Towards Description Logic on Concept Lattices*

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Abstract. In this position paper we start with a motivation of our study of modal/description logics with values in concept lattices. Then we give a brief survey of approaches to lattice-valued modal and/or description logics. After that we study some methods of context symmetrization, because the description logic on concept lattices is defined for symmetric co ntexts only. We conclude with a list of problems related to comparison of different lattice-valued modal/description logics, different variants of context symmetrization and resulting description logics, decidability and axiomatization of these logics.

1 Introduction

Let us start with an example that can explain our interest to study of polymodal and/or description logics with values in concept lattices. For it let us first fix some moment of time and let (1) URL be the set of all Uniform Resource Locator that are valid (exist) at this moment, (2) Key be the set of all Key-words in any existing language that are conceivable in this time, (3) F, S and T be binary relations on $URL \times Key$ that are implemented in some (non-real we assume) search engines First, Second and Third at the same moment of time that we fixed above.

Then let *Sh&Ga* be the set of all web-sites (their URL's hereinafter) that a search engine First finds by two key-words *Shilov* and *Garanina*; In terms of Formal Concept Analysis (FCA) [4] $Sh\&Ga = \{Shilov, Garanina\}'$ in the following formal context $\mathbb{F} = (URL,\ KW,\ F)$. Similarly, let Gr — be the set of all web-sites that Second finds by searching for a single key-word *Grebeneva*; in FCA terms one can write $Gr = \{Grebeneva\}'$ in the next one formal context $\mathbb{S} = (URL,\ Key,\ S)$.

Assume that we need to know all sites $Sh\&Ga \setminus Gr$ that are found by First by key-words *Shilov* and *Garanina* but that (according to Third) does not contain any word that is common for all sites that are found by Second for the key-word *Grebeneva*. In terms of set theory expanded by FCA-derivatives the desired set can be written as $URL_{Sh\&Ga} \setminus URL''_{Gr}$, where '$'$'

* This research is supported by Integration Grant *Mathematical and methodological aspects of intellectual Information Systems* n.15/10 of Siberian Branch, Russian Academy of Science.

represents derivative in the formal context $\mathbb{T} = (URL,\ Key,\ F)$. Recall that $URL_{Sh\&Ga} = \{Shilov, Garanina\}'$ in $\mathbb{K}_G$, $URL_{Gr} = \{Grebeneva\}'$ in $\mathbb{K}_Y$. So we can not write the following equality

$$URL_{Sh\&Ga\setminus Gr} = \{Shilov, Garanina\}' \setminus \{Grebeneva\}''',$$

but have to write another one

$$URL_{Sh\&Ga\setminus Gr} = \{Shilov, Garanina\}^{\downarrow_F} \setminus \{Grebeneva\}^{\downarrow_S\uparrow_T\downarrow_T},$$

where '$\downarrow_F$' represents the lower derivative in the context $\mathbb{F}$, '$\downarrow_S$' — the lower derivative in the context $\mathbb{S}$, and '$\downarrow_T$' and '$\uparrow_T$' — the lower and upper derivatives in the context $\mathbb{T}$. We believe that it would be nice to process queries like this one but (unfortunately) modern search engines can not do it; a part of the reason for this inability is due to lack of theory for processing such multi-context queries.

At the same time polymodal and/or descriptive logics (DL) [1] provide language for presentation of queries as above. In particular, if T^d denotes the inverse of the binary relation T, then $Sh\&Ga \setminus Gr$ may be represented in syntax of a polymodal logic by the following formula

$$[F](Shilov\&Garanina)\ \&\ \neg[T][T^d][S]Grebeneva$$

or in syntax of a description logic as the following concept term

$$\forall F.(Shilov \sqcap Garanina)\ \sqcap\ \neg\forall T.\forall T^d.\forall S.Grebeneva.$$

An interpretation of FCA constructs in DL has been studied in [7]. In these studies DL has been extended by FCA-derivatives and provided with Kripke semantics; as a result all concept terms are interpreted by sets of objects, not by concepts (or their extents), a lattice-theoretic structure of formal concepts (that is so important advantage of FCA) is lost.

