Decomposition of Intervals in the Space of Anti-Monotonic Functions

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Abstract. With the term 'anti-monotonic function', we designate specific boolean functions on subsets of a finite set of positive integers which we call the universe. Through the well-known bijective relationship between the set of monotonic functions and the set of anti-monotonic functions, the study of the anti-monotonic functions is equivalent to the study of monotonic functions. The true-set of an anti-monotonic function is an antichain. If the universe is denoted by N, the set of anti-monotonic functions is denoted by $AMF(N)$. This set can be partially ordered in a natural way. This paper studies enumeration in the resulting lattice of anti-monotonic functions. We define intervals of anti-monotonic functions according to this order and present four properties of such intervals, Finally we give a formula for the size of a general interval and a recursion formula for the n-th number of Dedekind.

Keywords: Dedekind numbers, anti-monotonic functions, antichains, complete distributive lattices

1 Intervals of Anti-Monotonic Functions

A Boolean valued function on the subsets of a set N, in short a Boolean function, is said to be anti-monotonic iff it is true for at most one of any two sets A and B for which $A \subsetneq B \subseteq N$ holds. A Boolean function is said to be monotonic iff the fact that it is true for a set B implies that it is true for any subset of B. There is a natural bijection between the set of monotonic functions and the set of anti-monotonic functions through the maximal true sets (e.g. [3]). Note furthermore that there is a bijection between anti-monotonic functions and antichains: the true sets of an anti-monotonic function form an antichain, i.e. a set of subsets of N such that no member set is included in another member set. In this paper we will use the antichain to denote an anti-monotonic function. We will denote the set of anti-monotonic functions on the set $N \subset \mathbb{N}$ by $AMF(N)$. We will use Greek letters to denote elements of $AMF(N)$ and Roman capitals to denote subsets of N, e.g. for $A \not\subseteq B, B \not\subseteq A$, $\alpha = \{A, B\} \in AMF(N)$. Unless otherwise stated, $N = \{1, \ldots n\}$ will be the set of the first n positive integers, and in this case we will occasionally use the notation $AMF(n) \equiv AMF(N)$. The size of $AMF(N)$ or $AFM(n)$ is the n-th number of Dedekind [8]. This size is known for values of n up to $n = 8$ [4]. Asymptotic expansions have been developed building on the size of the middle layer [6, 5].

Example 1. For $N = \{1, 2\}$, the Boolean function $2^N \to \mathbb{B}$, $\{\emptyset \to false, \{\emptyset\} \to false, \{1\} \to true, \{2\} \to true, \{1, 2\} \to false\}$ is anti-monotonic. The antichain of true sets is given by $\{\{1\}, \{2\}\}$. This antichain will be used to denote the anti-monotonic function.

For finite N, anti-monotonic functions form a finite distributive lattice with the join, meet and partial order given by

$$\alpha \vee \beta \ = \ max(\alpha \cup \beta) \tag{1}$$
$$\alpha \wedge \beta \ = \ max\{A \cap B | A \in \alpha, B \in \beta\} \tag{2}$$
$$\alpha \leq \beta \ \Leftrightarrow \ \forall A \in \alpha : \exists B \in \beta : A \subseteq B \tag{3}$$
$$(\Leftrightarrow \alpha \vee \beta = \beta \Leftrightarrow \alpha \wedge \beta = \alpha),$$

where for a general set S of subsets of N, $max(S)$ is the set containing only the largest sets in S according to $\subseteq$. A comprehensive textbook on Boolean functions is [2]. A recent study on counting non-equivalent monotone Boolean functions is found in [1]. Our antichains correspond to the notion of minimal sets playing an important role in the latter paper. A first analysis of intervals and decomposition is in [9]. We will make extensive use of the intervals in this lattice. For two antichains $\alpha, \beta \in AMF(N)$, the closed interval with bounds α and β is given by

$$[\alpha, \beta] = \{\chi \in AMF(N) | \alpha \leq \chi \leq \beta\}. \tag{4}$$

Analogous definitions hold for (half)open intervals. Note that these intervals are empty in case $\alpha \not\leq \beta$, in particular in case of non comparable α and β.

