Continuity, Connectivity and Regularity in Spatial Diagrams for N Terms

Amirouche Moktefi
Francesco Bellucci
Ragnar Nurkse School of Innovation and Governance
Tallinn University of Technology
Tallinn, Estonia
amirouche.moktefi@ttu.ee
bellucci.francesco@gmail.com

Ahti-Veikko Pietarinen
Department of Philosophy, History, Culture and Art Studies
University of Helsinki &
Ragnar Nurkse School of Innovation and Governance
Tallinn University of Technology
Tallinn, Estonia & Helsinki, Finland
ahti-veikko.pietarinen@helsinki.fi

***Abstract*—This paper discusses the role of continuity, connectivity and regularity in the design of spatial logic diagrams for *N* terms. Three specific diagrammatic schemes are discussed: Venn diagrams, Marquand tables and Karnaugh maps.**

***Keywords*— diagrams; Euler; Venn; Marquand; Karnaugh; continuity ; connectivity; regularity.**

I. Introduction: Contuinity and Connectivity

The aim of this paper is to discuss the role of the topological properties of continuity and connectivity in the design of spatial diagrams for *N* terms. Spatial diagrams have been and are still widely used in logic. They have been popularized by Leonhard Euler who used them thoroughly in his *Letters to a German Princess* (1768). There are no conditions as to the shape of the spaces as long as they are formed by continuous surfaces within closed curves. Early logicians used mostly circles but squares have been also regularly used, especially when the number of terms increases.

The very idea of spatial diagrams is simple: to represent a class of individuals with a space where those individuals are gathered. That's how Euler introduced his diagrams: "As a general notion contains an infinite number of individual objects, we may consider it as a space in which they are all contained. Thus, for the notion of *man* we form a space […] in which we conceive all men to be comprehended" [1, p. 339]. For instance, if we consider the circle *A* in [Fig. 1] to represent the class of *men*, then it is understood that *every man* is comprehended within that circle. This mode of representation deserves further exploration as to what cognitive and semeiotic processes are at work when it comes to representing a class with a space.

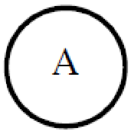

Fig. 1

Charles S. Peirce considered that spatial diagrams are "veridically iconic, naturally analogous to the thing represented, and not a creation of conventions" [2, p. 316]. Umberto Eco disputes this view and argues that the representation of classes with spaces is rather purely conventional because belonging to a class is not a spatial fact "except the fact that I might be defined to belong to the class of all those who are located in a certain place" [3, pp. 228-229]. Now, the interesting point is that it is with this very understanding that Euler introduced his spatial diagrams as we indicated above. As such, the individuals that form a class are to be imagined as if they were all assembled within that single space. Hence, what Eco considers to be an exception would rather be the general rule. The fact that these individuals cannot be really assembled does not matter. Classification itself is a purely mental operation and there is no need for classes or spaces to really exist. All that is required is to have an accurate diagram that provides a visual aid.

Having a continuous space simplifies the expression of what is represented and provides a representation that could be visually better grasped. However, saving continuity becomes difficult when the number of terms represented increases. There, it often happens that a class *A* is represented with a discontinuous space. For instance *A* could be represented with several sub-spaces representing each a subdivision of *A*. In such situations, diagrams are better drawn in such a way as to make those subdivisions connected. As such, they can be grouped into one continuous space standing for the entire class *A* as shown in [Fig. 2]. Hence, the connectivity of the subdivisions is what makes the whole space continuous.

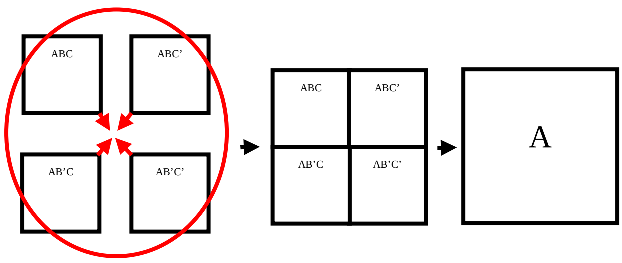

Fig. 2

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Keeping the subdivisions connected might prove to be difficult in diagrams with more than 3 or 4 classes. In the following, we will discuss how three designers of spatial diagrams (Venn, Marquand and Karnaugh) handled the issues of continuity and connectivity in diagrams for *N* terms. Drawing such diagrams for more than 3 terms was not required within syllogistic where arguments were reduced to series of syllogisms. Such problems were easily solved with traditional Euler diagrams. One represents classes with circles, then the logical relations of the classes are represented by the topological relations of the circles. However, the development of Boolean algebra changed the picture. Logicians had to face problems where they were offered an indeterminate number of premises with an indeterminate number of terms and were asked to extract the conclusion that follows by eliminating undesired or superfluous terms.