A variant of description logic (namely $\mathcal{ALC}$, Attribute Language with Complements) with values in concept lattices was defined in [8] but without a concept negation; the concept negation was defined only in concept lattices for symmetric contexts (i.e. in contexts where sets of objects and attributes are equal and the binary relation is symmetric). It implies that if we would like to define concept lattice semantics for the DL concept term above, we have to symmetrize contexts $\mathbb{F}$, $\mathbb{S}$ and $\mathbb{T}$ in some manner. In this position paper we present some preliminary results of our studies of ways of context symmetrization, formulate and discuss some topics that need more research.

2 Lattice-valued Modal and Description Logics

Modal and Description Logic are closely related but different research paradigms: they have different syntax and pragmatic, but very closely related semantics (in spite of different terminology). Lattice-valued modal logics were introduced in

[3, 2] by M.C. Fitting. They were studied in the cited papers from a proof-theoretic point of view. Later several authors attempted study of these logics from algebraic perspective [5, 6, 9]. Basic definitions related to modal logics on lattices and completeness theorems can be found in [5].

Description Logic (DL) is a logic for reasoning about concepts. But there is also an algebraic formalism developed around concepts in terms of concept lattices, namely Formal Concept Analysis (FCA). In this section we recall in brief definition of description logic $\mathcal{ALC}$ on concept lattices of (symmetric) contexts and some properties that follow from this definition, please refer [8] for full details. We use notation and definitions for Description Logics from [1][1]. For the basics and notation of Formal Concept Analysis, please, refer to [4].

Semantics of description logics on concept lattices comes from lattice-theoretic characterization of 'positive' (i.e. without negation) concept constructs (for close world semantics) that is given in the following proposition [8].

Proposition 1. *Let (Δ, Υ) be a terminological interpretation and $P(\Delta) = (2^{\Delta}, \varnothing, \subseteq, \Delta, \cup, \cap)$ be the complete lattice of subsets of Δ. Then semantics of $\mathcal{ALC}$ positive concept constructs $\top$, $\bot$, $\sqcup$, $\sqcap$, $\forall$, and $\exists$ enjoys the following properties in $P(\Delta)$: (1) $\Upsilon(\top) = \sup P(\Delta)$, and $\Upsilon(\bot) = \inf P(\Delta)$; (2) $\Upsilon(X \sqcup Y) = \sup(\Upsilon(X), \Upsilon(Y))$, and $\Upsilon(X \sqcap Y) = \inf(\Upsilon(X), \Upsilon(Y))$; (3) $\Upsilon(\forall R.\, X) = \sup\{S \in P(\Delta) : \forall s \in S \forall t \in \Delta((s,t) \in \Upsilon(R) \Rightarrow t \in \Upsilon(X))\}$, (4) $\Upsilon(\exists R.\, X) = \sup\{S \in P(\Delta) : \forall s \in S \exists t \in \Delta((s,t) \in \Upsilon(R)\ \&\ t \in \Upsilon(X))\}$.*

Conceptual interpretation is a formal context provided by an interpretation function.

Definition 1.
Conceptual interpretation is a four-tuple (G, M, I, Υ) where (G, M, I) is a formal context, and an interpretation function $\Upsilon = I_{CS} \cup I_{RS}$, where CS and RS are standard concept and role symbols, and (1) $I_{CS} : CS \to \mathfrak{B}(G, M, I)$ maps concept symbols to formal concepts, (2) $I_{RS} : RS \to 2^{(G\times G)\cup(M\times M)}$ maps role symbols to binary relations. A formal context (G, M, I) or conceptual interpretation (G, M, I, Υ) is said to be homogeneous (symmetric) if $G = M$ (and binary relation I is symmetric in addition).

