Knowledge of knots: shapes in action

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Abstract: Logic is to natural language what knot theory is to natural, everyday knots. Logic is concerned with some cognitive performances; in particular, some natural language inferences are captured by various types of calculi (propositional, predicate, modal, deontic, quantum, probabilistic, etc.), which in turn may generate inferences that are arguably beyond natural logic abilities, or non-well synchronized therewith (eg. ex falso quodlibet, material implication). Mathematical knot theory accounts for some abilities - such as recognizing sameness or differences of some knots, and in turn generates a formalism for distinctions that common sense is blind to. Logic has proven useful in linguistics and in accounting for some aspects of reasoning, but which knotting performaces are there, over and beyond some intuitive discriminating abilities, that may require extensions or restrictions of the normative calculus of knots? Are they amenable to mathematical treatment? And what role is played in the game by mental representations? I shall draw from a corpus of techniques and practices to show to what extent compositionality, lexical and normative elements are present in natural knots, with the prospect of formally exploring an area of human competence that interfaces thought, perception and action in a complex fabric.

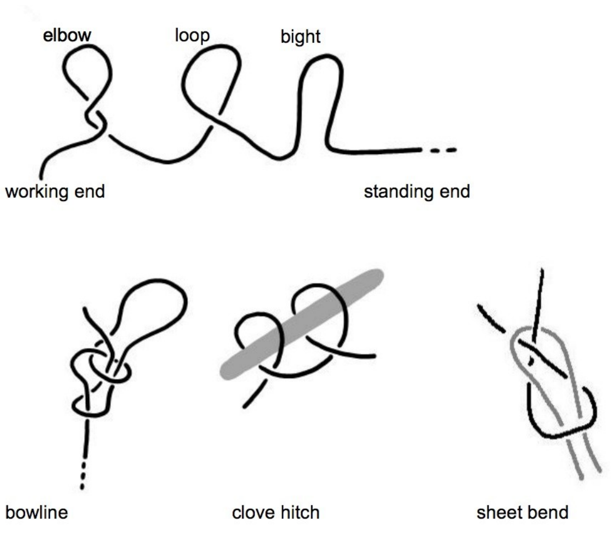

Fig 1. Some of the items we are going to discuss in this paper, listed here to assist the reader.

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The shoelace knot is the most common mildly complex knot everyone learns to tie. Most of us can tie it with closed eyes; our fingers somewhat know what to do.

I discovered recently that it can be tied in at least two more ways, over and above what I thought to be the canonical way. My youngest daughter was taught in school that one can tie a Plain Knot on two bights ("bunny ears"), and I found a number of tutorials on the web showing how to create half loops on your fingers to tie the knot in a single, swift movement – something I learned to do myself with huge intellectual pleasure.

I also decided as of late that I was able to analyse the shoelace knot. It is, actually, a composed knot: a *Plain Knot* followed by a *running Half Hitch* on a *bight*. These are semitechnical notions that I learned from sailing practice. More technically, bordering topological notions, thus cautiously, we can say that the shoelace knot decomposes into here a sequence of a "genuine" knot and an "unknot".

Knowing how it decomposes made me a bit ambitious. Can the shoelace knot be improved upon? I somewhat succeeded in getting rid of the Plain Knot (which I dislike, like many sailors) and ensuring some stability by tiying two Half Hitches on the bight (the latter one is once more a *running* Half Hitch, which provides easy unfastening).

The improvement is intellectually pleasant. Although we should handle matemathical notions with care, I'd say that I managed to replace a hybrid of a knot plus an unknot with something that is purely an unknot.

I think most of us appreciate that there are at least two action *atoms* in tying shoelaces. One could produce the Plain Knot without tying the running Half Hitch, and conversely. One may even understand something more – even if, I surmise, very few have ever tried this: tying first the Half Hitch, then the Plain Knot; i.e., execute the two steps of the shoelace knot in reverse order. Now, I predict that you will be surprised by the outcome: pulling the two ends, you end up with the Plain Knot! Exactly as it would happen when you pull the two ends of the shoe's knot. In both cases, the unknot disappears, and the knot stays.

The lesson from this simple example is that even if you have some understanding of the compositional structure of an action such as tying the shoe knot, you do not thereby have an access to the end result of just any knotting procedure that involves the elements of the composition. The consequences of the atomic actions you perform are not easy to predict; not even for experts.

My purpose here is to trace the perimeter of a small research program. There are many knotting performances that one might want to explain. People tie knots, even complex ones, learn to tie knots, talk about knots, draw them, understand knot diagrams, teach knots, at various levels of expertise and conscious understanding. There is a rich set of explananda. Moreover, the examples above suggest that some decomposition, some structure is available to knotters and guides their action. The main research question is thus: what is the structure of the underlying competence that accounts for these performances?

Knots in topology

As we search for structure, we note that the theoretical landscape is not empty. Knots are topologically interesting objects and a mathematical theory of knots has developed, providing descriptive and inferential tools to solve a number of problems. For our purposes, the main aspects of the topological account of knots are the distinction between knots and the unknot (Fig. 2), and the study of knot equivalencies. A further aspect of less concern is the peculiar classification of knots that is delivered by topology.

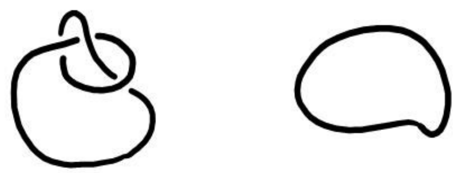

Fig. 2. Topological knot and topological unknot.

What use can be made of topological knot theory? Does it provide the appropriate framework for capturing the structure of competence? In topology, a knot is an equivalence class of knot representations, which are the planar projections (the shadows) of closed loops. The same knot can have countless representations; even when one reduces drastically the complexity of the representation (e.g. by limiting to a small integer <10 the number of crossings in the representation), the problem of identifying the members of each class has proven difficult and to date there is no algorithm that delivers a satisfactory classification [19].

Camps and Uriagereka [6] and Balari [3] have linked the complexity in knotting to that of syntax. They suggested that evidence about early human knotting practices are indirect evidence for early language use, thus proposing a link between knotting and linguistic performance. If the same computational power is assumed to underlie both knotting abilities and natural language, then evidence about knotting practice in the archeological record is evidence for at least the presence of the computational power for natural language in the brains of those who left that record. A crucial point in the argument is the recourse to topological knot theory to sustain the claim that knotting abilities require the computational power necessary for grammars, or that they share computational resources with language. We may suspend our judgement on the goal of using evidence about knotting as evidence about language (see the critiques by Lobina [15]; discussion in Balari et al [4], replies in Lobina and Brenchley [16]). At the same time we are still interested in explaining knotting competence. I would just be methodologically flexible as to the theoretical instrument we should employ. Indeed, there is no clear reason for thinking that the underlying competence is best captured by topological knot theory. There are both a generic and a specific reason for skepticism. The generic reason is purely cautionary: We have a long list of formalisms that somewhat mimick cognitive performances but in the end turn out to be quite independent from the latter and not good models thereof. Logical systems are both under and overshooting relative to people's inferential abilities. Queue theory models ideal queueing and not people's behaviour. Real-life buyers and sellers are not very well framed by rational choice theory. Coming to the point, topological classifications are misaligned with commonsense classifications [7].

