

Ziggy: Very Interactive Trigonometry

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Abstract. In this paper we describe a highly interactive touch-based application to teach the basics of trigonometry to secondary school students. The application, called Ziggy, lets students “touch” and “push” triangles, dynamically modifying the shape and size, and observe the effect on the angles, sides and the trigonometric ratios. We describe the pedagogical rationale behind Ziggy, the user interface and provide details on the implementation. An early version of Ziggy has been tested in small-scale experiments in the classroom.

1 Introduction

this paper describes a highly interactive touch-based application to teach the basics of trigonometry to secondary school students. In basic trigonometry sides and angles are calculated using the Pythagorean theorem, the 180° rule and trigonometric relationships, like the sine, cosine and tangent. For many learners basic trigonometry proves to be very difficult (Blackett & Tall, 1991).

Traditional education in trigonometry in the Netherlands focuses on procedural skills. Students learn to calculate sides and angles by applying the SOH-CAH-TOA mnemonic (Reichard, 2005). SOH stands for Sine equals Opposite over Hypotenuse, CAH is Cosine equals Adjacent over Hypotenuse, and TOA is Tangent equals Opposite over Adjacent. Lessons in trigonometry usually focus on recognizing right-angled triangles (possibly in complex figures), distinguishing the different sides, and using the mnemonic to identify which trig function can be used to calculate a requested side or angle. This way, students learn to produce correct answers, but are not encouraged to develop a real insight in trigonometry problems. Often students fail to realise trig functions are, in fact, ratios and they do not recognise the relationship between the graphical depiction of a triangle and the corresponding values. Relating shapes and numbers is hard for students, who often view calculation as the only way to obtain knowledge about triangles (Blackett & Tall, 1991). Interactive applications may help to bridge this gap by showing how changing the shape affects the values.

A general challenge in mathematics is its abstract nature (Zengin, Furkan, & Kutluca, 2012). This makes reasoning about mathematical objects and functions difficult. Successful students often make use of perceptual and visual patterns to solve problems (Kellman, Massey, Roth, Burke, Zucker, Saw, Aguero, & Wise, 2008). However, identifying visual patterns is not always straightforward and real-world motion might help students to recognise these patterns more readily (Ottmar, Landy, & Goldstone, 2012). Touch devices seem to be well-suited realising this idea.

This paper presents an interactive application in which students can learn and explore trigonometry on their touch devices. The use of tablets in the classroom has gained popularity in Dutch

secondary education. Several schools have classes in which the iPad replaces all traditional learning material (text and exercise books, calculators, notebooks). However, there is little learning material available which makes use of the unique features of touch devices and has a sound didactical foundation. The application described here, called Ziggy³, contains instructional material and exercises related to basic trigonometry in accordance with the learning goals as prescribed by the Dutch government for all secondary school students.

Whereas in traditional paper exercises the focus is on procedural skills, the interactive exercises in Ziggy emphasize the dynamic relationship of the trigonometric functions. We expect that direct interaction with triangles can foster a deeper sense of the underlying principles. Touch control makes it possible to transform mathematical objects and functions into graspable objects, that can be manipulated, transformed and explored in an intuitive manner. Mathematical entities become actual things making them less abstract.

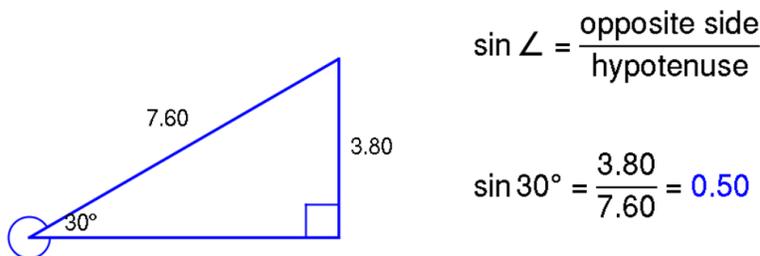
We aim to develop a comprehensive environment to teach basic trigonometry to secondary school students. This encompasses integrating interactive instructional material to explain what basic trigonometry is and a touch-based interface to solve trigonometry problems using algebraic manipulation. We have been unable to find a system specifically designed to teach basic trigonometry. There is of course a significant body of work into interactive environments to support the broader topic area of geometry (e.g., GeoGebra, Cinderella, Geometer's Sketchpad). The strength of these environments lies in the ability to create learning materials on geometry and related subjects. However, to our knowledge, they do not currently provide comprehensive instruction on basic trigonometry similar to the standard curriculum, with explanations and exercises, and suitable for use by students in secondary schools. Allowing students to interactively solve trigonometry problems requires interactive algebraic manipulation which has been part of Computer Algebra Systems and mathematics editors for some time. An overview of editing algebraic expressions can be found in Nicaud & Bouhineau (2008). We use the idea of equivalence drag and drop, "consisting of moving a sub-expression inside a global expression and preserving the equivalence of the global expression" (Nicaud & Bouhineau, 2008), extensively and integrate it with the ability to drag symbols from the pictorial representation of a trigonometry problem to algebraic equations.