Semantics of $\mathcal{ALC}$ positive concept constructs $\top$, $\bot$, $\sqcup$, $\sqcap$, $\forall$, and $\exists$ as well as semantics of negative construct $\neg$ are defined in [8] as follows.

Definition 2.
Let (G, M, I, Υ) be a conceptual interpretation, $\mathbb{K}$ be a formal context (G, M, I), and $\mathfrak{B} = (\mathbb{K})$ be the concept lattice of $\mathbb{K}$. The interpretation function Υ can be extended to all role terms in a terminological interpretation $((G \cup M), \Upsilon)$ in the standard manner so that $\Upsilon(R)$ is a binary relation on $(G \cup M)$ for every role term R. The interpretation function Υ can be extended to all positive $\mathcal{ALC}$ concept terms as follows. (1) $\Upsilon(\top) = \sup \mathfrak{B}$ and $\Upsilon(\bot) = \inf \mathfrak{B}$; (2) $\Upsilon(X \sqcup Y) = \sup(\Upsilon(X), \Upsilon(Y))$, and $\Upsilon(X \sqcap Y) = \inf(\Upsilon(X), \Upsilon(Y))$; (3) Let $\Upsilon(X) =$

[1] But we use Υ instead of '$\cdot$' for terminological interpretation function for readability.

$(Ex', In') \in \mathfrak{B}$. Then (a) $\Upsilon(\forall R.\ X) = \sup_K\{(Ex, In) \in \mathfrak{B} : \forall o \in Ex\ \forall a \in In\ \forall o' \in G\ \exists a' \in M\ ((o, o') \in \Upsilon(R) \Rightarrow o' \in Ex',\ (a, a') \in \Upsilon(R), \text{ and } a' \in In')\}$, (b) $I(\exists R.\ X) = \sup_K\{(Ex, In) \in \mathfrak{B} : \forall o \in Ex\ \forall a \in In\ \exists o' \in G\ \forall a' \in M\ ((a, a') \in \Upsilon(R) \Rightarrow (o, o') \in \Upsilon(R),\ o' \in Ex', \text{ and } a' \in In')\}$. In addition, if $\mathbb{K}$ is a symmetric context and $\Upsilon(X) = (Ex, In) \in \mathfrak{B}$, then $\Upsilon(\neg X) = (In, Ex)$.

The following proposition [8] states that for any conceptual interpretation every positive $\mathcal{ALC}$ concept term is an element of concept lattice; in addition, if an interpretation is symmetric then this fact holds for all $\mathcal{ALC}$ concept terms.

Proposition 2. *For any conceptual interpretation (G, M, I, Υ), for every positive $\mathcal{ALC}$ concept term X, semantics $\Upsilon(X)$ is an element of $\mathfrak{B}(G, M, I)$. For any symmetric conceptual interpretation (D, D, I, Υ), for every $\mathcal{ALC}$ concept term X, semantics $\Upsilon(X)$ is an element of $\mathfrak{B}(D, D, I)$.*

3 Ways to Build a Symmetric Context

The above proposition 2 leads to the following idea: to define semantics of $\mathcal{ALC}$ with values in an arbitrary concept lattice by isomorphic embedding of the background context into a symmetric one in such a way that for the positive fragment of $\mathcal{ALC}$ the original semantics and the induced semantics equal each other. Below we examine some opportunities how to "symmetrize" a given context, i.e. to build a symmetric context from an arbitrary given background context. Below we are going to study how to build a symmetric context from a given one by set-theoretic and algebraic manipulations with a binary relation of the context. Without loss of generality we may assume that the background context is reduced [4] and has disjoint sets of objects and attributes.