2 Disconnected Intervals

Given two intervals $[\rho_1, \rho_2]$ and $[\rho_1', \rho_2']$ in $AMF(N)$, we have

$$\{\chi \vee \chi' | \chi \in [\rho_1, \rho_2], \chi' \in [\rho_1', \rho_2']\} = [\rho_1 \vee \rho_1', \rho_2 \vee \rho_2']. \tag{5}$$

We will use the notation

$$[\rho_1, \rho_2] \vee [\rho_1', \rho_2'] \equiv [\rho_1 \vee \rho_1', \rho_2 \vee \rho_2']. \tag{6}$$

A fundamental property in the decomposition of intervals is related to the concept of (dis)-connectedness. Two intervals are said to be disconnected if the decomposition in equation 6 is unique. Two intervals are connected if they are not disconnected. If two intervals are disconnected, we will call the join of these intervals *direct*:

Definition 1. *The interval* $[\rho_1 \vee \rho'_1, \rho_2 \vee \rho'_2]$ *is the direct join of two intervals* $[\rho_1, \rho_2]$ *and* $[\rho'_1, \rho'_2]$ *in* $AMF(N)$ *if the intervals are disconnected. The direct join is denoted by* $[\rho_1, \rho_2] \otimes [\rho'_1, \rho'_2]$.

Example 2. For $N = \{1, 2, 3\}$, we have $[\{\{1\}\}, \{\{1, 3\}\}] \vee [\{\{2\}\}, \{\{2, 3\}\}] = [\{\{1\}, \{2\}\}, \{\{1, 3\}, \{2, 3\}\}]$. The element $\{\{1\}, \{2\}, \{3\}\} = \{\{1\}\} \vee \{\{2\}, \{3\}\} = \{\{1\}, \{3\}\} \vee \{\{2\}\}$ shows that the two intervals on the lefthand side are connected. In the case of $[\{\{1\}, \{3\}\}, \{\{1, 3\}\}] \vee [\{\{2\}, \{3\}\}, \{\{2, 3\}\}] = [\{\{1\}, \{2\}, \{3\}\}, \{\{1, 3\}, \{2, 3\}\}]$, we see that the underlying intervals are disconnected and $[\{\{1\}, \{2\}, \{3\}\}, \{\{1, 3\}, \{2, 3\}\}] = [\{\{1\}, \{3\}\}, \{\{1, 3\}\}] \otimes [\{\{2\}, \{3\}\}, \{\{2, 3\}\}]$.

3 Decomposition Theorem

The decomposition of intervals is based on the following Theorem 1, which is actually valid in a general distributive lattice.

Lemma 1. *For two anti-monotonic functions* $\alpha, \beta \in AMF(N)$ *we have*

$$[\alpha \wedge \beta, \alpha \vee \beta] = [\alpha \wedge \beta, \alpha] \otimes [\alpha \wedge \beta, \beta]. \tag{7}$$

Proof. It is clear that for $\chi_\alpha \in [\alpha \wedge \beta, \alpha]$ *and* $\chi_\beta \in [\alpha \wedge \beta, \beta]$*, we have* $\chi_\alpha \vee \chi_\beta \in [\alpha \wedge \beta, \alpha \vee \beta]$*. Moreover, since* $\chi_\beta \wedge \alpha \leq \beta \wedge \alpha$*, we have* $\chi_\alpha = (\chi_\alpha \vee \chi_\beta) \wedge \alpha$ *and similarly* $\chi_\beta = (\chi_\alpha \vee \chi_\beta) \wedge \beta$*. For* $\chi \in [\alpha \wedge \beta, \alpha \vee \beta]$ *we have* $\chi \wedge \alpha \in [\alpha \wedge \beta, \alpha], \chi \wedge \beta \in [\alpha \wedge \beta, \beta]$ *and*

$$(\chi \wedge \alpha \vee \chi \wedge \beta) = \chi \wedge (\alpha \vee \beta) = \chi. \tag{8}$$

More generally we have

Theorem 1. *For three anti-monotonic functions* $\alpha, \beta, \rho \in AMF(N)$ *such that* $\rho \in [\alpha \wedge \beta, \alpha \vee \beta]$ *we have*

$$[\rho, \alpha \vee \beta] = [\rho, \alpha \vee \rho] \otimes [\rho, \beta \vee \rho]. \tag{9}$$

Proof. Note that $(\alpha \vee \rho) \wedge (\beta \vee \rho) = (\alpha \wedge \beta) \vee \rho = \rho$ *so that the interval* $[\rho, \alpha \vee \beta] = [\rho, (\alpha \vee \rho) \vee (\beta \vee \rho)]$ *satisfies the conditions of Lemma 1. Any* $\chi \in [\rho, \alpha \vee \beta]$ *has the unique decomposition* $\chi = (\chi \wedge (\alpha \vee \rho)) \vee (\chi \wedge (\beta \vee \rho))$*.*

Theorem 1 can be strengthened as

Theorem 2. *For three anti-monotonic functions* $\alpha, \beta, \rho \in AMF(N)$ *such that* $\rho \in [\alpha \wedge \beta, \alpha \vee \beta]$ *we have*

$$[\rho, \alpha \vee \beta] = [\rho \wedge \alpha, \alpha] \otimes [\rho \wedge \beta, \beta]. \tag{10}$$

