ we name maximum strong connectivity fuzzy subgraph or fuzzy strong component connectivity if there is no other subgraph $\tilde{G}'' = (X'', \tilde{U}'')$ for which $\tilde{G}' \subset \tilde{G}''$, and $\rho(\tilde{G}') \leq \rho(\tilde{G}'')$.

Definition 8. A subset vertices $\bar{B}_\alpha \subset X$ is called fuzzy antibase with the degree $\alpha \in [0, 1]$, if some vertex $y \in \bar{B}_\alpha$ may be accessible from any vertex $x \in X / \bar{B}_\alpha$ with a degree not less α and which is minimal in the sense that there is no subset $\bar{B}' \in \bar{B}_\alpha$, having the same accessible property.

Let's designate through $\tilde{R}(\bar{B})$ a fuzzy set of vertices, from which the subset \bar{B} is accessible, i.e.:

$$\tilde{R}(\bar{B}) = \bigcup_{x_i \in \bar{B}} \tilde{\Gamma}^-(x_i),$$

Here, $\tilde{\Gamma}^-(x_i)$ is a fuzzy reciprocal transitive closure of the vertex x_i . Then the set \bar{B}_α is fuzzy antibase with a degree α in only case, when the conditions are carried out:

$$\tilde{R}(\bar{B}_\alpha) = \{ \langle \mu_j/x_j \rangle | x_j \in X \& (\forall j=1, n) (\mu_j \geq \alpha) \}, \quad (3)$$

$$(\forall B' \subset \bar{B}_\alpha) [\tilde{R}(B') = \{ \langle \mu'_j/x_j \rangle | x_j \in X \& (\exists j=1, n) (\mu'_j < \alpha) \}]. \quad (4)$$

The condition (3) designates, that any vertex either is included into set \bar{B}_α , or is accessible from some vertex of the same set with a degree not less α . The condition (4) designates that any subset $\bar{B}' \in \bar{B}_\alpha$ does not have the property (3).

The following property follows from definition of fuzzy antibase:

Property 1. In fuzzy antibase \bar{B}_α there are no two vertices which are entered into same strong connectivity fuzzy subgraph with degree above or equal α .

Let $\{\mu_1, \mu_2, \dots, \mu_L\}$ is a set of all values of membership function which are attributed to edges of graph \tilde{G} . Then the following properties are true:

Property 2. In any fuzzy circuit-free graph exists exactly L fuzzy antibases with degrees $\{\mu_1, \mu_2, \dots, \mu_L\}$.

Property 3. In any fuzzy circuit-free graph there is just one fuzzy antibase with degree α .

Property 4. If in a fuzzy circuit-free graph an inequality $\alpha_1 < \alpha_2$ is executed, then inclusion $\bar{B}_{\alpha_1} \supset \bar{B}_{\alpha_2}$ is carried out.

Let's note interrelation between fuzzy antibases and strong connectivity fuzzy subgraphs. Following properties are true.

Property 5. If subset \bar{B}_α is fuzzy antibase with degree α , then there is such subset $X' \subseteq X$, that $\bar{B}_\alpha \subset X'$, and fuzzy subgraph $\tilde{G}' = (X', \tilde{U}')$ has degree of strong connectivity not less α .

Property 6. If subset \bar{B}_α is fuzzy antibase with degree α , then there is not such subset $X' \subseteq X$, that $X' \subseteq \bar{B}_\alpha$ and fuzzy subgraph $\tilde{G}' = (X', \tilde{U}')$ has degree of strong connectivity α .

The following consequence follows from property 6:

Consequence 1. That in fuzzy graph \tilde{G} there was fuzzy antibase with degree α , consisting of only one vertex, it is necessary that fuzzy graph \tilde{G} was strong connectivity with degree α .

Property 7. Let $\gamma(x_i, x_j)$ is an accessible degree of vertex x_j from vertex x_i . Then the following statement is true:

$$(\forall x_i, x_j \in \bar{B}_\alpha) [\gamma(x_i, x_j) < \alpha]. \quad (5)$$

Differently, the accessible degree of any vertex $x_j \in \bar{B}_\alpha$ from any other vertex $x_i \in \bar{B}_\alpha$ is less than meaning α .

Let a set $\tau_k = \{X_{k1}, X_{k2}, \dots, X_{ki}\}$ be given, where X_{ki} is a fuzzy k -vertex antibase with the degree of α_{ki} . We define as $\alpha_k = \max \{\alpha_{k1}, \alpha_{k2}, \dots, \alpha_{ki}\}$. In the case $\tau_k = \emptyset$ we define $\alpha_k = \alpha_{k-1}$. Volume α_k means that fuzzy graph \tilde{G} includes k -vertex subgraph with the accessible degree α_k and doesn't include k -vertex subgraph with an accessible degree more than α_k .