Even closer to the point, the specific reason is that topological knot theory is concerned with knot equivalencies, where knots are defined over close loops in 3d-space. Ecological knots, on the other hand, are the result of transformations that take you from a situation in which there is no knot, to a situation in which there is a knot. You do not tie topological knots, because you cannot.

An intuitive demonstration of the gap between ecological knots and topological knots is at hand. You can take a close loop and tie a Plain Knot on it (Fig. 3)

Fig. 3. A closed loop, and a knot tied on it. Not a knot for topology.

Even more dramatic are examples from real practice, for instance the cases of the Half Hitch and of the Clove Hitch. These are two most used knots. The Half Hitch, the quickest way to fix a piece of rope around an object, is fundamental in tapestry and knitting.[1] The Clove Hitch is a basic knot in sailing and farming. The fact is, neither is, topologically speaking, a knot.[2] If we resort to the graphical convention of topology, we can represent them (Fig. 4) as trivial twists in a closed loop.

Fig. 4. Half Hitch (left) and Clove Hitch (middle, right) are invisible to topology.

The Clove Hitch's advantage is precisely in the fact that it is *not* a topological knot. This means that you can tie it in the middle of a piece of rope, without having to manipulate the ends of the rope (Fig. 5).

1 Tutorial received on Sep 18, 2011, at the Gobelins Tapestries in Paris.

2 A word of caution. If you tie a Half Hitch at the end of a loop, you end up with a Plain Knot (a trefoil knot, which is different from an unknot). Textbooks of knotting practice tend not to distinguish between “pure” Half Hitches and Plain Knots. But this is a terminological issue. The important thing is that a Half Hitch is part of a Plain Knot.

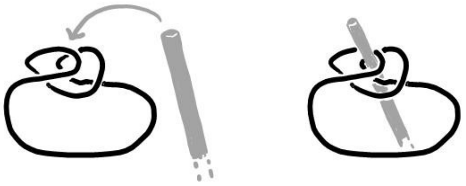

Fig. 5. Tying a Clove Hitch on a pole.

Two further aspects are that you cannot use hitches as stop knots and that you need an object around which tying them. As we shall see, there is an important notion of dependency at work here.

Our problem is thus pretty straightforward. If we find a sense for the claim that the drawing on the right hand side of Fig. 5 represents a knot, and if we accept that this knot is invisible to topology, then we need to find an alternative, non topological (or not only topological) account for our intution that a knot is represented here.

The point of contact between topological and ecological knots concerns a small subset of explananda that have to do with recognition and categorization. A very specific ecological task is that of checking if a certain knot is the correct one (did the shipboy execute the Bowline correctly?) In this case one categorizes and assesses an equivalence, much in the same way in which topologists categorize and assess equivalences of topological knots. But, as I mentioned, this is but one of the tasks to be explained.

The Camps and Uriagereka paper [2006] makes an interesting, not uncommon assumption about the performance. It describes the execution of the Plain Knot in a way that mimicks the way the knot diagram is drawn, not the way the knot is normally tied. What is the difference? The “drawing” style consists in taking the working end of the rope and make it travel about as if it was the engine of a moving train. In real life, on the other hand, one creates a small bight, and retrieves the working end. This should interest us. What descriptions of knotting practice are to be used as good explananda? I concede that in the initial phases of learning one may use the moving train metaphor. But after a while some other gestures take over.

To sum up, the central set of problems is thus twofold:

What is the performance we want to explain?

What is the structure of the underlying competence?

The first part of what follows will be devoted to looking for interesting cases of performance, such as the understanding of knot equivalencies, description of knotting practice, etc. This is an uncharted territory.[1] The second part of the paper proposes a framework for dealing with the explanation of the performances. Having rejected mathematical topology as a model of competence, I shall draw on the theory of Graphic Schemes [18] in order to propose a two-step approach to knowledge of knots in sensori-motor terms, and plead for a type of mental topological representations that are process-sensitive. Knots are living memories of processes, and we need some concepts to explore their structure and constituents. We are after mid-level conceptualizations: close enough to common sense, to ecological knotting, but such as to allow for formalization. If we want to look beyond the formal toolkit of topology we do not have much of a choice. We need to start from some semi-intuitive, semi-technical ideas. Knots from topological theory will henceforth be called 'topological knots'. Ecological knots will be just knots.

But what are knots? Let's proceed stepwise. Metaphysically, we consider knots as physical configurations of rope (be they construed as individuals, “disturbances” [14] or properties). Not all configurations of rope are knots, of course. Besides, we take knots as configurations for which a certain relation to space is essential (if you travel in a tunnel, it does not matter for you if it is knotted or not), at least insofar as it allows for movement of the knot along the rope. Knots are stable configurations of rope that are gounded on friction, but not all such stable configurations of rope are knots. For instance, rope that is wrapped around a pole may be stable and grounded on friction, but it dies not constitute a knot. Some *stabilized crossing* of rope must occur, under contextual tension. On the opposite end, a large rope jam may not count as knot for natural language: it is just a large jam. Finally, knots are what we may call *active shapes*, shapes that trap some energy. The

1 For this reason, I am a bit skeptical about the conclusions one may draw about cognition of knots from the results of experiments that measure certain responses of people to the perception of knots. We do not know yet what aspects of the performance are to be explained. Cf. [22 ,10]

mental representation of knots would thus be that of shapes *that store an action*. We shall rely on some intuitive understanding of the notion in what follows, within the limits set by these examples.

What are the explananda?

We start from the explananda. The following is a mix of platitudes, personal reports, and established evidence.

(1) People tie knots. This is our starting point. Knots are extremely useful artifacts. They have various functions that rely on a basic principle, preventing rope from slipping by exerting pressure on different parts of the rope and, if they involve an object, of the object they may be tied on. Crowell [11] provides an informal digest of some of the few papers in knots physics up to 2011, in particular the seminal work of Bayman on hitches [5]. In order to work, knots must be tied in such a way as to create *nips*, friction points between parts of rope. Some parts of rope should be made to pass in loops or over other parts so that nips are formed. Typically the standing end of a piece of rope supports a load, which may serve the function of assuring a tension (this is the case with hitches). The working end of a piece of rope is in general used for tying the knot.