Proof. Any $\chi \in [\rho, \alpha \vee \beta]$ *satisfies* $\chi = (\chi \wedge \alpha) \vee (\chi \wedge \beta)$*, with* $\chi \wedge \alpha \in [\rho \wedge \alpha, \alpha], \chi \wedge \beta \in [\rho \wedge \beta, \beta]$*. Any* $\chi = \chi_\alpha \vee \chi_\beta$ *with* $\chi_\alpha \in [\rho \wedge \alpha, \alpha], \chi_\beta \in [\rho \wedge \beta, \beta]$ *is in* $[\rho, \alpha \vee \beta]$*. Furthermore* $\chi \wedge \alpha = (\chi_\alpha \wedge \alpha) \vee (\chi_\beta \wedge \alpha)$ *where* $\chi_\alpha \wedge \alpha = \chi_\alpha$ *(since* $\chi_\alpha \in [\rho \wedge \alpha, \alpha]$*) and* $\chi_\beta \wedge \alpha \leq \beta \wedge \alpha$ *(since* $\chi_\beta \in [\rho \wedge \beta, \beta]$*) so that* $\chi_\beta \wedge \alpha \leq \rho \wedge \alpha \leq \chi_\alpha$*, and we conclude* $\chi_\alpha = \chi \wedge \alpha$*. Equivalently, we obtain* $\chi_\beta = \chi \wedge \beta$ *proving the uniqueness of decomposition.*

Corollary 1. *For any two anti-monotonic functions α, ρ, the intervals $[\rho, \rho \vee \alpha]$ and $[\rho \wedge \alpha, \alpha]$ are isomorphic lattices.*

Proof. *Since $\rho \wedge \alpha \leq \rho$, we can apply Theorem 2 to find $[\rho, \rho \vee \alpha] = [\rho, \rho] \otimes [\rho \wedge \alpha, \alpha]$. This implies that $[\rho \wedge \alpha, \alpha] \rightarrow [\rho, \rho \vee \alpha] : \chi \rightarrow \rho \vee \chi$ defines an isomorphism with inverse $[\rho, \rho \vee \alpha] \rightarrow [\rho \wedge \alpha, \alpha] : \chi \rightarrow \alpha \wedge \chi$.*

Corollary 2. *For two anti-monotonic functions $\rho_1, \rho_2 = \vee_{i \in I} \alpha_i$ with $\forall i, j \in I : \alpha_i \wedge \alpha_j \leq \rho_1$, we have*

$$[\rho_1, \rho_2] = \otimes_{i \in I} [\rho_1, \rho_1 \vee \alpha_i] \tag{11}$$

$$= \otimes_{i \in I} [\rho_1 \wedge \alpha_i, \alpha_i]. \tag{12}$$

Proof. *The proof follows from a simple iteration over the indices $i \in I$, applying Theorems 1 and 2 for each component α_i, $i \in I$.*

In the following, we will use the notation $o_{\rho,\gamma}$ for any two anti-monotonic functions $\rho \geq \gamma$ to denote the largest χ for which $\chi \wedge \rho = \gamma$. A general partition of an interval is given by Theorem 3.

Theorem 3. *For anti-monotonic functions $\rho_1 \leq \rho \leq \rho_2$*

$$[\rho_1, \rho_2] = \cup_{\gamma \in [\rho_1, \rho]} [\gamma, o_{\rho,\gamma} \wedge \rho_2]. \tag{13}$$

The intervals $[\gamma, o_{\rho,\gamma} \wedge \rho_2]$ for $\gamma \leq \rho$ are disjoint and nonempty.

Proof. *For each $\gamma \in [\rho_1, \rho]$ consider the set $S_\gamma = \{\chi \in [\rho_1, \rho_2] | \chi \wedge \rho = \gamma\}$. These sets are disjoint. Since for each $\chi \in [\rho_1, \rho_2]$, we have $\chi \wedge \rho \in [\rho_1, \rho]$, the union of these sets is the whole interval. γ is a lower bound on S_γ. Since $(\chi_1 \wedge \rho = \gamma$ and $\chi_2 \wedge \rho = \gamma) \Rightarrow (\chi_1 \vee \chi_2) \wedge \rho = \gamma$, the set has exactly one maximal element. We denote this element in the case of $\rho_2 = \{N\}$ by $o_{\rho,\gamma}$. Since, in addition, $\chi_1 \wedge \rho = \gamma, \chi_2 \wedge \rho = \gamma \Rightarrow (\chi_1 \wedge \chi_2) \wedge \rho = \gamma$, the set of all solutions to the equation in the lattice is closed under $\wedge$ and $\vee$ and hence equals the full interval $[\gamma, o_{\rho,\gamma}]$. For general ρ_2, S_γ is the intersection with $[\rho_1, \rho_2]$ which is given by $[\gamma, o_{\rho,\gamma} \wedge \rho_2]$.*

The function $o_{\rho,\gamma}$ defined for any $\rho \geq \gamma \in AMF(N)$ is the top of the interval $[\gamma, o_{\rho,\gamma}] = \{\chi | \chi \wedge \rho = \gamma\}$. It is given by

$$o_{\rho,\gamma} = \widetilde{\tilde{\gamma} \setminus \tilde{\rho}} \tag{14}$$

where $\tilde{}$ denotes the dual in the lattice $AMF(N)$.

