Mathematical Model of Banking Firm as Tool for Analysis, Management and Learning

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Abstract. An essential concern for banking firms is the problem of assets and liabilities managing (ALM). Over last years a lot of model tools were offered for solving this problem. We offer the novel approach to ALM based on transport equations for loan and deposit dynamics. Given the bank's initial state, and various deposit inflow scenarios the model allows provide simulations including stress-testing, and can be used for assessment of liquidity risk, for examine loan issue decisions to choose reasonable solution, and in the learning purposes.

Keywords. Asset- liability management, Differential equations, Liquidity risk, Duration

Key terms. Banking, Mathematical Modelling, Decision making

1 Introduction

A banking firm is rather a complex system within the context of management problem. It is caused by a considerable number of financial flows and the funds, having a various origin and differing by dynamic and probabilistic characteristics, and at the same time forming the unified system. Stable functioning of the system is provided due to hierarchy, external (prudential supervision) and internal regulators and restrictions, and feedbacks.

Among the mathematical models of banking firms it is possible to separate two basic groups. There are models of optimization of assets portfolio (static, single and multi-period) using linear and dynamic programming mainly [1-2], and models of assets and liability management (ALM), using methodology and the technique of the stochastic differential equations [3-5].

One of the problem solved by models ALM is management of various risks (especially credit risk and interest-rate risk), including the problem of default probability decrease.

In connection with computer engineering development, from the middle of 70th years of the last century the computer models of banks focused on problems of planning and decision-making support systems began to appear. However such projects had no further development [6-8].

Then we will turn our attention to one of the possible approaches to bank modelling as a dynamic system, which can be called hybrid. The main tasks which the model developed must solve are the analysis and management of liquidity and stress-testing of a bank. In addition, it can be used for optimization of assets profile.

Aggregation of elements of balance sheet can be varied according to the objectives of modelling and principles developing of state variables vector. We will use the following simplified schematic (Tab. 1).

Fixed assets of bank we will ignore, taking into account only financial flows. Obviously, balance sheet equation takes place:

$$A = S+B+Q+X=Y+C+M=L\ , \tag{1}$$

where equity (capital) of a bank C is a balancing variable.

For detailed modelling of credit risks, loans issued can be divided by categories of the debtors having various reliabilities. Division of deposits on time and demand is necessary for calculation of instant liquidity. It is ignored in considered below version of the model for simplicity.

Table 1. The aggregated balance sheet of commercial bank.

<table>
<tr><th colspan="2">Assets (A)</th><th colspan="2">Liabilities (L)</th></tr>
<tr><td rowspan="3">Loans issued (X):</td><td>Business</td><td rowspan="2">Debt (Y)</td><td>Time deposits</td></tr>
<tr><td>Private customers (buyer`s credits, mortgage etc.)</td><td>On-demand deposits and current accounts</td></tr>
<tr><td>Other banks</td><td colspan="2" rowspan="3">Inter-bank credits (M)</td></tr>
<tr><td rowspan="2">Securities</td><td>Shares (Q)</td></tr>
<tr><td>Bonds (B)</td></tr>
<tr><td rowspan="2">Reserves (S)</td><td>Cash</td><td colspan="2" rowspan="2">Equity, including retained profit of last periods (C)</td></tr>
<tr><td>Rest fund, loan loss reserves etc.</td></tr>
</table>

Formally it is possible to mark three groups of operations in the balance-sheet table:

- Reallocation of assets between separate items
- Reallocation of liabilities between separate items
- Identical change of assets and liabilities at one period

Though the bank opens a position in liabilities with grant of a loan (opening of a credit line) at one time, from the formal point of view this operation is resolved into reallocation of asset`s items.

Similarly, if the deposit remains unclaimed in maturity date it either is prolonged, or is transferred in demand deposits (with no interest accruing or with the minimum percentage) according to contact conditions. Actually, in this case there is a realloca-tion of liability`s items.

At last, when interest on loans (or other types of income or expenses) are received (or repaid), at one time it is changed both assets, and liabilities, own capital of bank increases or decreases.