(2) People can untie knots – and know when a knot is so jammed, it cannot be untied, as did Alexander the Great when, according to legend, he decided it was better to cut the Gordian Knot than to try and untie it. Some simple rules for untying are: Running knots are untied by pulling the working end. Non-running knots are tied by pulling a bight.

Knowledge about knotting and knowledge about unknotting are not necessarily aligned. It looks as if one will be able to untie any knot, whereas tying specific knots requires a certain amount of training. There is, of course, an asymmetry here, related to the complexity of the task. What one is normally requested to create is a *specific* knot (say a Bowline, or a Cleat Hitch, or a Sheet Bend). One is not requested to create an unorganized knotted structure (which one may easily do by simply piling a number of simple knots and pulling the working end randomly through whichever loops are formed). Knotting and unknotting appear to require different algorithms. However, tying knowledge is useful in untying a knot. I remember that I can easily untie a Bowline; I know (but we shall see that this is no trivial knowledge) that a Bend Sheet is a Bowline. I immediately find a way to untie a Bend sheet.

(3) We have normative intuitions about knots. In Ashley's apt words, “A knot is never 'nearly right'; it is either exactly right or it is hopelessly wrong, one or the other; there is nothing in between” [2, p.18].

(4) There is an understanding of the distinction between permanent and transitory knots. In ordinary life many knots are not permanent (shoelace knots, mooring knots, knots for climbing) and must be so designed and executed that one be able to easily untie it. Other knots, such as knots for parcels, for tapistry, for fisher nets and weaving are designed to be permanent. Most natural knots that one must quickly dispose of are unknots (the Clove Hitch). Most natural knots that one should not dispose of (stop knots) are topological knots. There is an understanding of what kind of knot is suitable for different purposes, and thus of the functional properties of each knot.

(5) People show clear degrees of expertise in tying knots. This point is less trivial, but no less true. I acquired a certain expertise; before that, I admired other people's expertise. Expertise manifests itself in speed and accuracy of the performance, in recognitional/parsing abilities, in assessment of other people's performances, in style of execution, in the ability of generalizing, in “parsing” knots one has not seen before.

(5.1) People with a limited knot repertoire face a number of knotting problems that they routinely fail to solve. A classical example is the tying up of a parcel. Without knowledge of appropriate knots and techniques one will inevitably end up with a loose rope. Another example, concerning the understanding of rope properties, is the systematic kinking of water hoses when coiling them.

(5.2) The standard knot repertoire of the large majority of adults who do not have a professional or leisurable interest in knots is very small, of two-three knots, including the Plain Knot and the shoelace knot. (Personal poll, >20 individuals.) It appears that those who learn more knots are either professionals (sheperds, sailors) or people with a hobbystic interest (e.g. fishermen).

In general, knotters take pragmatic shortcuts. They ask, What is a knot good for? In a real life scenario, instead of connecting two pieces of ropes through the Sheet Bend, that handbooks suggest as the appropriate solution to this problem, people link two Bowlines. This is because one seldom connects two ropes, one knows how to tie a Bowline, one does not remember the Sheet Bend, and one needs to solve a problem on the spot.

(5.3) The algorithm for tying difficult, complex knots may be forgotten after a while. (Personal observation of practice.)

(5.4) Knots appear to be cross cultural. There is a large record of knotting practices for many different purposes over and above tying objects. Knots are used as marks for measuring on ropes (whence the measuring unit 'knot'). The archeological record shows probable braids in the hair of Cro-Magnon ivory heads (upper Paleolithic, -25000). Knotted carpets date back to -3000. The Inca used since -4500 and until +650 a positional number system (Quipu or Khipu, meaning 'knot' in Quechua) based on ropes and knots [23]. Different knots had different syntactic roles. Basically (but there are complications) a knot denotes a unit; series of knots represent a number between zero (no knot) to nine (nine knots). The end of a numeral was denoted by a Figure-of-Eight Knot. The value of a Many-Turn Long Knot was given by the number of turns. What matters for our purposes is the use of different knots, two of which are pretty standard (the Plain Knot and the Figure of Eight Knot). The fact that the same knots are used in different and distant cultures can be the result of cultural transmission, but more simply can be just a consequence of the fact that the space of possible solutions is not much populated at the “easy” end.

(5.5) Animals do not appear to be able to tie knots, with the remarkable exception of great apes in captivity [13]. There is some reason to suppose that this cultural habit is imported from humans. “Takanoshi Kano, a bonobo specialist, notes: “. . . I wonder wild apes may meet need to make a knot, and also you should notice that knot-like objects for apes to untie do not exist in wild situations” (p. 626-627). Herzfeld and Lestel studied the behavior of Wattana, an orangutan at the Paris Ménagerie of Jardin des Plantes. Using hands, feet and mouth Wattana tied half-Hitches, simple knots and even shoelace knots, and created some assemblages. *“Her knots were not restricted to single ones; she also made double and triple knots. Some of them were even more complex, for she passed the ends back and forth through the loops already formed. She also sometimes wrapped a string around another string held between her two feet, passing the string back and forth, making loops and then passing one end of the string through one or another of the loops already made before pulling it taut. One might call this a sort of “interlacing”, a form of weaving*” (p. 631) Two facts are worth mentioning. First, Wattana used knots as projectiles (they increase locally the mass of rope). This indicates that there are practical, noncognitive attractors in knot tying that may not be part of any planning. Arguably, Wattana has made, and made use of, an interesting discovery in naïve physics. Second, Tübo, a fellow young male, untied some of Wattana's knots (p. 643). Knotting acquired social relevance.

(6) Children start tying knots at age 3-4, have a long learning phase, and a slow performance for some years (informal poll of kindergarten teachers). Strohecker [20] is a study of instruction of children in an experimental setting.

(7) Language. People teach knots by showing them but also by accompanying the ostension with a description of the algorithm that generates the knot (playing at the interface between action and the conceptual system.) It is also possible to describe the knot, i.e. the configuration of tied up rope, the structure of the knot – both in a view-dependent and in a view-independent way. Incidentally, when a manual explains a knot, it normally talks about the movement of the rope, not about the hand movement.

(8) People can see *some* knot equivalencies/differences by just visually inspecting knots. Topologists for that matter, are skilled at that. Expertise plays an important role here. In learning to solve graphical knot equivalencies, topologists make use of the Reidemeister moves (Fig. 6).

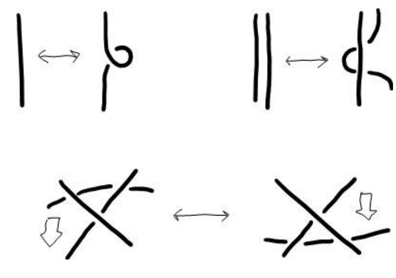

Fig. 6. The three Reidemeister moves.