Definition 5. A fuzzy set

$$\tilde{B}^- = \{\langle \alpha_1 / 1 \rangle, \langle \alpha_2 / 2 \rangle, \dots, \langle \alpha_n / n \rangle\}$$

is called a fuzzy set of antibases of fuzzy graph \tilde{G} .

Property 8. The following proposition is true:

$$0 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n = 1.$$

The fuzzy set of antibases defines the greatest degree of service of all territory by the k centres of service ($k \in \{1, 2, \dots, n\}$). Thus, it is necessary to determine a fuzzy set of antibases for a finding of the greatest degree.

3 Method for finding of fuzzy set of antibases

We will consider the method of finding a family of all fuzzy antibases with the highest degree. The given method is an analogue method for definition of all minimal fuzzy dominating vertex sets [5] and it is a generalization of the Maghout's method for crisp graphs [6].

Assume that a set \bar{B}_α is a fuzzy base of the fuzzy graph \tilde{G} with the degree α . Then for an arbitrary vertex $x_i \in X$, one of the following conditions must be true.

- a) $x_i \in \bar{B}_\alpha$;
- b) if $x_i \notin \bar{B}_\alpha$, then there is a vertex x_j such that it belongs to the set \bar{B}_α with the degree $\gamma(x_i, x_j) \geq \alpha$.

In other words, the following statement is true:

$$(\forall x_i \in X)[x_i \in \bar{B}_\alpha \vee (x_i \notin \bar{B}_\alpha \rightarrow (\exists x_j \in \bar{B}_\alpha) \gamma(x_i, x_j) \geq \alpha)]. \quad (6)$$

To each vertex $x_i \in X$ we assign Boolean variable p_i that takes the value 1, if $x_i \in \bar{B}_\alpha$ and 0 otherwise. We assign a fuzzy variable $\xi_{ij} = \alpha$ for the proposition $\gamma(x_i, x_j) \geq \alpha$. Passing from the quantifier form of proposition (4) to the form in terms of logical operations, we obtain a true logical proposition:

$$\Phi = \&_i (p_i \vee (\bar{p}_i \rightarrow (\vee_j (p_j \& \gamma_{ij})))) .$$

Taking into account interrelation between implication operation and disjunction operation ($\alpha \rightarrow \beta = \bar{\alpha} \vee \beta$), we receive:

$$\Phi = \&_i (p_i \vee p_i \vee \vee_j (p_j \& \gamma_{ij})) .$$

Supposing $\xi_{ii} = 1$ and considering that the equality $p_i \vee \vee_j p_j \& \xi_{ij} = \vee_j p_j \xi_{ij}$ is true

for any vertex x_i , we finally obtain:

$$\Phi = \&_i (\vee_j (p_j \& \gamma_{ij})) . \quad (7)$$

We open the parentheses in the expression (7) and reduce the similar terms the following rules:

$$a \vee a \& b = a; a \& b \vee a \& \bar{b} = a; \xi' \& a \vee \xi'' \& a \& b. \quad (8)$$

Here, $a, b \in \{0, 1\}$, $\xi' \geq \xi''$, $\xi', \xi'' \in [0, 1]$.

Then the expression (7) may be rewrite as:

$$\Phi = \bigvee_{i=1, \overline{k}} (p_{1_i} \& p_{2_i} \& \dots \& p_{k_i} \& \alpha_i). \quad (9)$$

Property 9. If in expression (9) further simplification on the basis of rules (8) is impossible, then everyone disjunctive member i defines antibase with the highest degree α_i .

Proof. Let's consider, that further simplification is impossible in expression (9). Let, for definiteness, disjunctive member

$$(p_1 \& p_2 \& \dots \& p_k \& \alpha), k < n, \alpha \in (0,1] \quad (10)$$

is included in the expression (9).

Let's assume, that the subset $X' = \{x_1, x_2, \dots, x_k\}$ is not antibase with degree α . Then there is some vertex, for example $x_{k+1} \in X/X'$, for which the statement $(\forall i = \overline{1, k})(\gamma(x_{k+1}, x_i) < \alpha)$ is true. In other words, the accessible degree of any vertex of subset X' from vertex x_{k+1} is less than value α .