Note 1. Note that the theorems so far, including the proofs, did not refer explicitly to anti-monotonic functions. In fact, they only relied on the properties of the operators $\wedge$ and $\vee$ and are seen to be valid in any complete distributive lattice. The following sections specifically refer to the definition of an anti-monotonic function as a function on subsets of a superset. Although we believe, the following properties, especially Theorem 4, can be generalized as well, we for now restrict the discussion to the space $AMF(N)$.

In what follows, disconnectedness of intervals turns out to be related to a corresponding property of the top of the interval.

Definition 2. *Given two anti-monotonic functions $\rho, \alpha \in AMF(N)$ with $\rho \leq \alpha$. Two sets $A, B \in \alpha$ are said to be connected with respect to ρ if and only if $\{A \cap B\} \not\leq \rho$. Connectedness of such sets is denoted by $C_\rho(A, B)$. $C_\rho(.,.)$ defines a graph with the sets of α as vertices. The vertices of each connected component of this graph correspond to a subset of α and thus to an anti-monotonic function. We will refer to these anti-monotonic functions as the connected components of α with respect to ρ and denote the set of such components by $C_{\rho,\alpha}$.*

We now have

Corollary 3. *For anti-monotonic functions $\rho_1, \rho_2 \in AMF(N)$ with $\rho_1 \leq \rho_2$*

$$[\rho_1, \rho_2] = \otimes_{\chi \in C_{\rho_1,\rho_2}} [\rho_1 \wedge \chi, \chi]. \tag{15}$$

Proof. The proof follows immediately from Corollary 2.

Corollary 3 leads to Algorithm 1 for the total decomposition of an interval. Examples 3, 4 and 5 illustrate how the algorithm works.

Example 3. Consider the interval $[\{\emptyset\}, \{\{1,2\},\{3\}\}]$. Since we have $\{\{1,2\} \cap \{3\}\} = \{\emptyset\} \leq \{\emptyset\}$ so that $C_{\{\emptyset\},\{\{1,2\},\{3\}\}} = \{\{\{1,2\}\},\{\{3\}\}\}$, and
$[\{\emptyset\}, \{\{1,2\},\{3\}\}] = [\{\emptyset\}, \{\{1,2\}\}] \otimes [\{\emptyset\}, \{\{3\}\}]$.

Example 4. Consider the interval $[\{\{4\}\}, \{\{1,2,4\},\{3,4\}\}]$. Since we have $\{\{1,2,4\} \cap \{3,4\}\} = \{\{4\}\} \leq \{\{4\}\}$ so that $C_{\{\{4\}\},\{\{1,2,4\},\{3,4\}\}} = \{\{\{1,2,4\}\},\{\{3,4\}\}\}$, and
$[\{\{4\}\}, \{\{1,2,4\},\{3,4\}\}] = [\{\{4\}\}, \{\{1,2,4\}\}] \otimes [\{\{4\}\}, \{\{3,4\}\}]$.

Example 5. Consider the interval $[\{\{4\},\{6\}\}, \{\{1,2,4\},\{3,4\},\{3,5,6\}\}]$. Since we have
$\{\{1,2,4\} \cap \{3,4\}\} = \{\{4\}\} \leq \{\{4\},\{6\}\}$, $\{\{1,2,4\} \cap \{3,5,6\}\} = \{\emptyset\} \leq \{\{4\},\{6\}\}$,
$\{\{3,4\} \cap \{3,5,6\}\} = \{\{3\}\} \not\leq \{\{4\},\{6\}\}$,
so that $C_{\{\{4\},\{6\}\},\{\{1,2,4\},\{3,4\},\{3,5,6\}\}} = \{\{\{1,2,4\}\},\{\{3,4\},\{3,5,6\}\}\}$, and
$[\{\{4\},\{6\}\}, \{\{1,2,4\},\{3,4\},\{3,5,6\}\}] = [\{\{4\}\}, \{\{1,2,4\}\}] \otimes [\{\{4\},\{6\}\}, \{\{3,4\},\{3,5,6\}\}]$.