The configurations linked by double arrows in Fig. 6 are local moves that do not change the corresponding topological knot and can be interchanged in a graphic representation of a knot. Slightly more formally, the three Reidemeister moves are sufficient to connect any two diagrams that represent the same type of knot (they are "shadows" of 3D movements in the knots). The Reidemeister Theorem states that "If one knot can be transformed into another knot by continuous manipulation in space, the same result can be obtained by a manipulation whose projection consists uniquely of Reidemeister moves and trivial manipulations of the diagram in the plane" [19, p. 41]. One simple hypothesis is that after a learning phase topologists interiorize the Reidemeister moves (a discussion of topological knots in cognition in De Toffoli and Giardino [12]).

(9) People may be blind to some knot equivalencies. I want to offer two cases concerning ecological knots.

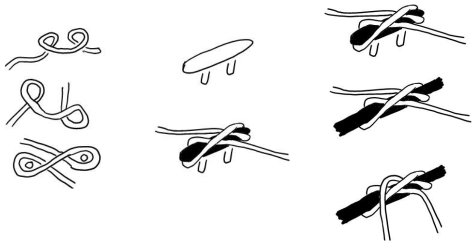

Fig. 7. The equivalence of Cleat Hitch and Clove Hitch.

The Cleat Hitch and the Clove Hitch are the same unknot (Fig. 7). The difference is in the fact that the Cleat Hitch takes advantage of the geometry of the cleat.

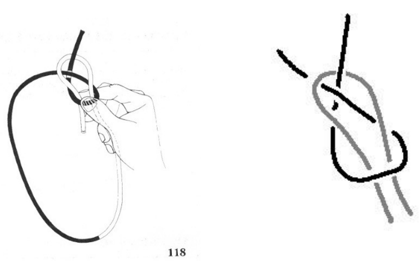

Fig. 8. The equivalence of Bowline and sheet bend. (Left figure from Asher [1], who uses two different colors for the two ends of the rope in order to point the similarity).

The Bowline and the sheet bend are the same knot (Fig. 8). The only difference is that the Bowline is tied on a single piece of rope, whereas the sheet bend is used for tying together two disconnected pieces of rope.

Although the knots involved are relatively simple, and although the equivalences have been noted in some texts ([2], [1]), knotters and many knot handbooks are largely unfamiliar with these equivalencies. I was instructed by one of my knotting teachers about the Cleat Hitch/Clove Hitch equivalence, and still find it a bit surprising. The Cleat/Clove Hitch equivalence is in a sense a purely topological equivalence; we all sense that the shape of the object has something to do with the difference. More about this later.

Some dimensions for measure could be tentatively introduced here, in reference to the population of experienced topologists. Knots can be graded according to intrinsic complexity. But they can be graded according to the subjective difficulty in parsing them as well. Thus, even the unknot (by definition, the simplest case) can be presented in ways that make it hard to parse (Fig. 9).

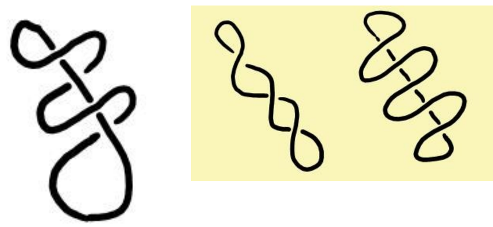

Fig. 9. Left, the unknot, under a difficult presentation. Right, two easy-to parse representations of the unknot.

Sossinsky [19] reports that only advanced algebraic techniques made it possible to show that two particular knot representations, that were considered for more than a century to belong to different knots, turned out to be in the same equivalence class.

Looking beyond knots, people have some sub-personal and personal access to topological equivalences presented visually [9] (some caveats in [8]). But people do not have access – neither personal, nor subpersonal – to relatively simple topological equivalences. Casati and Varzi [7] presented a number of cases of topologically equivalent objects that are seen as having quite different holes in it, even pretty simple ones (Fig. 10).

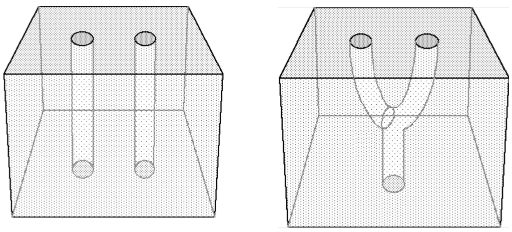

Fig. 10. The two cubic items are topologically equivalent, but they do not appear to be deformable into each other without cutting or gluing.

Topologists must train themselves to assess topological equivalencies (in particular in the case of knots, but not limited to that case) and the Reidemeister moves are meant to be an aid to pen and pencil reasoning. They so do by providing a framework to decompose any intuitive move and thus treat it mathematically.

(10) As a particular case of the previous point, expertise can be context-bound. Draftsmen who specialize on faces may be poor at drawing trees [18]. Skilled knot topologists may overlook mistakes in representations of sailors's knots. Fig P1.e of Sossinsky [19] wrongly represents a sheet bend – the "knot" will definitely untie if pulled (Fig. 11).

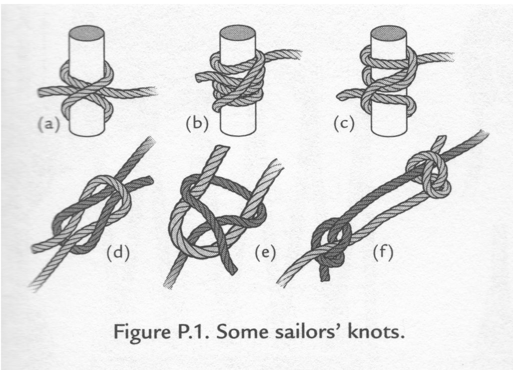

Fig. 11. Reproduced from Sossinsky [19]. The purported "sheet bend" represented in (e) will not hold.

(11) The same knot can be tied in different ways. There is the train-way (movement of the working end to create the whole knot structure) but often the tying does not require pulling of the working end, or requires it only partially. I gave at the beginning an example with the shoelace. Textbooks often present several procedural variants for the most common knots such as the Bowline and the Clove Hitch [2].

It is important to observe that these variants are not easily predictable, and realization of the equivalence in their result often comes as a surprise (or, if not, as an interesting theorem).

(12) Metric knowledge. Knotters have an understanding of how much rope is needed to tie a knot ("Will it suffice?", "You took too much/too little")

(13) Handedness: Michel and Harkins [17] found that "observational learning of manual skills [knot tying] is significantly enhanced when the student and teacher are concordant in handedness". Some video tutorials for knots present a subjective viewpoint on the hands, and those that do not may warn about the "mirror" effect created by looking at a video.

(14) People make systematic mistakes or encounter systematic difficulties in tying certain types of knot (eg. turning the final loop in tying a Cleat Hitch.)