This paper is organized as follows. Section 2 contains a description of Ziggy as a learning environment for basic trigonometry, and Section 3 provides a brief overview of the implementation. To test if touch control and the interactivity that Ziggy offers contributes to students understanding of trig functions, a prototype of Ziggy was implemented in a classroom setting. Section 4 gives preliminary results of this research. Section 5 contains conclusions and suggestions for further work.

2 Learning Environment

In this section we describe how the interrelatedness of the graphical representation of trigonometry and the underlying mathematical equations has been realised in on touch devices. We illustrate the design, functionality and user interface of Ziggy.

³ Ziggy is an acronym for "Zeer interactieve goniometrie", the Dutch translation of "Very interactive trigonometry". It is available at <http://ziggy.gw.utwente.nl>.



- _____ The angle in this triangle stays 30°
- _____ x Try to make the triangle larger or smaller.
- _____ What happens to the sides? And what is the effect on the sin?

Fig. 1. Screen layout for examples. At the top are menu's for navigation, left is a graphical representation of an example, right are equations related to the example, instructions are at the bottom. Interactive version of the figure: ziggy.gw.utwente.nl/mui_fig1.html.

Ziggy is a web-based application, optimized for touch devices, which lets students interact with triangles and learn the basics of trigonometry. The application combines several features aimed at obtaining more in-depth knowledge of trigonometry and supporting the learning process of the student. The most important of these features are the interactive angles, the approach to solve exercises by manipulating equations directly and the immediate feedback.

Interactive angles. Figure 1 illustrates the interface of Ziggy. On the left is a graphical representation of a trigonometry example or exercise. The area on the right is used for mathematical equations related to the example, and the instructions are at the bottom.

The example in the figure explains the sine. The triangle only contains values directly relevant to the sine: a right angle, the value of the angle for which the sine is computed and the lengths of the opposite side and hypotenuse. The equations on the right provide the definition of the sine function.

A semi-circle on an angle indicates it is draggable. We have considered several visualisations to show the draggability of an angle and this visualisation does not obscure the angle itself, a full circle would, and is large enough such that it can be dragged with the finger. In the figure, the student can drag the 30° angle to the left or right to make the triangle grow or shrink respectively. The values displayed along the sides are updated dynamically while the student is dragging as are the corresponding values in the equation. In the example, the angles are fixed and the student can deduce that changing the size of the triangle does affect the length of the sides, but the ratio between these lengths remains the same.

In other examples triangles do change shape (and ratio) when manipulated by a student. Figure 2 provides an illustration for exercises. In order to calculate an unknown value students choose the appropriate rule by clicking on one of the buttons in the upper right corner of the screen. An

empty equation appears on the right and students solve the exercise by dragging values from the triangle to the empty slots in the equation. This equation is interactive as well, so values change dynamically when students manipulate the triangle. This offers students the opportunity to witness the effect of a change in shape on the values of the sine, cosine or tangent.