Algorithm 1 Decompose the interval $[\rho_1, \rho_2]$

Require: $\rho_1 \leq \rho_2$
Ensure: Result is a set of intervals, $\{I_1, I_2, \ldots, I_k\}$, such that $[\rho_1, \rho_2] = \otimes_{i=1\ldots k} I_i$
  function DECOMPOSEINTERVAL(ρ_1, ρ_2)
    Compute the set C_{ρ_1,ρ_2} of connected components according to Definition 2.
    return $\{[\rho_1 \wedge \gamma, \gamma] | \gamma \in C_{\rho_1,\rho_2}\}$
  end function

In what follows, we will use the following notations

Definition 3. *Let $\rho_1, \rho_2 \in AMF(N)$, $\rho_1 \leq \rho_2$. We use the notion of level λ_l in the interval $[\rho_1, \rho_2]$ to denote maximal anti-monotonic functions consisting of elements of a specific size l, and we introduce the $(.)^+$ and $(.)^-$ operators to transform functions from one level to a neighboring level, as follows:*

$$\lambda_l = \{A \subseteq N | \rho_1 \vee \{A\} \in]\rho_1, \rho_2], |A| = l\} (\forall l \geq 0), \quad (16)$$
$$\alpha^+ = \{X \in \lambda_{l+1} | \forall x \in X : X\backslash\{x\} \in \lambda_l \Rightarrow X\backslash\{x\} \in \alpha\}, (\forall \alpha \subseteq \lambda_l, \forall l \geq 0), \quad (17)$$
$$\alpha^- = \{X \in \lambda_{l-1} | \exists A \in \alpha : X \subseteq A\}, (\forall \alpha \subseteq \lambda_l, \forall l > 0). \quad (18)$$

Note that in Definition 3, in (17) $\alpha^+ \subseteq \lambda_{l+1}$ and in (18) $\alpha^- \subseteq \lambda_{l-1}$,

4 Decomposition of a General Interval into Computationally Easy Intervals

We will now use decomposition to compute the size of any interval. Theorem 4 builds on the following Lemma.

Lemma 2. *Let $\rho_1, \rho_2 \in AMF(N)$, $\rho_1 \leq \rho_2$ and $\chi \in [\rho_1, \rho_2]$. Then χ has the following unique decomposition:*

$$\chi = \rho_1 \vee \chi_0 \vee \chi_1 \vee \chi_2 \vee \ldots, \quad (19)$$
$$\text{where } \forall l \geq 0 : \chi_l \subseteq \lambda_l,$$
$$\chi_{l+1}^- \leq \chi_l,$$
$$\chi_{l+1} \leq \chi_l^+.$$

Proof. *We start from the decomposition in sets of specific levels:*

$$\chi = (\chi \cap \rho_1) \vee (\chi \cap \lambda_0) \vee (\chi \cap \lambda_1) \vee (\chi \cap \lambda_2) \vee \ldots (\chi \cap \lambda_{s-1}) \vee (\chi \cap \lambda_s) \quad (20)$$

where s is the size of the largest set in χ. We now set $\chi_l = \emptyset$ for $l > s$. Further, let $\chi_s = \chi \cap \lambda_s$ and note that the decomposition does not change if we add χ_s^- to the sets of level $s-1$.

$$\chi = (\chi \cap \rho_1) \vee (\chi \cap \lambda_0) \vee (\chi \cap \lambda_1) \vee (\chi \cap \lambda_2) \vee \ldots (\chi \cap \lambda_{s-1} \vee \chi_s^-) \vee \chi_s. \quad (21)$$

This suggests the recursive definition

$$\forall l \in \{0, \ldots, s-1\} : \chi_l = (\chi \cap \lambda_l) \vee \chi_{l+1}^- \quad (22)$$

leading to

$$\chi = (\chi \cap \rho_1) \vee \chi_0 \vee \chi_1 \vee \chi_2 \vee \ldots \chi_{s-1} \vee \chi_s \quad (23)$$

and since $\chi \geq \rho_1$:

$$\chi = \rho_1 \vee \chi_0 \vee \chi_1 \vee \chi_2 \vee \ldots \chi_{s-1} \vee \chi_s. \quad (24)$$

The inequalities in (19) now follow immediately from (22) and Definition 3. Given the decomposition (19), it follows immediately that $\chi_s = \chi \cap \lambda_s$. $\chi_{s-1} \geq \chi_s^-$

implies $\chi_s^- \subseteq \chi_{s-1}$. Furthermore, necessarily $\chi \cap \lambda_{s-1} \subseteq \chi_{s-1}$ so that we find $\chi_{s-1} \geq \chi_s^- \vee (\chi \cap \lambda_{s-1})$. Since any set in χ_{s-1} not dominated by a set in χ_s is necessarily in χ, we have $\chi_{s-1} \leq \chi_s^- \vee (\chi \cap \lambda_{s-1})$ and equality follows. Recursive application of this reasoning proves uniqueness.