(15) Generalizability. To some extent, once one has learned to tye a given knot, one can generalize (to thicker ropes, to specular knots, to different supports, to constrained tying, e.g. with a single hand). There are limits, though (once more, expertise is often context-bound). I learned a certain sequence for the Clove Hitch ("superpose rings in a "non intuitive way"), but this only holds for a rope's standing end that is presented on the right hand-side of the right hand. It is difficult for me to do the same for the left hand. I learned how to tie a Bowline with two hands, but I may need to do it with only one hand. I am better at tying the Cleat Hitch on the starboard side than on the port side. Under constraints (rain), I happened to have to tie a sheet to the roof using a Clove Hitch: no visibility, wet glasses, only one hand available, use of teeth, "generalizing", starting from the memorized sequence, no visual control. It helped that I had memorized the sequence eyes shut. Success in some of these performances would speak in favor of some generality in mental representation of knots. More often than not, success does not appear to be at hand, thus indicating rigidity of the representation.

(16) Retrievability: We have some understanding

of the knotting sequence, given perception of the final result of the knot. At some point I realized that a Clove Hitch is the result of tying two Half Hitches (Fig. 12).

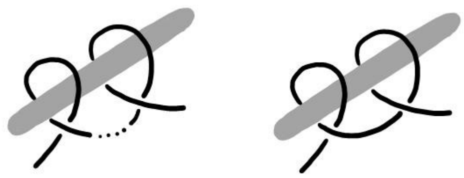

Fig. 12. Tying two Half Hitches in a sequence produces a Clove Hitch. Functionally, it is not a simple addition: the second step "closes" the Clove Hitch.

In the case at issue, the visual asymmetry of the final product masks the iteration within the sequence. We see the working end and the standing end "leave" in two different directions. But if we follow the movement of one of the two ends, we can appreciate the iteration.

Topologists can appreciate the compositionality of knots; a standard task is the decomposition of a knot into prime knots, i.e. knots that cannot be further decomposed. Once more, it is not obvious that this performance has an ecological counterpart. A topological knot can be decomposed into two trefoil knots; but a sequence of two plain knots on the same piece of rope does not automatically count as *one* knot.

Some imperfect understanding of compositionality may make one imagine impossible operations. I dreamed for a while of an "inverse" knot of a given knot, such that by combining the two and pulling ends I would end up with the unknot.

(17.1) Graphical competences 1. Drawing knots, given knowledge of a knot and of the knotting process (as opposed to copying a knot from life), is not trivial. Personal experience (Fig. 13) suggests that the best way to draw a knot (without copying it from life) is to retrace the movement that tied it. (Draftsmen who prepare drawings for manuals are likely to copy tied knots.)

(17.2) Graphical competences 2: deciphering diagrams. Over and above topological diagrams, diagrams are widely used in knot textbooks. Some diagrams appear to be more useful or more readable and effective than others. Although it is difficult to provide a measure, we can point out some elements.

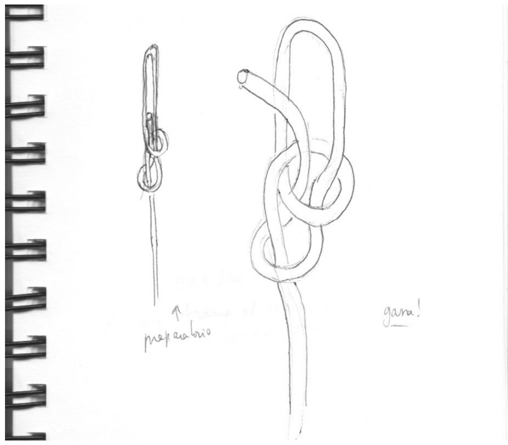

Fig. 13. Author's drawing of a Bowline from memory (left, preparatory sketch. 15.06.2012)

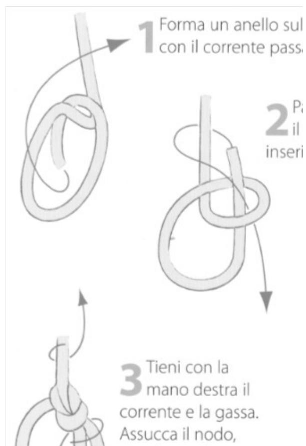

Fig. 14. The "little train" graphic method, based on the path followed by the working end.

Fig. 15. The “grab the bight” method. Circles, or hooks, indicate what to hold and where to move it in the next step.

The “little train” rendition method (Fig. 14) follows the topological construction of the knot and is in general of little assistance. The “grab the bight” rendition method (Fig. 15) models actual motor shortcuts that create the knot structure. Thumb rules for diagrams are derived by general indications about how to avoid cluttering graphic rendition, applied to the specifics of knots (Tufte [21]). Diagrams must represent intersections in order to convey the structure of the knot, thus intersection noise should be avoided. Tangents suggest intersections and are thus forbidden; information about intersections should be kept; irrelevant intersections ought to be avoided (Fig. 16). Graphic conventions about intersections (rope is not “cut” at the intersection, but is seen to continue under it) rely on Gestalt factors, such as the law of good continuation (“What is the continuation of what?”) which also underlies perception of physical knots .

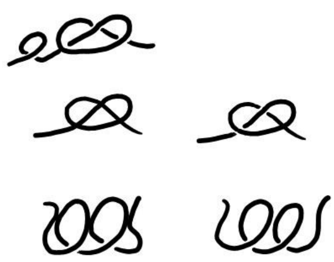

Fig. 16. On the left hand side column are examples of poor graphical renditions for knots. Top, an irrelevant loop is drawn. Middle, intersections are ambiguous. Bottom, tangents clog the image. Improvements appear in the right hand side column.

(18) We do not only categorize knots and have names for them; we also have names for knot parts. These are not only technical names such as 'bight', 'loop', 'elbow'. (See Fig. 1.). People use terms such as 'bunny ears' in teaching how to tie shoelaces (for a bight), or refer to the 'hole' or 'furrow' in describing a Bowline (for a loop). These terms, invented in order to name the parts of the knot or of the rope that contribute to knot structure, are metaphorical or analogical. The action repertoire for knotting, that includes complex interactions with rope and object, is fine-grained, and the scarcity of dedicated terminology is compensated for by metaphorical introductions.[1]

As we have seen, there is a large set of different but partly interdependent recognitional, practical and linguistic performaces to be explained. What are the ingredients of the explanans?

Knotting competence

Tying knots is a sequential action that uses repeated moves to create configurations of rope. Our first goal is to spell out the mental lexicon for the basic operations one performs on ropes when tying. Besides, knots are a wonderful case study for embodied and object-dependent cognition, as their realization depends on continuous object and sensorimotor feedback. The proper representation may involve not only the structure of the knot on a piece of rope, but the structure of the complex that includes rope and object.

In what follows I rely on Pignocchi [18] on the organization of the learning sequence of draftsmen. In learning to draw, children – but also adults – move from simple scrawls, the results of motor experiments, to more and more complex skeletal representations, that they are then able to integrate in images with an articulate content (Fig. 17).