2 Model with Certain Terms of Loans and Attracted Funds

The main difficulty in modelling of assets and liabilities dynamics is concerned with necessity taking into account terms of loans and deposits. Due to these the state variables must depend on two parameters - current time (t) and current "age" (τ) or the remained term to maturity (T-τ). That is why dynamics of the issued loans can be described by following transport equation:

$$\frac{\partial x}{\partial t} + \frac{\partial x}{\partial \tau} = u(t,\tau) \tag{2}$$

In addition $X(t) = \int_0^T x(t,\tau)d\tau$ - total amount loans issued,

$X^*(t) = \int_0^T x(t,\tau)e^{-\delta\tau}d\tau$ - present value of loans, T – term of loans.

Movement of time deposits is described similarly:

$$\frac{\partial y}{\partial t} + \frac{\partial y}{\partial \tau} = v(t,\tau) \tag{3}$$

$Y(t) = \int_0^T y(t,\tau)d\tau$ - total amount of time deposits,

$Y^*(t) = \int_0^T y(t,\tau)e^{-\delta\tau}d\tau$ - present value of time deposits, T – term of deposits.

Variables *u(t, τ)* and *v(t, τ)* denote the flows of issued loans (temporary outflow of financial resources of bank) and deposits (temporary inflow) distributed by time taking into account amortization (interest payment or installment credits). Accordingly, total inputs of loans *U(t)* and deposits *V(t)* (or its present values $U^*(t)$ and *V*(t)*) is described as:

$$U(t) = \int_0^T u(t,\tau)d\tau\ ,\ U^*(t) = \int_0^T u(t,\tau)e^{-\delta\tau}d\tau$$

$$V(t) = \int_0^T v(t,\tau)d\tau\ ,\ V^*(t) = \int_0^T v(t,\tau)e^{-\delta\tau}d\tau$$

Solution of the equations (2-3) can be represented in the closed form:

$$x(t,\tau) = \int_0^t u(\xi, \tau - t + \xi) d\xi + \varphi(\tau - t)$$

$$y(t,\tau) = \int_0^t v(\xi, \tau - t + \xi) d\xi + \psi(\tau - t)$$

where $\varphi(\tau)$ and $\psi(\tau)$- initial distributions of loans and deposits by "age", or may be obtained by use corresponding equations with finite differences.

Dynamics of reserves (*S*) and equity (*C*) is described by the equations including stochastic members which consider random nature of change in value of shares and possible loans losses:

$$dS = [U(t) - V(t) + \rho_X X - \rho_Y Y + \rho_B B - \rho_M M - Z(t)]dt + \mu Q dt + \sigma Q dW_t - x_t dJ_t$$

$$dC = [\rho_X X - \rho_Y Y + \rho_B B - \rho_M M - Z(t)]dt + \mu Q dt + \sigma Q dW_t - x_T dJ_t$$

where dW_t – increment of Wiener stochastic process, dJ_t – increment of compound Poisson process with exponential distributed size of jumps (loan losses), $Z(t)$ – operation expenses and payment for dividends; $x_T(t)$ – repayment of a loans in maturity date, ρ_X, ρ_Y, ρ_B, ρ_M – accordingly interest on loans, deposits, bonds income, cost of credits; μ - average portfolio return of trading securities, σ- volatility of securities portfolio.

Investments in liquid assets - shares $Q(t)$ and bonds $B(t)$ can be considered as some parameters of management and to be calculated, proceeding from structure of assets chosen or planned by bank taking into account loan demand. Similarly, the volume of received loans $M(t)$ can be select depending on bank`s requirement in financial resources.

It is necessary to add the equations of dynamics of duration to the equations of movement of assets and liabilities to model liquidity risk taking into account change of interest rates

If *r* – the annual interest rate, so in this case Macaulay duration for an asset $x(t,\tau)$ is defined by expression:

$$D_x(t) = T - \frac{1}{X^*(t)} \cdot \int_0^T \tau \cdot x(t,\tau) e^{-\delta\tau} d\tau ,$$

where $\delta = ln(1+r)$.

Similarly duration of another financial flows y(t,τ), u(t,τ), v(t,τ) are calculated. It is possible to show that dynamics of duration is described by any of presented below the equations which is chosen according to liquidity research tasks.

$$\frac{dD_x}{dt} = \left[D_u(t) \frac{U^*(t)}{X^*(t)} - 1 \right] - \lambda(t) D_x - \delta D_x$$

$$\frac{dD_x}{dt} = (D_u(t) - D_x) \frac{U^*(t)}{X^*(t)} - \left[1 - D_x \frac{x_T(t)}{X^*(t)} \right] - \delta D_x$$

$$\frac{dD_x}{dt} = \lambda(t)[D_u(t) - D_x] - \left[1 - D_u(t)\frac{x_T(t)}{X^*(t)}\right] - \delta D_x$$

where $\left(\frac{dX^*}{dt}\right)\frac{1}{X^*} \equiv \lambda(t)$

Model (2-3) has been transformed to system of difference equations and realized as computer program [9]. The user independently chooses one of two operating modes of the program: calculation in case of predefined planning horizon, or calculation with possible correction of parameters, setting physical speed of calculation.