Theorem 4. *For $\rho_1, \rho_2 \in AMF(N)$ with $\rho_1 \leq \rho_2$, we have*

$$|[\rho_1, \rho_2]| = \sum_{\alpha_1 \subseteq \lambda_1} \sum_{\alpha_3 \subseteq \alpha_1^{++}} \sum_{\alpha_5 \subseteq \alpha_3^{++}} \ldots 2^{|\lambda_0| - |\alpha_1^-| + |\alpha_1^+| - |\alpha_3^-| + |\alpha_3^+| - |\alpha_5^-| \ldots} \quad (25)$$

$$= \sum_{\alpha_0 \subseteq \lambda_0} \sum_{\alpha_2 \subseteq \alpha_0^{++}} \sum_{\alpha_4 \subseteq \alpha_2^{++}} \ldots 2^{|\alpha_0^+| - |\alpha_2^-| + |\alpha_2^+| - |\alpha_4^-| \ldots}. \quad (26)$$

Proof. *Note that the number of non trivial summations in (25) and (26) is always finite: there is a maximal level for any finite interval, above this level α^{++} will be empty and the contribution in the power of 2 will be zero. Given the decomposition in Lemma 2, and a list of specific levels $l_1 < l_2 < \ldots < l_k$ where $\sigma_{l_i} \subseteq \lambda_{l_i}$ are given such that*

$$\forall l_i, l_{i+1} : \sigma_{l_{i+1}}^{-d_i} \leq \sigma_{l_i}$$
$$\textit{and } \forall l_{i-1}, l_i : \sigma_{l_i} \leq \sigma_{l_{i-1}}^{+d_{i-1}}$$

where $d_i = l_{i+1} - l_i$ and $\alpha^{+/-d} = (\ldots((\alpha^{+/-})^{+/-})\ldots)^{+/-}$ (d operators $(.)^{+/-}$), one can ask for the set of χ decomposing according to (19) such that $\forall i : \chi_{l_i} = \sigma_{l_i}$. This set has a lower bound $\chi_b = \rho_1 \vee \sigma_{l_0}^{-l_0} \vee \sigma_{l_0}^{-(l_0-1)} \ldots \vee \sigma_{l_0} \vee \sigma_{l_1}^{-d_0} \vee \sigma_{l_1}^{-(d_0-1)} \ldots$ and an upper bound $\chi_t = \rho_1 \vee \lambda_0 \vee \lambda_1 \ldots \vee \lambda_{l_0-1} \vee \sigma_{l_0} \vee \sigma_{l_0}^+ \vee \sigma_{l_0}^{+2} \ldots \vee \sigma_{l_0}^{+(l_1-l_0-1)} \vee \sigma_{l_1} \vee \sigma_{l_1}^+ \ldots$. In fact, all elements in the interval $[\chi_b, \chi_t]$ satisfy this requirement. In the case of all odd, respectively all even, levels given, summing the sizes of all such intervals over all possible specifications σ_{2l+1}, respectively σ_{2l}, results in the expansions of the Theorem.

Theorem 4 allows to compute intervals in $AMF(N)$ for $|N| = 7$, and computes all intervals for $|N| = 6$ in milliseconds.

Example 6. As a simple application of Theorem 4, consider intervals of the form $I_N = [\{\emptyset\}, \binom{N}{2}]$ where, for convenience, we use the notation $\binom{N}{k}$ to denote the set of subsets of size k of a set $N = \{1, 2, ..., n\}$. I_N is seen to have only two nonempty levels. Indeed, $\lambda_0 = \{\}$, λ_1 is the set of all singletons of elements of N and $\lambda_2 = \binom{N}{2}$. Since $\lambda_0 = \{\}$, for each $\alpha_1 \subseteq \lambda_1$ we have $\alpha_1^- = \{\}$, while $\alpha_1^+ = \binom{span(\alpha_1)}{2}$ [1]. We find

$$|I_N| = \sum_{\alpha_1 \subseteq \lambda_1} 2^{|\lambda_0| - |\alpha_1^-| + |\alpha_1^+|} = \sum_{S \subseteq N} 2^{\binom{|S|}{2}}$$
$$= \sum_{i=0}^{n} \binom{n}{i} 2^{\binom{i}{2}}, \quad (27)$$