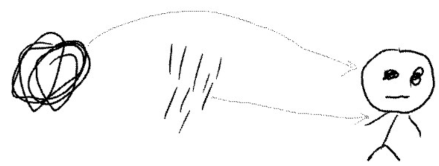

Fig. 17. Basic scrawls that merge into more and more complex graphic schemes in learning

1 Semi-technical knot terminology (“working end/standing end”) appears to be recent; it is used by people who teach knots with words. As a contrast, the terminology of types of ropes used in sailing (mooring line, sheet, etc.) is probably older, as it is used to distinguish ropes with different functions.

Mastery of Atomic Graphic Schemes (AGS) controls the production of simple scrawls. Scrawls are represented as mentally undivided, accomplished in a single gesture. Once Atomic Graphic Schemes are stabilized, they can either be reused in more complex, Molecular Graphic Schemes (MGS), that are chains of AGS, or be slightly modified to fulfill other representational purposes. The repeated execution of MGS has in some cases the effect of making them to some extent automatic, thereby turning them into new, richer AGS.

The account has a number of theoretical advantages. For instance, it explains the difficulty in generalizing and the topic-boundedness of expertise. Draftsmen specialize: those who are good at drawing flowers are not thereby good at drawing faces. The theory also explains the peculiar stylistic traits of draftsmen, which depend on the idiosyncracies of AGS. The theory keeps “Darwinian” and “Lamarckian” aspects of creativity, introduced by Johnson-Laird, in balance. Little random variations in executing AGSs or MGSs may appear satisfactory and get stabilized by repetition (Darwininan aspect). General constraints on how to hold and move a pencil and on what counts as a representation control the exploration of new AGS (Lamarckian aspect). The account further predicts that at least some gesture that produced the drawing are perceptually retrievable.

The working hypothesis of the present article is that the theory of AGS provides a plausible model of knotting competence and of its development. Accordingly, one would learn some Atomic Knotting Schemes (AKS), reuse them in (compositional) Molecular Knotting Schemes (MKS) that, with practice, become or are treated like new atomic lexical entries. In learning an AKS, one associates a sequence of movements and a visual (or visuo-tactile) result. The peculiarities of learning, innovation, generalization and transmission would be explained by using the resources of the AKS-MKS framework. For instance, random variation in AKS can get stabilized by repetition; general constraints on how to tie knots control the search of new knots and condition the consolidation of MKS (balance between Darwinian and Lamarckian aspcts). Finally, gestures behind knot production would be retrievable.

In developing AGS one relies on existing abilities. Holding an object like a pencil, or tracing a line in the sand with one's fingers, are proto-graphic activities. Likewise, in the creation of AKS one relies on pre-existing abilities. Whoever has used a piece of rope (say, to walk a dog) knows that coiling it twice around one's hand renders the grip firmer. When we pull something, we often take advantage of fixed poles to reduce our effort. When coiling rope around a bar (e.g. around a tree) we easily discover that a Half Hitch configuration is extremely effective. These are proto-knotting activities and knowledge that get integrated in the simplest AKS. Creating MKS, on the other hand, involves the deployment of compositional abilities.

A side hypothesis is that one will, or will not, be able to tie an unknown knot by looking at the result, according to one's repertoire of AKS and MKS.

But what are the ingredients of AKS and MKS? In the following sections we describe some of the hypothetical ingredients of the mental computations involving AKS and MKS. Some of these are sub-atomic, such as the ability to generate and see certain relations between movements of the hand and configurations of rope. Others are of higher level, such as the chaining of AKS in long sequences.

Parts of knots, of rope

Some parts of the rope become salient and are used as beacons for orienting the knotting process. The corresponding concepts may be lexicalized, or may be activated by analogies. In the traditional way to teach the Bowline, a segment is dubbed the “tree”, the working end is renamed “the rabbit”, and the loop is a “furrow” or a “rabbit's hole”.

“A mnemonic used to teach the tying of the bowline is to imagine the end of the rope as a rabbit, and where the knot will begin on the standing part, a tree trunk. First a loop is made near the end of the rope, which will act as the rabbit's hole. Then the "rabbit" comes up the hole, goes round the tree right to left, then back down the hole. ” (Bowline, Wikipedia entry, retrieved on 09.01.2012)

Terminology in knotting practices is semi-technical (Fig. 18). It is not to be assumed that knotters know it, nor that it lexicalizes some mental concepts knotters have. Most likely it has been fixed by writers of knot books for teaching purposes.

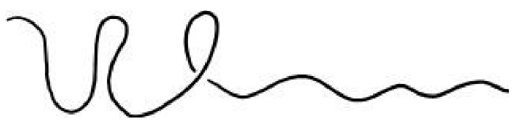

Fig. 18. From left to right: working end, bight, loop, standing end.

Loops and torsions

Asher [1] brings to the fore some important physico-geometrical properties of twisted rope. If you look along the axis of a piece of rope while twisting it e.g. clockwise, you will see that the rope undergoes a torsion; call this a right-handed torsion. Right-handedness is here an intrisic, viewpoint independent characteristic of the rope (if you look at the rope from the opposite direction, it will appear to you right-handed as well.) In order to release the torsion, you can do either of two things: twist the rope counterclockwise, or coil it counterclockwise. If you look at the coiled rope along the axis of the coil, you will see that moving away from you the coil is left-handed.

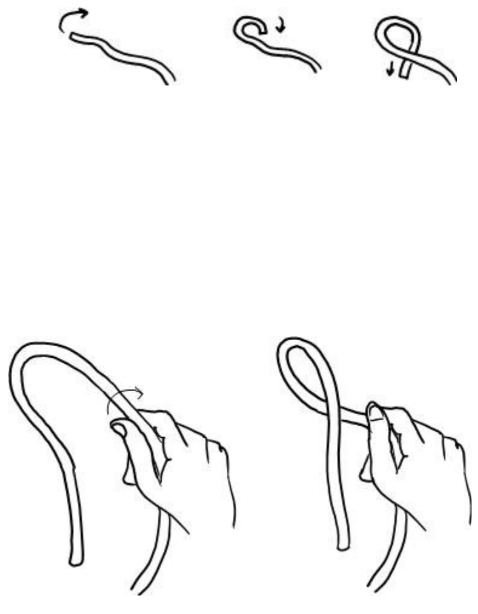

Fig. 19. Two ways to create loops.

Thus, you can produce a loop

1) by shifting the working end (Fig. 19, top).

2) By passing the thumb over the index finger to induce a torsion of the rope (fig 19, bottom).

A rope can be modeled as a series of rigid coaxial discs with a limited freedom of movement around their axis. The internal circumference of a loop is shorter than its external circumference. Each disk is then asked to rotate a bit in order to find room for the matter compressed in the internal part of the loop. Conversely, torsions automatically generate loops (Fig. 20).