Let $\mathbb{K} := (G, M, I)$ be a reduced context where $G \cap M = \varnothing$ and $\mathbb{K}^d = (M, G, I^-)$ be its dual context. Let also use the following notation for binary relations (on M and/or G): (1) $\varnothing$ be the empty binary relation, (2) $\times$ be a total binary relation, (3) E be the identity binary relation, (4) E^c be the complement for E. We would like to combine the cross-tables of $\mathbb{K}$ and the dual context $\mathbb{K}^d$ into the symmetric one in the following way: $\begin{array}{c|cc} & G & M \\ \hline G & ? & I \\ M & I^{-1} & ? \end{array}$. Let us represent the above cross-table in a shorter way as $\begin{array}{c|c} ? & \mathbb{K} \\ \hline \mathbb{K}^d & ? \end{array}$ and denote the corresponding symmetric context by $\mathbb{K}_\circ := (G_\circ, M_\circ, I_\circ)$. Recall that $\mathfrak{B}(G, M, I)$ is the concept lattice of the context $\mathbb{K}$, $\mathfrak{B}(G_\circ, M_\circ, I_\circ)$ is the concept lattice of the context $\mathbb{K}_\circ$. Let us use the standard notation '$'$' for derivatives in the background context $\mathbb{K}$ but (for distinction) the notation '$^\circ$' for derivatives in the symmetric one. We are going to fill question quadrants by different combinations of $\varnothing$, $\times$, E and E^c. Below we study 9 of these 16 combinations.

Case 1. $\begin{array}{c|c} \varnothing & \mathbb{K} \\ \hline \mathbb{K}^d & \varnothing \end{array}$. It is the disjoint union of $\mathbb{K}$ and $\mathbb{K}^d$. The concept lattice $\mathfrak{B}(\mathbb{K}_\circ) = \mathfrak{B}(\mathbb{K} \,\dot\cup\, \mathbb{K}^d)$ is a *horizontal sum* [4], i.e. the union of two sublattices $\mathfrak{B}(\mathbb{K})$ and $\mathfrak{B}(\mathbb{K}^d)$, such that $\mathfrak{B}(\mathbb{K}) \cap \mathfrak{B}(\mathbb{K}^d) = \{\bot, \top\}$.

Case 2. $\begin{array}{c|c} \times & \mathbb{K} \\ \hline \mathbb{K}^d & \varnothing \end{array}$. The concept lattice of this context is isomorphic to the *vertical sum* [4] of the concept lattices $\mathfrak{B}(\mathbb{K}^d)$ and $\mathfrak{B}(\mathbb{K})$ (where the concept lattice $\mathfrak{B}(\mathbb{K}^d)$ is upper than $\mathfrak{B}(\mathbb{K})$).

Case 3. ($\varnothing$,$\times$). This case is the same like a previous, but we have to swap components of the vertical sum.

Case 4. $\begin{array}{c|c} \times & \mathbb{K} \\ \hline \mathbb{K}^d & \times \end{array}$. We have here the direct sum $\mathbb{K} + \mathbb{K}^d$ of contexts $\mathbb{K}$ and $\mathbb{K}^d$ [4] and the concept lattice of the sum is isomorphic to the product of the concept lattices $\mathfrak{B}(\mathbb{K}) \times \mathfrak{B}(\mathbb{K}^d)$. A pair (A, B) is a concept of the direct sum $(\mathbb{K} + \mathbb{K}^d)$ iff $(A \cap G, B \cap M)$ is a concept of $\mathbb{K}$ and $(A \cap M, B \cap G)$ is a concept of $\mathbb{K}^d$. It implies that isomorphism is given by $(A, B) \mapsto ((A \cap G, B \cap M), (A \cap M, B \cap G))$.