[1] Here the span of an anti-monotonic function is the set of elements occurring in true sets of the function, i.e. $span(\alpha) = \bigcup_{X \in \alpha} X$

which is the well known formula for the number of labeled graphs with at most n nodes (Sloane series A006896 [7]). This identity becomes obvious when we apply the alternative expression:

$$|I_N| = \sum_{\alpha_2 \subseteq \lambda_2} 2^{|\lambda_1| - |\alpha_2^-|} \tag{28}$$

$$= \sum_{graphs\ g\ on\ n\ vertices} 2^{n - |vertices\ in\ g|}$$

$$= \sum_{i=0}^{n} |graphs\ covering\ \{1,2,...,i\}| \binom{n}{i} 2^{n-i}. \tag{29}$$

5 A Recursion Formula for the Size of the Complete Space

The previous sections were concerned with the structure of arbitrary intervals. In this section, we present a formula for the size of $AMF(n+k), k \geq 0$ summing over the space $AMF(n)$. The formula is used to generate an efficient algorithm to compute the size of $AMF(n+2)$ from $AMF(n)$. We start from the following observation, using the operator $\times$ defined as [2]

$$\forall \chi \in AMF(N), S \subset \mathbb{N}, S \cap N = \emptyset : \chi \times \{S\} = \{X \cup S | X \in \chi\}. \tag{30}$$

Example 7. For $\chi = \{\{1\}, \{2,3\}, \{3,4\}\}$ and $S = \{5,6,7\}$ we have according to this definition $\chi \times \{S\} = \{\{1,5,6,7\}, \{2,3,5,6,7\}, \{3,4,5,6,7\}\}$.

We can now prove

Lemma 3. *Given $n, k > 0$, $N = \{1, \ldots, n\}$ and $K_n = \{n+1, \ldots, n+k\}$, for each $\chi \in AMF(N \cup K)$ there exists exactly one sequence $\{\chi_{\{S\}} | S \subseteq K_n\}$ of functions in $AMF(N)$ such that*

$$\chi = \bigvee_{S \subseteq K_n} \chi_S \times \{S\}, \tag{31}$$

$$\forall S \subseteq K_n \ : \ \chi_S \in AMF(N), \tag{32}$$

$$\forall S, S' \subseteq K_n \ : \ S \subseteq S' \Rightarrow \chi_S \geq \chi_{S'}. \tag{33}$$

Proof. *For each $S \subseteq K_n$ define $\chi_S = \{X \backslash K_n | X \in \chi, S \subseteq X\}$*

Corollary 4. *For finite $N, K \subseteq \mathbb{N}, N \cap K = \emptyset$ as in Lemma 3, the size of $AMF(N \cup K)$ is equal to the number of homomorphisms $(2^K, \subseteq) \rightarrow (AMF(N), \geq)$.*

[2] This is a restricted definition of the $\times$ operator discussed extensively in [9]. This paper introduces an effective enumeration technique which can be used in to sum over the space $AMF(N)$.

Now consider the restricted homomorphisms $(2^K\backslash\{\emptyset, K\}, \subseteq) \to (AMF(N), \geq)$, i.e fix χ_S for any $S \notin \{\emptyset, N\}$. Any such restricted homomorphism can be completed by components $\chi_0 \geq \bigvee_{k\in K} \chi_{\{k\}}$ and $\chi_N \leq \bigwedge_{k\in K} \chi_{N\backslash\{k\}}$. We define coefficients P_{N,K,ρ_0,ρ_N} as follows

Definition 4. *For finite $N, K \subseteq \mathbb{N}, N\cap K = \emptyset$, and for $\rho_0, \rho_N \in AMF(N), \rho_0 \geq \rho_N$, P_{N,K,ρ_0,ρ_N} is the number of homomorphisms $f : (2^K\backslash\{\emptyset, K\}, \subseteq) \to ([\rho_N, \rho_0], \geq)$ such that $\bigvee_{k\in K} f(\{k\}) = \rho_0$ and $\bigwedge_{k\in K} f(N\backslash\{k\}) = \rho_N$*

and we find

Theorem 5. *For finite $N, K \subseteq \mathbb{N}, N \cap K = \emptyset$,*

$$|AMF(N \cup K)| = \sum_{\rho_0 \geq \rho_N \in AMF(N)} |[\emptyset, \rho_N]| P_{N,K,\rho_0,\rho_N} |[\rho_0, \{N\}]|. \quad (34)$$

Proof. *Any restricted homomorphism f can be extended by elements of the intervals $[\emptyset, \rho_N]$ and $[\rho_0, \{N\}]$, and any extension results in a different function in $AMF(N \cup K)$.*

The P-coefficients are in general hard to compute. In the special case of $|K| = 2$ however, the following property leads to a simple algorithm.