The program is interactive as the user can change values of some key parameters in the process of calculation, without interrupting its work. As key parameters are chosen: a fraction of cash invested in various kinds of assets, revenues (interest rates), a duration of demand deposits, credit demand, inflow of deposits, crediting scenarios (distribution of loans by time).

Dynamics of inflows and outflows of cashes; diagram of change of durations of assets and liabilities; distributions of loans and deposits, and also input flow by time are displayed on the screen of computer.

A stress-testing is provided in the program. The user can choose the period of stress-testing and such stresses-scenarios as decrease in inflow of deposits, decrease in duration of deposits (the scenario of outflow of deposits); decrease in accessible volume of attracted funds on the interbank market.

3 Model with Fixed Terms of Lending and Borrowing

Model (2-3) presented above is rather difficult in numerical realization and does not allow to consider some important facts, for example, dependence on interest rates from different terms of lending or borrowing. Therefore we will consider simplified modification of previous model under supposing that terms of loans and deposits are fixed.

It is possible to fix the most typical terms according to the classification used in the bank reporting, in spite of the fact that terms of loans (or deposits) can be arbitrary. Both loans and time deposits are structured by terms as follows: till 30 days, from 31 till 90 days, from 91 till 180 days, from 181 days till 1 year, from 1 year till 3 years, over 3 years.

Accordingly, it is possible to establish several typical periods T_k for each of them time transactions are described by the partial differential equation of the first order

$$\frac{\partial x}{\partial t} + \frac{\partial x}{\partial \tau} = a(\tau, x) \qquad (4)$$

with a boundary condition $x(t,0)=u(t)$ and the initial condition $x(0,\tau)=\varphi(\tau)$. Initial and boundary condition should be consistent, that is $u(0)=\varphi(0)$.

Here t - current time, $0 \le t < \infty$, τ - elapsed time since the moment of settlement of transaction ("age" of an loan or deposit), $0 < \tau \le T$, $a(\tau,x)$- value of "amortization" of an loan or deposit (inflation, installment credit etc.).

Similarly (2-3), the variable *x(t, τ)* is the allocated variable characterizing some credit tools, accounted in assets or in liabilities (loans for limited period, time deposits, interbank lending or borrowing, coupon bonds or other assets and liabilities with the fixed term of repayment).

Further it will be assumed that

$$a(\tau, x) = -\varepsilon x \tag{5}$$

i.e. repayment of credits occurs proportionally to their volume with coefficient ε, which is not dependent on age. It can be used and other schemes (when credit repayment begins not at once and (or) occurs in advance established equal shares.

It is easy to verify that the solution of the equation (4) looks like a travelling wave

$$x(t,\tau) = u(t-\tau)\exp(-\varepsilon x) \tag{6}$$

For consistency an initial and boundary conditions at $t<\tau\le T$ it is necessary to predetermine *u(t)* on an interval $t\in[-T,0)$.

From (4) - (6) follows

$$x(0,\tau) = \varphi(\tau) = u(-\tau)\exp(-\varepsilon\tau) \tag{7}$$

and after replacement τ for $-t$,

$$u(t) = \varphi(-t)\exp(\varepsilon t) \text{ under } -T\le t<0 \tag{8}$$

The total value of the considered loan (or deposit) are obtained by integration on age

$$X(t) = \int_0^T x(t,\tau)\,d\tau \tag{9}$$

Substituting (6) in (9), we have

$$X(t) = \int_0^T u(t-\tau)\exp(-\varepsilon\tau)\,d\tau \tag{10}$$

Integrating (4), we obtain the ordinal differential equation

$$\frac{dX}{dt} = u(t) - \varepsilon X - x(t,T) = u(t) - \varepsilon X - u(t-T)\exp(-\varepsilon T) \tag{11}$$

As assets with different terms of repayment are in portfolio of assets or liabilities, so it is possible to replace scalar variable *X(t)* in (11) with vector. Vector`s components are financial tools with different terms of repayment

$$\frac{dX_k}{dt} = u_k(t) - \varepsilon_k X_k - x_k(t,T_k) = u_k(t) - \varepsilon_k X_k - u_k(t-T_k)\exp(-\varepsilon_k T_k) \tag{12}$$

For simplicity further we will suppose $T_k=k$, where k - the term, expressed in months.