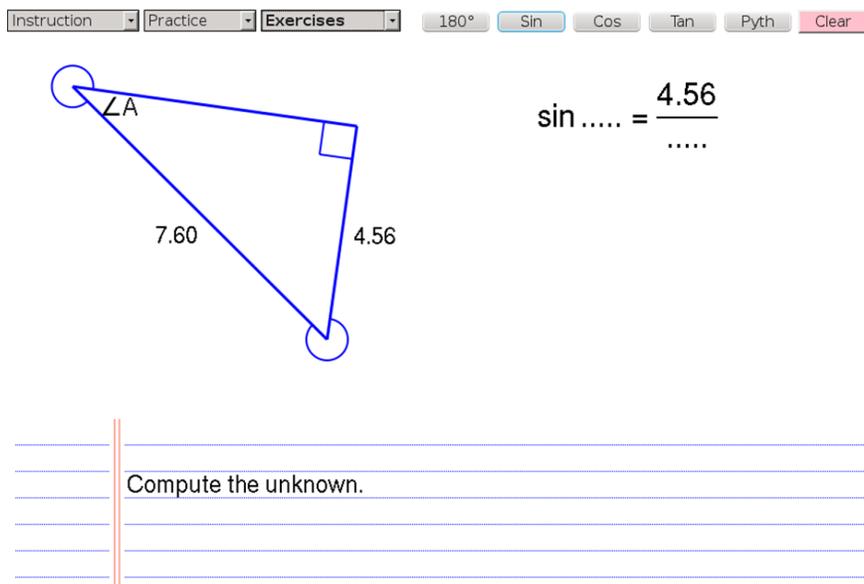


Fig. 2. Screen layout for exercises. At the top-right are buttons for selecting a formula to solve the exercise shown in the triangle. Students solve the problem by dragging values from the triangle into the equation. See ziggy.gw.utwente.nl/mui_fig2.html for an example of a randomised exercise.

Solving exercises by direct manipulation. The method for solving exercises in Ziggy is partly based on the pushing symbols method introduced in Algebra Touch (Ottmar et al., 2012; Regular Berry software, 2012). In Algebra Touch students solve algebraic equations by rearranging variables and values: pushing symbols to different locations while respecting the laws of mathematics.

Trigonometry, in contrast to algebra, relies heavily on graphical representations. Symbols (variables and values) do not only appear in equations, they also appear in the graphical representation. Algebra problems comprise solving or simplifying a *given* equation. In trigonometry students solve problems by first *selecting* the correct equation and then filling this equation with the symbols from the graphical representation.

Figure 2 is used as an example of how the pushing symbols idea is applied in Ziggy. First, the student selects one of the five equations at the top right. The structure of the equation appears, in this case the sine function. Next, the student fills in the dots by dragging and dropping symbols from the graphical representation to the equation. Values in the equation stay associated with their origin. The student has dragged the opposite side of $\angle A$ to the numerator (4.56) in the equation. When the student thereafter drags one of the angles the length of the opposite side will change, and so will the numerator. Once the student has completed dragging symbols from the triangle to the equation, the exercise has been reduced to an algebra problem and the Algebra Touch approach is applied.

$$\cos 37^\circ = \frac{a}{8.05} \quad \cos 37^\circ = \frac{a}{8.05} \quad \cos 37^\circ * 8.05 = a$$

Fig. 3. An illustration of the pushing symbols technique introduced in Algebra Touch and also used in Ziggy. The formula on the left is solved by dragging the denominator (8.05) across the “=” (middle). When the student moves it to location where the mathematical laws are maintained a hole is made for the value and the student can drop it (right).

Figure 3 illustrates how equations are solved. In the figure we have one unknown (variable a). So, to solve the equation it is first necessary to isolate this unknown. This is accomplished by multiplying both sides by the denominator (8.05). In Algebra Touch and Ziggy this is achieved by pushing 8.05 to the other side of the =, effectively multiplying both sides by 8.05. This change from division to multiplication happens as soon as 8.05 passes the =. Students see the change to the structure of the equation when pushing a symbol, as illustrated in the figure. The equation has now been rearranged such that a is isolated and it can be solved by clicking on the operators. Tapping $\cos 37^\circ$ results in replacing this expression by its value, and tapping * results in the left and right-hand side being multiplied to reveal the value of a .

Immediate feedback. Ziggy immediately shows feedback when students make mistakes trying to solve exercises. This immediate feedback, suggested by math teachers, is meant to minimize confusion and ensure students learn the rules quickly and accurately without developing misconceptions. Students are not allowed to drag values to a mathematically incorrect location in an equation. For example, angles cannot appear in locations where a side is expected. Exercises usually consist of a triangle with two given values and an unknown value. For example when $\angle A$ is the unknown, and the opposite side and hypotenuse are given, sine is the only rule which applies. When a student selects one of the other rules, Ziggy displays the message “wrong rule”. And when values are incorrectly entered into a formula, the definition is given, for example “sin = opposite / hypotenuse”.