We present the expression (7) in a kind:

$$\begin{aligned} \Phi = & (1p_1 \vee \xi_{12}p_2 \vee \dots \vee \xi_{1n}p_n) \& (\xi_{21}p_1 \vee 1p_2 \vee \dots \vee \xi_{2n}p_n) \& \dots \& \\ & (\xi_{k+1,1}p_1 \vee \xi_{k+1,2}p_2 \vee \dots \vee \xi_{k+1,k}p_k \vee 1p_{k+1} \vee \dots \vee \xi_{k+1,n}p_n) \& \dots \\ & \& (\xi_{n1}p_1 \vee \xi_{n2}p_2 \vee \dots \vee 1p_n). \end{aligned} \quad (11)$$

In expression (11) all coefficients $\xi_{k+1,i} < \alpha, \forall i = \overline{1, k}$. Therefore, in expression (9) all disjunctive members which do not contain variables $p_{k+1}, p_{k+2}, \dots, p_n$, necessarily contain coefficients smaller value α . From here follows, that the disjunctive member (10) is not included in the expression (9). The received contradiction proves, that subset $X' = \{x_1, x_2, \dots, x_k\}$ is antibase with degree α .

Let's prove now, that the disjunctive member (10) is minimum member. We will assume the return. Then following conditions should be carried out:

a) subset $X' = \{x_1, x_2, \dots, x_k\}$ is antibase with degree $\beta > \alpha$;

or

b) there is a subset $X'' \subset X'$ which is antibase with degree α .

Let the condition a) is satisfied. Then the next statement is true:

$$(\forall x_j, j = \overline{k+1, n})(\exists x_i, i \in \overline{1, k} \mid \gamma(x_j, x_i) \geq \beta).$$

Let's present expression Φ in the form of (11). If to make logic multiplication of each bracket against each other without rules of absorption (8) we will receive n^2 the disjunctive members containing exactly n of elements, and, on one element from each bracket of decomposition (11). We will choose one of n^2 disjunctive members as follows:

- From the first bracket we will choose element $1p_1$;

- From the second bracket - element $1p_2$; ...;

- From k^{th} bracket - element $1p_k$;
- From $(\kappa+1)^{\text{th}}$ bracket we will choose element $\xi_{k+1,i_1} p_{i_1}$ such, that index $i_1 \in [1, k]$, and volume $\xi_{k+1,i_1} > \beta$;
- From $(\kappa+2)^{\text{th}}$ bracket - element $\xi_{k+2,i_2} p_{i_2}$, for which index $i_2 \in [1, k]$, and volume $\xi_{k+2,i_2} > \beta$, etc.;
- From n^{th} bracket - element $\xi_{n,i_{n-k}} p_{i_{n-k}}$, for which index $i_{n-k} \in [1, k]$, and volume $\xi_{n,i_{n-k}} > \beta$.

Using rules of absorption (8), the received disjunctive member can be led to kind $(p_1 \& p_2 \& \dots \& p_k \& \beta')$, in which the volume $\beta' = \min\{\xi_{k+1,i_1}, \xi_{k+2,i_2}, \dots, \xi_{n,i_{n-k}}\} \geq \beta > \alpha$ and which will be necessarily absorbed disjunctive member (10). (Decomposition (9) the disjunctive member cannot include the received contradiction (10)) proves impossibility of case a).

Let's assume now, that the condition is satisfied. Let, for definiteness, $X'' = \{x_1, x_2, \dots, x_{k-1}\}$. Considering expression Φ in the form of decomposition (8), we will choose a disjunctive member as follows:

- From the first bracket we will choose element $1p_1$,
- From the second bracket - element $1p_2, \dots$,
- From $(\kappa-1)^{\text{th}}$ bracket - element $1p_{k-1}$,
- From k^{th} bracket we will choose element $\xi_{k,i_1} p_{i_1}$ such, that index $i_1 \in [1, k-1]$, and volume $\xi_{k,i_1} \geq \alpha$,
- From $(\kappa+1)^{\text{th}}$ bracket - element $\xi_{k+1,i_2} p_{i_2}$, for which index $i_2 \in [1, k-1]$, and volume $\xi_{k+1,i_2} \geq \alpha$, etc.,
- From n^{th} bracket - element $\xi_{n,i_{n-k+1}} p_{i_{n-k+1}}$, for which index $i_{n-k+1} \in [1, k-1]$, and volume $\xi_{n,i_{n-k+1}} \geq \alpha$.

Using rules of absorption (8) the received disjunctive member can be led to kind $(p_1 \& p_2 \& \dots \& p_{k-1} \& \alpha)$, in which size $\beta = \min\{\xi_{k,i_1}, \xi_{k+1,i_2}, \dots, \xi_{n,i_{n-k+1}}\} \geq \alpha$ and which will be necessarily absorbed by a disjunctive member (10). (Decomposition (9) the disjunctive member cannot include the received contradiction (10)) proves impossibility of case b).

Property 9 is proved.

The following method of foundation of fuzzy antibases may be proposed on the base of property 9:

- We write proposition (7) for given fuzzy graph \tilde{G} ;
- We simplify proposition (7) by proposition (8) and present it as proposition (9);
- We define all fuzzy antibases, which correspond to the disjunctive members of proposition (9).