Time tools (issued loans, bonds, interbank credits, time deposits) from the mathematical point of view are similar, that is why we will consider them in the context of one and only construction, giving the general designation: X_k - to time tools in assets and Y_k – in liabilities. Then the previous model can be presented as:

$$\frac{dX_k}{dt} = u_k(t) - \varepsilon_k X_k - x_k(t,k) = u_k(t) - \varepsilon_k X_k - u_k(t-k)\exp(-\varepsilon_k k) \tag{13}$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t + f(t)dt \tag{14}$$

$$\frac{dQ}{dt}=\sum_k\frac{dY_k}{dt}-\sum_k\frac{dX_k}{dt}+\frac{dZ}{dt}+\sum_k\rho_k X_k-\sum_k\eta_k Y_k-g(t)-f(t) \qquad (15)$$

$$\frac{dY_k}{dt}=v_k(t)-\varepsilon_k Y_k-y_k(t,k)=v_k(t)-\varepsilon_k Y_k-v_k(t-k)\exp(-\varepsilon_k k) \qquad (16)$$

$$\frac{dZ}{dt}=w(t)-\frac{Z}{D_z} \qquad (17)$$

where *w(t)* - inflow of on-demand deposits , $v_k(t)$ - inflow of time deposits and borrowed funds; *f(t)* - purchase (+) or sale (-) trading securities (t/s); *g(t)* – operation costs on carrying out of activities of bank; μ - securities portfolio return; σ - volatility of securities portfolio; W_t- Wiener stochastic process; η_k - interest on the time deposits and borrowed funds; ρ_k - interest on issued loans; D_z - duration (characteristic turnover time) on-demand deposits.

It is easily to obtain the equation of dynamics of equity by differentiation of balance equality and corresponding substitutions (13) - (17). As follows,

$$\frac{dC}{dt}=\sum_k\rho_k X_k-\sum_k\eta_k Y_k+\frac{dS_t}{dt}-f(t)-g(t) \qquad (18)$$

For simplicity it is supposed complete withdrawal of deposits after term in this version of model. However it is easy to take into account possibility of prolongation of the deposit or its transfer in category on-demand deposits. It is considered that dividends are not paid.

Besides, credit risks (default risk, or a delay of payments) are not considered, that also it is possible to take into account by entering of corresponding adjustments.
It is considered that interests on the attracted funds and the received credits are paid according to accrual. However it is easy to set and other scheme in which interests are accumulated on depositary accounts and are paid after term of deposit.

Let $\alpha_k = X_k/X$ and - $\beta_k = Y_k/Y$ - structure of time loans and deposits.

Besides, for simplicity we will assume that there are no investments in trading securities. Then dynamics of the capitals are described by the equation:

$$\frac{dC}{dt}=X\sum_k\rho_k\alpha_k-Y\sum_k\eta_k\beta_k-g(t), \qquad (19)$$

It is giving evident representation about sensitivity of dynamics of capital to changes of main parameters of assets and liabilities.

Main objective of shareholders and bank management is the increase in capital:

$$\frac{dC}{dt}\to\max \qquad (20)$$

subject to restrictions on financial resources and risks (credit and market, loss of liquidity, bankruptcy).

4 Conclusions

The approach to mathematical modelling of cash flow moving in asset and liability accounts of the commercial bank based on the partial differential equations is novel and has no analogues in the literature. At the same time, the given approach is quite logic as reflects process of change of actives simultaneously in time and on "age". Depending on particular theoretical or practical problems the given approach can be realized in the various modifications, two of which are presented in the article.

As the preliminary testing has shown, the computer program created by use model (2-3) allows provide various simulations, including stress-testing, and can be used in the educational purposes to provide the best understanding of the dynamic processes taking place in banking firm.

It is necessary the further development of the offered modelling approach such as improvement of program tool and also, as required, model detailed elaboration to use these models as part of decision support system for asset and liability management in commercial bank. The modified model (13-18) has been proposed for these goals.

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