Case 5. $\begin{array}{c|c} E & \mathbb{K} \\ \hline \mathbb{K}^d & E \end{array}$. Let $(X, Y) \in \mathfrak{B}(\mathbb{K}_\circ)$ be a concept and let $X = A \dot\cup B$ where $A \subseteq G$, $B \subseteq M$. We have the following cases:
(1) $B = \varnothing$. Let $X = A$. If $|A| = 1$, then $(X, Y) = (\{a\}, \{a\} \cup A')$, and $(X, Y) = (A, A')$ otherwise.
(2) $A = \varnothing$. Let $X = B$. If $|B| = 1$, then $(X, Y) = (\{b\}, \{b\} \cup B')$, and $(X, Y) = (B, B')$ otherwise.
(3) $|B| = 1$ and $A \neq \varnothing$. Let $X = A \cup \{b\}$. If $\{b\} \in A'$ and $|A| = 1$, then $(X, Y) = (\{a\} \cup \{b\}, \{a\} \cup \{b\})$, and if $\{b\} \in A'$ $|A| > 1$, then $(X, Y) = (A \cup \{b\}, \{b\})$.
(4) $|B| > 1$ and $A \neq \varnothing$. Let $|B| > 1$, then $X = A \cup B$. If $B \subseteq \{a\}'$ and $|A| = 1$, then $(X, Y) = (\{a\} \cup B, \{a\})$.

Case 6. $\begin{array}{c|c} \varnothing & \mathbb{K} \\ \hline \mathbb{K}^d & E \end{array}$. This case is a very similar to the previous one.
(1) $B = \varnothing$. $(X, Y) = (A, A')$.
(2) $A = \varnothing$. If $|B| = 1$ then $(X, Y) = (\{b\}, \{b\} \cup B')$ else $(X, Y) = (B, B')$.
(3) $|B| = 1$. If $\{b\} \in A'$, we have $(X, Y) = (A \cup \{b\}, \{b\})$.
(4) $|B| > 1$. All the concepts in this case will be either $\top$ or $\bot$.

Case 7. (E,$\varnothing$). This case is similar to the previous.

Case 8. $\begin{array}{c|c} \times & \mathbb{K} \\ \hline \mathbb{K}^d & E \end{array}$. Let us use subcases as in the case 5.
(1) $B = \varnothing$. $(X, Y) = (A, G \cup A')$.
(2) $A = \varnothing$. If $|B| = 1$, then $(X, Y) = (\{b\}, \{b\} \cup B')$, and $(X, Y) = (B, B')$ otherwise.
(3) $|B| = 1$. If $\{b\} \in A'$, then $(X, Y) = (A \cup \{b\}, \{b\} \cup \{b\}')$, else $(X, Y) = (A \cup \{b\}, \{b\}')$.
(4) $|B| > 1$. $(X, Y) = (A \cup B, B')$.

Case 9. (E,$\times$). This case is similar to the case 8.

4 Conclusion

Now we are ready to formulate several topics and problems that we consider natural and important for further research.

In [5] the definition for modal logics with values in a given finite distributive lattice L is presented. This definition is easy to expand on polymodal logics. In section 2 we represented the definition for description logic $\mathcal{ALC}$ (that can be considered as a polymodal version of $\mathbf{K}$) with values in concept lattices of symmetric contexts. Assume that $\mathbb{K}$ is a finite symmetric context; then $\mathfrak{B}(\mathbb{K})$ is a finite lattice, but it may not be a distributive lattice. Question: Assuming that $\mathfrak{B}(\mathbb{K})$ is a finite distributive lattice, whether $\mathcal{ALC}$ with values in $\mathfrak{B}(\mathbb{K})$ is a polymodal $\mathfrak{B}(\mathbb{K})$-$\mathbf{ML}$?

In section 2 we represented the definition for description logic $\mathcal{ALC}$ with values in concept lattices of symmetric contexts and for positive fragment of $\mathcal{ALC}$ with values in arbitrary concept lattices. Questions: (1) Is decidable or axiomatizable the positive fragment of $\mathcal{ALC}$ with values in concept lattices? (2) Is decidable or axiomatizable $\mathcal{ALC}$ with values in concept lattices of symmetric contexts?

In the section 3 we examine 9 of 16 variants of context symmetrization. Topics for further research are following: (1) to study the remaining 7 cases of context symmetrization and isomorphic embedding with E^c in one or two free quadrants; (2) to examine under which embedding from these in these 16 the induced semantics of the positive fragment of $\mathcal{ALC}$ is equal to the original semantics.

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