Example 6. Find all fuzzy antibases for the fuzzy graph 2 presented in Fig.2:

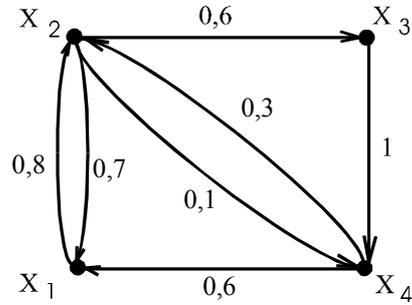


Fig. 2. Fuzzy graph 2.

The vertex matrix for this graph has the following form:

$$R = \begin{pmatrix} 0 & 0,8 & 0 & 0 \\ 0,7 & 0 & 0,6 & 0,1 \\ 0 & 0 & 0 & 1 \\ 0,6 & 0,3 & 0 & 0 \end{pmatrix}.$$

On the basis of the made above definition of fuzzy accessible vertex we can construct an accessible matrix N , containing accessible degrees for all pairs of vertices:

$$N = \|\gamma_{ij}\|_n = \bigcup_{k=0}^{n-1} R^k.$$

Here, $\gamma_{ij} = \gamma(x_i, x_j)$, $x_i, x_j \in X$, R^k – the vertex matrix power k for graph. Matrix R^0 is an identity matrix:

$$R^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We raise the contiguity matrix to 2, 3 powers. Uniting them, we find an accessible matrix:

$$N = R^0 \cup R \cup R^2 \cup R^3 = \begin{pmatrix} 1 & 0,8 & 0,6 & 0,6 \\ 0,7 & 1 & 0,6 & 0,6 \\ 0,6 & 0,6 & 1 & 1 \\ 0,6 & 0,6 & 0,6 & 1 \end{pmatrix}.$$

The corresponding expression (7) for this graph has the following form:

$$\Phi = (p_1 \vee p_2 0,8 \vee p_3 0,6 \vee p_4 0,6) \& (p_1 0,7 \vee p_2 \vee p_3 0,6 \vee p_4 0,6) \& \\ \& (p_1 0,6 \vee p_2 0,6 \vee p_3 \vee p_4) \& (p_1 0,6 \vee p_2 0,6 \vee p_3 0,6 \vee p_4 0,6).$$

Multiplying parenthesis 1 and 2, parenthesis 2 and 4 and using rules (8) we finally obtain:

$$\Phi = p_1 0,6 \vee p_1 p_4 0,7 \vee p_1 p_2 p_4 \vee p_2 0,6 \vee p_2 p_4 0,8 \vee p_3 0,6 \vee p_4 0,6.$$

It follows from the last equality that the graph \tilde{G} has 7 fuzzy antibases, and fuzzy set of antibases is defined as:

$$\tilde{B}^- = \{ \langle 0,6/1 \rangle, \langle 0,8/2 \rangle, \langle 1/3 \rangle, \langle 1/4 \rangle \}.$$

The fuzzy set of antibases defines the next optimum allocation of the service centres: If we have 3 or more service centres then we must place these centres into vertices 1, 2, and 4. The degree of service equals 1 in this case. If we have 2 service centres then we must place these centres into vertices 2, and 4. The degree of service equals 0,8 in this case. If we have only one service centre then we can place it in any vertex. The degree of service equals 0,6 in last case.

3 Conclusion

The task of definition of optimal allocation of the service centres as the task of definition of fuzzy antibases of fuzzy graph was considered. The definition method of fuzzy antibases is the generalization of Maghout's method for nonfuzzy graphs. It is necessary to mark that the suggested method is the method of ordered full selection, because these tasks are reduced to the task of covering, i.e. these tasks are NP-complete tasks. However, this method is effective for the graphs which have not homogeneous structure and not large dimensionality.

References

1. Christofides, N.: Graph theory. An algorithmic approach. Academic press, London (1976)
2. Malczewski, J.: GIS and multicriteria decision analysis. John Willey and Sons, New York (1999)
3. Kaufmann, A.: Introduction a la theorie des sous-ensembles flous. Masson, Paris (1977)
4. Bershtein, L.S., Bozhenuk, A.V.: Fuzzy graphs and fuzzy hypergraphs. In: Dopico, J., de la Calle, J., Sierra, A. (eds.) Encyclopedia of Artificial Intelligence, Information SCI, pp. 704--709. Hershey, New York (2008)
5. Bershtein, L.S., Bozhenuk, A.V.: Maghout method for determination of fuzzy independent, dominating vertex sets and fuzzy graph kernels. J. General Systems. 30: 45--52 (2001)
6. Kaufmann, A.: Introduction a la combinatorique en vue des applications. Dunod, Paris (1968)