3 Implementation

In this section we give some details of the implementation with an emphasis on techniques used to create the highly interactive interface presented in the previous section. Ziggy is entirely written in JavaScript and runs in all major web-browsers using touch control when the device allows it.

Representing examples and exercises. Ziggy contains many examples used for instruction as well as randomly generated practice exercises. For the instructional examples it is necessary, for the designer or teacher, to be able to specify which values associated with a triangle are visible to the student, which angles are interactive and what happens when a student drags an angle.

To be able to specify the examples we developed a JSON (JavaScript object notation) representation. A triangle is implicitly represented using the fundamental notions of trigonometry: points, segments (connecting two points) and angles (between two segments). A basic triangle is specified as follows (\angle is the \angle symbol):

```

id: 'basic_triangle',
points: [ { id: 'A', label: 'A' },
          { id: 'B', label: 'B' },
          { id: 'C', label: 'C' } ],
segments: [ { id: 'AB', label: 'c', points: ['A', 'B'] },
             { id: 'BC', label: 'a', points: ['B', 'C'] },
             { id: 'AC', label: 'b', points: ['A', 'C'] } ],
angles: [ { id: 'ABC', label: '\u2220 B', segments: ['AB', 'BC'] },
          { id: 'BCA', label: '\u2220 C', segments: ['BC', 'CA'] },
          { id: 'CAB', label: '\u2220 A', segments: ['CA', 'AB'] } ]

```

This basic triangle defines the three points, segments and angles, and assigns identifiers and labels to them. For a specific example details about the visualisation, values for the angles and lengths of sides are required. The specification below inherits from the basic triangle and represents the triangle shown in Figure 1.

```

id: 'explain_sin_try',
inherit_from: 'basic_triangle',
points: [ { id: 'C', x: 7.582, y: 1, visible: false },
          { id: 'B', x: 1, y: 1, visible: false },
          { id: 'A', x: 7.582, y: 4.8, visible: false }
        ],
segments: [ { id: 'AC', show: 'value' },
             { id: 'AB', show: 'value' },
             { id: 'BC', show: 'none' } ],
angles: [ { id: 'ABC', show: 'value', draggable: true },
          { id: 'BCA', show: 'value', draggable: false },
          { id: 'CAB', show: 'none', draggable: false } ],
constraints: [ { type: 'align', how: 'horizontal', points: ['B'] },
               { type: 'align', how: 'vertical', points: ['A', 'C'] },
               { type: 'fixed_angle', value: 30, angle: 'BCA' } ]

```

It is normally most convenient to define the spatial layout of a triangle by giving the x- and y-coordinates of the end points. This immediately constrains the values of the sides and the angles. The *show* attribute of segments and angles defines how the length of a segment or the degrees of an angle is visualised. Possible values for *show* are: none, value, label or both (label and value). Finally, *constraints* define which relations must be maintained while the student drags interactive angles.

The triangles and the ability to drag the angles are implemented using the model-view-controller (MVC) paradigm. The *model* represents the points, segments and angles. The *views* are the visual objects in the graphical representation as well as the values and labels of the segments and the angles. The model and view objects are automatically created and associated with each other from the JSON specification. The *controller* is the event handler used for dragging. When a student drags an angle, the model is updated by calculating the new positions and values for the model objects. These values are then propagated to the view objects.

Pushing symbols. Although the pushing symbols technique as used in Algebra Touch is conceptually and intuitively attractive, it is not obvious how the technique can be extended when equations become more complex. Algebra Touch deals with the basic mathematical operators (+, -, *, /). For each of these rearranging one of the symbols has a clear mathematical definition. In

Ziggy, we also have to deal with trigonometric functions and, in the case of the Pythagorean theorem, powers which have to be solved by taking roots. For both functions and powers, the pushing symbols technique is not possible in the same way as it is for the basic operators. We opted for tapping, rather than pushing, when an expression can be simplified unambiguously. For example, $3^2 + 4^2 = c^2$ is solved by tapping on 3^2 and 4^2 resulting in $9 + 16 = c^2$. Next, tapping $+$ to get $25 = c^2$. Then, tapping $=$ results in $\sqrt{25} = c$, and finally tapping the root gives $5 = c$.

User interface. Several common-sense user interface principles have been applied. All values shown in the graphical representation of a triangle are draggable. In an initial version the end points of segments had labels, and given that points have no role in problem solving, dragging was not possible which confused students. Dragging on touch devices is only comfortable when the objects are large enough. Moreover, putting a finger on top of a symbol causes it to be hidden from view. Therefore, symbol dragging starts, the visualisation “jumps up” such that it is no longer hidden under the finger and the student can see what is being dragged.