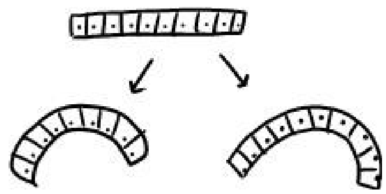

Fig.20 Ropes are things such that torsions generate loops and conversely.

These are physico-geometric properties of rope, that can be machine produced and machine measured. Two invariants surface: the right-handed torsion, that generates a left-handed coil, and the left-handed-torsion, that generates a right-handed coil (and conversely). As a consequence, the global shape of a part of the knot (of the coil) *stores some implicit information about the potential torsion of the rope*.

When learning to tie knots, a person performs twisting and coiling; these invariants are associated with sensorimotor primitives. Twisting rope provides haptic feedback. One *feels* the torsion, i.e. one feels that the rope tries to get back to its original shape. The tension at the tip of your fingers is *released* when a coil is formed. To form the coil, you just have to move your hands close to each other. Conversely, at the end of one coil, one realizes that one has generated a torsion, which can be eliminated by untwisting the rope. (Neglect of this operation has produced many a kinked water hose.)

To sum up, the basic rules are:

Twisting and joining causes coiling

Coiling and separating produces twists.

The basic knowledge of any knotter concerns the interaction of the physical and geometrical properties of rope. But although a knotter may implicitly know (feel) the torsion-to-loop interaction, she may be blind to the converse interaction. Beginners must be told that when coiling rope, for each loop they have to produce a torsion, otherwise loops will mess up.

Knowledge of knots is first and foremost storage in memory of these elementary sensorimotor regularities. The final *visual* shape of the coil is associated with a certain movement that produces or releases a torsion.[1]

A number of ecological knots are created by generating coils and making them interact. The Clove Hitch, when constructed in the middle of rope, without using the working end, is the result of the superposition of two coils. Creating coils is a prerequisite for executing these knots in an efficient way. We have seen that some most commmon knots are unknots, topologically speaking: the Clove Hitch, the Half Hitch. Let me add one more, less common knot to the lot, the Sheepshank (Fig. 21). The Sheepshank is the result of the pairwise intersection of three coils.

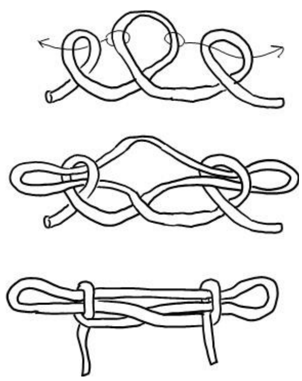

Fig. 21. The Sheepshank is actually an unknot, that only survives because of tensions.

These (un)knots, incidentally, have the advantage that they can be tied in the middle of rope, without access to the ends of the rope, by simple interaction of loops. On top of loops/coils, one produces bights when executing a knot (by holding running and standing ends, each in one hand, and having the hands get closer to each other.)

Pass-through

If some (un)knots require no access to the working end, "real" knots are in general tied by having the working end pass through a loop. A basic principle governs knot production.

Working-end-and-loop axiom. You can only tie a (real, i.e. non-unknot) knot by having the the working end pass through a loop.

This is a necessary but in no way sufficient condition, as you can tie unknots that way (e.g. the Clove Hitch, that can be tied directly on the standing end.)

The role of visual crossing

Intuitively, no matter how many times you coil a piece of rope around a pole, you won't thereby have a knot. But coil once and cross over, and you'll have a Half Hitch. Crossing is related to twisting and coiling. The structure of knots involves a passage through a loop, whereby the principle that:

the 2d projection of any knot will always involve a crossing.

Thus perceptual crossing (the presence of an x-junction in the image) is a necessary condition for being recognized as a knot (it is not a sufficient condition, as the unknot can present crossings)

Crossing is in general an important condition in assessing topological equivalencies. The "tied" double donaught is topologically equivalent to the "untied" double donaught, notwithstanding their visual difference, that suggests a topological distinction. The perceived difference is an x-junction.

Knowledge of the sidedness of a loop: guaranteeing stability

Another piece of intuitive knowledge concerns the interaction between the intrinsic orientation of the loop and the side of the loop from where the working

[1] The different ways of tying the shoelace knot desribed at the beginning generate different torsions, thus store energy in different ways in each knot. Accordingly, the generated knots are only superficially equivalent.

end must enter if one wants to get a stable knot structure (i.e. the knot will form and not collapse into the unknot.) When starting the Bowline or the Sheepshank or the Plain Knot, I know that the working end must enter from one, and not the other, side of the loop; or that a bight must pass on one, and not the other, side of the loop. I immediately see that the wrong side will not provide a stable structure. I know whether I am creating a "building step" or, sadly, an "empty step". One has intuitions about what will work.

In tying a Bowline, one can compute the correct sidedness at each step. Knowing that the working end should exit the loop where it entered the loop, one must first determine the correct way to enter the loop, and consequently the correct way to create a bight around the standing end (Fig. 22).

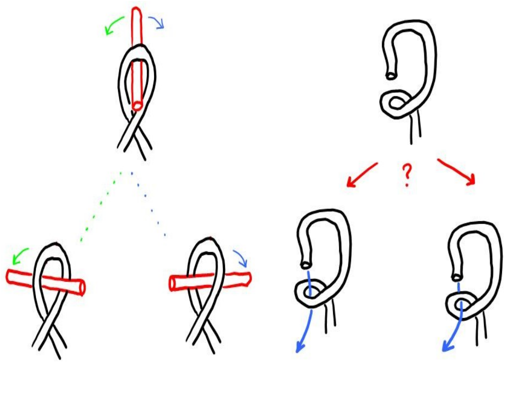

Fig. 22. Only the right bottom interaction of each case stabilizes the tension stored in the loop. (The stabilizer can be an external object, or a piece of the same rope used for the loop.)

Knot and object: external dependency

Many knots are used to create ties between objects, or to fix rope on an object. This invites discussion of a complex set of invariances and, consequently, of sensorimotor contingencies. The first, basic principle concerns unknots:

Unknot dependency: *An unknot can be tied only on an object.*

Clove Hitches, Cleat Hitches, Half Hitches are unknots whose survival depends upon the existence of an object they are tied around. They are *dependent knots*.

There are seeming counterxamples: you can tie a Clove Hitch on a portion of the very same rope you are using for the hitch. Now, although the "local" movements are those for creating a Clove Hitch, the end result is a Plain Knot. The seeming counterexample allows us to distinguish two senses of dependency:

self-dependency (e.g. you can tie a Clove Hitch on the standing end) and

other-dependency (e.g. you tie the Clove Hitch on an object that is not the rope itself.)