Property 1. For finite $N, K \subseteq \mathbb{N}, N \cap K = \emptyset, |K| = 2$, we have

$$P_{N,K,\rho_0,\rho_N} = 2^{|C_{\rho_N\backslash\rho_0,\rho_0\backslash\rho_N}|}. \quad (35)$$

Proof. Let $K = \{k_1, k_2\}$. The coefficient is the number of solutions to the simultaneous equations

$$\chi_{\{k_1\}} \vee \chi_{\{k_2\}} = \rho_0, \quad (36)$$
$$\chi_{\{k_1\}} \wedge \chi_{\{k_2\}} = \rho_N. \quad (37)$$

Let $A, B \subseteq \rho_0\backslash\rho_N$ such that $C_{\rho_N}(A, B)$, i.e. $\{A \cap B\} \not\leq (\rho_N\backslash\rho_0)$. Then A and B must be in at least one of $\chi_{\{k_1\}}$ or $\chi_{\{k_2\}}$ due to (36) and in at most one due to (37). On the other hand, any set A in $\rho_0\backslash\rho_N$ must be in either $\chi_{\{k_1\}}$ or $\chi_{\{k_2\}}$ and can not be in both.

We thus obtain the formula

$$|AMF(n + 2)| = \sum_{\rho_0 \geq \rho_n \in AMF(n)} |[\emptyset, \rho_n]||[\rho_0, \{N\}]| 2^{C_{\rho_n\backslash\rho_0,\rho_0\backslash\rho_n}}. \quad (38)$$

A Java implementation of Algorithm 2 , summing over non equivalent functions only for ρ_N in $AMF(6)$, allowed to compute $AMF(8)$ in 40 hours on a Macbook Pro. Note that the order of summation can be chosen such as to minimize the number of evaluations of the interval sizes $|[\rho_0, \{N\}]|$. Indeed, although the sizes of these intervals could be computed through the mapping Size of the representative of the class of the dual of ρ_0, these transformations still are computationally intensive.

Algorithm 2 Recursion formula using P-coefficients to compute $|AMF(n+2)|$ by enumeration of $AMF(n)$

Require: $n \in \mathbb{N}, n \geq 0$.
Ensure: Result is Dedekind Number $n+2$, this is the size of the space $AMF(n+2)$.
function DEDEKINDNUMBER(n+2)
Compute the set R of nonequivalent representatives in $AMF(n)$ of equivalence classes under permutation of N.
Compute the cardinalities of each of the equivalence classes as COUNT : $R->\mathbb{N}$.
For each $\rho \in R$, compute the size of $[\emptyset, \rho]$ as SIZE : $R->\mathbb{N}$.
$sum \leftarrow 0$
for $\rho_0 \in AMF(n)$ **do**
$partialSum \leftarrow 0$
for $\rho_N \in R, \rho_N \leq \rho_0$ **do**
$partialSum \leftarrow partialSum+$COUNT$(\rho_N)*$SIZE$(\rho_N)*2^{|C_{\rho_N \setminus \rho_0, \rho_0 \setminus \rho_N}|}$
end for
$sum \leftarrow sum + partialSum * |[\rho_0, \{N\}]|$
end for
return sum.
end function

6 Conclusions and Future Research

In this paper, we analyzed intervals in the space of anti-monotonic functions. Some structural properties were derived which allowed decomposition. We used the properties of intervals to derive a formula allowing efficiently computing the size of any interval in spaces with values of n up to 7. Finally, we derived an expansion of the size of the $(n+k)$th space based on an enumeration of the space n. The terms in this expansion are products of sizes of intervals multiplied by coefficients which we termed 'P-coefficients of order k'. P-coefficients of order 2 turn out to be efficiently computable, and the resulting formula, combined with a reduction to nonequivalent anti-monotonic functions, allowed computing the 8th number of Dedekind on a very standard laptop in less than two days.

The results in sections $1-3$ were obtained using the operators $\wedge$ and $\vee$ only and are thus valid for any distributive lattice. The proofs of the results in sections $4, 5$ explicitly relied on properties of sets. It is not hard to see that there are more general equivalents of these formulae. We plan to extend the analysis of intervals and derive the more general equivalents in a forthcoming paper.

The success in computing $|AMF(8)|$ on a basic laptop naturally leads to the question how far a more sophisticated hardware could take us towards computing $|AMF(9)|$. We are presently undertaking such an attempt, but new idea's will be needed to succeed. Computing $|AMF(9)|$ using P-coefficients of order 2 according to Algorithm 2 involves enumerating $AMF(7) \times$ *nonequivalent* $AMF(7)$ which is exactly $2414682040998 \times 490013148 = 1183225948328495041704$ terms. Each term would require computing a second order P-coefficient and multiplying two interval sizes of intervals in $AMF(7)$. There would be 490013148 such interval sizes to compute.

More promising is the study of the P-coefficients of higher order. Algorithm 2 for $n = 6$ equipped with a fast evaluator for order 3 P-coefficients could produce $|AMF(9)|$. Study of these higher order P-coefficients hence is on our research agenda.

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