In this section we have briefly described how Ziggy is implemented. Part of the implementation is directly related to the requirements imposed by the trigonometry domain. For example, the “symbolic” specification of examples makes it, at least in theory, possible for educators to design their own trigonometry examples. Other parts, in particular the representation of equations, and interactively manipulating these equations using the pushing symbols method, is independent of trigonometry and could be applied to other domains.

4 Ongoing Research

We evaluated Ziggy’s potential as a learning tool. In a pilot study, the application was introduced to seven third and fourth grade (14–15 years) students with varying math skills. They worked with the application for 15 minutes. All their actions were observed focusing on ease of use and potential shortcomings in the user interface. In an interview students were asked to judge the look and feel of the application, the ease of use, and the perceived effect regarding learning effects and motivation.

In a second study we compared education using Ziggy with traditional education. A prototype of Ziggy was implemented in a real classroom setting at a secondary school in the Netherlands. Participants were 28 third grade students who followed a track on pre-university education. The group consisted of 16 males and 12 female students from two different classes. A control group, which received traditional education about trigonometry was compared to an experimental group. This experimental group was an “iPad class” and had been using a tablet in the classroom since the beginning of the school year. Both groups participated in two 50 minute lessons in basic goniometry. Both lessons started with instruction on the five rules of basic trigonometry: Pythagorean theorem, the 180° rule, and the sine, cosine and tangent. This instruction was the same for both groups. In the control group static pictures were used to illustrate the underlying principles as opposed to the dynamic and interactive triangles of Ziggy in the experimental group. In the second part of the lessons students were offered a set of exercises in which an unknown value of a triangle had to be calculated using two given values and one of the five abovementioned rules. The Ziggy group used interactive and dynamic triangles, and solved the equations using the pushing symbols technique described. The control group were offered similar

exercises on paper, using static pictures of triangles, and they used pen and paper and a calculator to compute their answers.

Preliminary findings do not support our assumption that the use of Ziggy will increase learning effects. Both groups in the second study showed similar learning outcomes on the post-test. In fact, the control group performed slightly better than the Ziggy group on items aimed at testing understanding of the dynamic relations of trig functions. At the time of the study, Ziggy was still under development and results can be partly attributed to flaws in the prototype. For example, students in the Ziggy group were not able to look up information in the instruction pages during practice. We also feel that some of the support that Ziggy offers, might actually be too much. Students quickly thought they understood the material, and processing of underlying principles seemed to remain shallow. Though this support was requested by teachers, we propose to reduce the support as students progress to stimulate them to think for themselves instead of relying on Ziggy to correct mistakes. In a new version of Ziggy, exercises have been added which are especially designed to challenge students' understanding of dynamic relationships between trig functions and stimulate a deeper understanding of the material.

Results regarding usability, motivation and perceived usefulness of the application are very positive. Students and teachers like Ziggy and indicated they believe it will be more effective than traditional material. Ziggy is said to be fun, easy-to-use and well organized. Further research will aim to match learning results to these perceptions.

5 Conclusions

This paper has described Ziggy, a touch-based application to teach trigonometry to secondary school students. The application integrates a graphical representation of triangles and the formula associated with the basic trigonometry rules. This integration has made it possible to let students first of all actively experience and experiment with the relation between the shape and size of a triangle and the resulting values of the trigonometric functions: sine, cosine and tangent. Secondly, the problem solving design in which students drag variables and values from the graphical representation to the formula potentially fosters a deeper understanding of trigonometric functions.

Although the small-scale experiments we conducted with Ziggy did not reveal a learning gain compared to traditional approaches, we believe the design is pedagogically sound. Further experimentation, in particular focusing on the level and types of support and feedback, is required. The exercises in the experiments were to a large extent the dynamic equivalent of traditional exercises. With the computer-based implementation there is the opportunity to think of new types of exercises which are not possible on paper.

The Ziggy architecture consists of two major parts. First, there is the domain-dependent graphical representation of triangles and the underlying implementation of trigonometry. Second, there is the implementation of interactively manipulation mathematical equations and solving them by pushing symbols. One area for further research is to create graphical representations of other mathematics and science domains and integrate these with the pushing symbols approach.

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