Two-object topologies

This introduces the theme of two-object topologies.[1] The formal counterpart of two-object topologies involving knots is the study of *links*. Once more, the descriptive gap between link theory and ecological links is as wide as the gap between knot theories and an account of ecological knots. The arguments are the same that we used for topological knots. For instance, topological links are not the result of tying, whereas ecological links are tied.

We should distinguish the metaphysical properties of self/other-dependency from the functional properties of self-reference. Some knots are self referential: they are only used to store ropes, or to reduce the volume or length of rope.

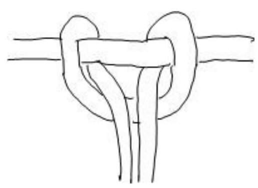

Fig. 23. A notoriously slippery knot, the Cow Hitch is used to "serve the rope", fixing it so as not to create a mess, and allowing for quick untying.

Self-dependency and self-reference are orthogonal to each other. The Cow Hitch (Fig. 23) and the Sheepshank are self-referential in that they are used to change the properties of the rope and not for fixing anything. But the Cow Hitch is other-dependent, whilst the Sheepshank is self-dependent. (Functionally, the Sheepshank is used to shorten the

1 Two-object topologies were used by Casati and Varzi [7] in order to account for some classificatory performances related to holed objects. Considering holes as completely filled, the topological properties of the contact surface between the host and the filler correlates fairly well with most commonsense categories of holes.

available portion of rope, or to use rope with a damaged portion.)

Object-based functional lexicon

We have seen that the lexicon for knots introduces terms to characterize parts of the rope and elements of knots. The part of rope, or the external object that stabilizes the loop (see again Fig. 22) is a stabilizer. Whether a part of rope or an external obejct can act as a stabilizer depends on its capability to counter the tension created by the loop. In the case of rope, this may in turn depend on the stabilizer part's tension, or on its weight (for instance, if the standing end is long enough).

In view of the importance of external objects in tying knots, we need some semi-technical terminology for supports (Fig. 24).

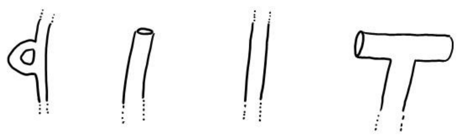

Fig. 24. Handlebody; pole; bar; cleat. Dashed lines indicate that the object's bounds are not within reach; the object is to be considered endless for knotting purposes (rope will not be allowed to slip out from there.) The cleat is topologically equivalent to the bar, but its shape conditions the knot. Starting from this simple taxonomy, endless compositional variations are available.

As we noticed, geometric features of knots are related to their causal properties (stop knots, stable knots, etc.) There is further an interaction of geometrical features of knots and physical properties of both ropes and things tied. For instance, one can create a figure-of-eight stop knot that is large enough so as not run through the handlebody. The elementary morphologies of Fig. 23 provide a lexicon of basic shapes. There is no upper bound to the complexity of object shapes one can use to create links. The basic lexicon helps characterizing the elementary interactions between rope and object. Tying a knot through a handlebody takes advantage of the topology of the handlebody to constrain the movement of the rope, and at the same time the handlebody requires that the working end passes through the hole in it. The pole is less constraining and at the same time allows for knotting procedures that do not involve the working end. Another relevant aspect of two-object interaction, in the case of knots, is that one of the two objects is in the norm underformable. This means that one can take advantage of its rigidity in the execution of the knotting sequence. A final object feature knotting takes advantage of is the *permanence* of the topology of the object. We do rely on the fact that objects (as opposed to rope) do not change their topology. (And indeed, we are susprised when this happens, for instance when topological properties of the object can migrate to the rope. If I tie up my arms by crossing them, and then grasp two ends of a piece of rope with my hands, and then open up my arms, I end up with a Plain Knot on the rope, and no knot on my arms. The knot has *moved* from my arms to the rope.)

The features in question defy classification; geometry is intertwined with function. Topologically there is no difference between the pole and the cleat, and although there is no topological difference between Clove and Cleat Hitch, as the execution is controlled by the object, it results in two utterly different procedures. Or, cosider ring and bar (an example of a bar would be a tall tree, around whose trunk one ties a knot). Functionally they could be considered equivalent: their end segments do not exist or are not accessible, so that the tying procedure requires a use of the working end. But the ring has the further property of keeping the knot in a certain place.

Knowledge of knots, the original loop, and the fundamental role of the Half Hitch

Half Hitches are ubiquitous components of knots (cf. once more Fig. 12, showing that the Clove Hitch is the result of tying two Half Hitches in sequence. Now, the Half Hitch is an unknot: it is a simple loop. According to Unknot Dependency, it can be tied only on an object. The object acts as a stabilizer of the loop. This object – according to Self-Dependency, can be be another part of the same rope. This has in general the consequence of turning an unknot into a knot. The Half Hitch then “becomes” a Plain Knot. This elementary dynamics is at the basis of most knotting. The Half Hitch stabilizes the tension generated in the creation of the loop. Knowledge of knot is thus mastering of operations that orchestrate the management of the energy stored with the creation of the original loop.

(Provisional) conclusions

We have seen that some ecological knots are, mathematically speaking, unknots, and thus that the topological theory of knots is at best a partial account of knotting abilities. We have further seen that the mental representation of knots should allow for limited generalization, understanding of knot equivalences and compositionality. Even if you have some understanding of the compositional structure of tying a simpe knot, you do not thereby have an access to the end result of just any knotting procedure that involves the elements of the composition. The consequences of the atomic actions you perform are not easy to predict; not even for experts. As knotting involves external objects essentially, the feedback loop that unites perception and action is essential for our understanding of them; knotting provides an ideal case for situated cognition and externalized mental procedures. It is early to provide a formal characterization of the principles at play – an algebra of knotting and knot understanding, as if it were, as opposed to an algebra of knots. Some existing models accounting for motor-perceptual performances (e.g. models for drawing) can be reused in the case of knotting, thus allowing us to distinguish an atomic level, with sub-atomic parts, and a molecular level. We were able to enlist some principles at the atomic or subatomic level, all involving sensorimotor representations: the relevance of parts of rope and their lexicalization; the duality of twisting and coiling and the contribution of both to the storing of action into the configuration of the knot; the interaction of loops and ends to create stable structure (good and bad 'sides' of the loop) and the consequence for the visual aspect of the knot, that must include x-junctions; the interactions of rope with object shape and topology and the lexical saliency of functional object features; the necessity of dealing with two-object representations; the object-dependency of unknots; the distinction between self-dependency and self-reference; and the fundamental role of loop stabilization in half hitches, that turns out to be the most important subatomic elements of knots.

The present article pleads for the investigation of process topology as opposed to static topology. Shapes are usually considered as static properties. But in the case of ecological knots, their features bear a trace of the process that led to them, that included planning, motor execution, and perceptual control, in the service of the management of the energy stored in the shape of a rope to create stable structures.

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