II. Venn Diagrams for N Terms

In 1880, the logician John Venn, who was a great admirer of Boole, invented a new type of diagrams where relations between classes are not directly exhibited by the circles [4]. One first draws a framework diagram where all combinations of terms are represented. For instance, for 2 terms *x* and *y*, one uses two circles to divide the universe into 4 compartments *xy, xy', x'y, x'y'* (where *x*' stands for not-*x*, and *y*' for not-*y*).

In order to represent propositions, one has to add marks to indicate the occupation or emptiness of the compartments. For instance, to represent the proposition "All *x* are *y*", one has to shade *xy'* to indicate its emptiness, as shown in [Fig. 3]. In order to handle more complex logic problems, Venn designed diagrams where *n* continuous curves divide the universe into 2^n compartments. For n = 3, one simply uses the famous three-circle diagram [Fig. 4]. For n = 4, Venn knew how to add a curve to his 3-term diagram in order to obtain a 4-term diagram [Fig. 5]. However, he preferred to use a new figure with four ellipses, as shown in [Fig. 6], because of its simplicity and symmetry [5, p. 116]. For n = 5, Venn failed in making ellipses intersect in the desired way. So, he suggested using the diagram shown in [Fig. 7].

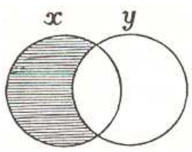


Fig. 3

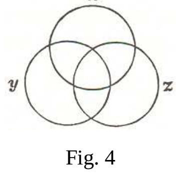


Fig. 4

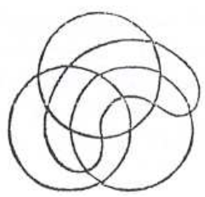
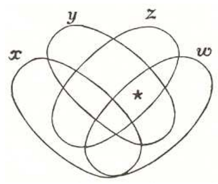


Fig. 5



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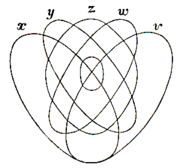

Fig. 7

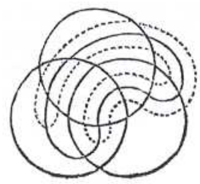

Fig. 8

In this 5-term diagram, the fifth term (*z*) is represented by an annulus. It follows that class not-*z* is discontinuous and is formed by two disconnected spaces. It must be noted that Venn knew, using an inductive method, how to represent 5-term continuous diagrams [Fig. 8]. However, Venn preferred to use the other diagram because of its symmetry, in spite of his dissatisfaction with its discontinuity. For more than 5 terms, Venn believed his diagrams would not offer the visual aid one would expect, even if they continue to be accurate: "Up to four or five terms inclusive, our plan works very successfully in practice; where it begins to fail is in the accidental circumstance that its further development soon becomes intricate and awkward, though never ceasing to be feasible" [5, p. 113]. When Venn faced such complex problems, he preferred to use tabular diagrams that were invented by the logician Allan Marquand [5, pp. 139-140, 373-376].

III. Marquand Tables for N Terms

Allan Marquand was one of Peirce's students at John Hopkins University. He introduced new diagrams that were designed to supersede Venn diagrams, Marquand says: "It is the object of this paper to suggest a mode of constructing logical diagrams, by which they may be indefinitely extended to any number of terms, without losing so rapidly their special function, viz. that of affording visual aid in the solution of problems" [6, p. 266].

Marquand used squares rather than circles. He first represents the logical universe with a square. The limitation of the universe, absent in Venn diagrams, makes it possible to represent with a closed surface the class where all terms are negated. Marquand tables should not be understood however as Venn diagrams to which we have added a square around to limit the universe. Indeed, the cognitive constructions of the diagrams differ. Venn puts together the individuals that form a given class *x* and leaves *outside* the individuals that are not *x*. Marquand rather divides the universe into 2-subclasses *x* and not-*x*, equally considered. Thus, Venn proceeds by classification while Marquand appeals to division.

After one has represented the universe with a square, it suffices to divide it into subdivisions corresponding to the different combinations of the terms involved in the argument. For two terms A and B, one gets [Fig. 9] (where *a* stands for the negation of A, etc.). This diagram shows how important it was to choose a rectilinear shape in order to get a symmetrical division of the universe. Making subdivisions of equal size is purely conventional for Marquand and Venn. However, it is obvious that for convenience and practicality, it is better to make the compartment of equal size. It must be remembered that Euler and Venn always favored symmetrical diagrams where classes were represented with congruent spaces (same shape and same size). For 4 terms, Marquand divides the square in the way represented in [Fig. 10].

It is important for our purpose to understand how the order of the combinations is obtained on each side. For instance, horizontally, Marquand divides first the square into two subclasses: A and not-A. Then each sub-class is itself divided into sub-divisions C and not-C. Hence, this dichotomy division produces the horizontal sequence AC, A*c*, *a*C, *ac* that can be observed on the top of the diagram. The vertical sequence is produced similarly. One immediately observes that several classes are not represented with continuous spaces: C, *c*, D and *d*.

	A	***a***
B		
b		

Fig. 9

	AC	**A*c***	***a*C**	***ac***
BD				
B*d*				
***b*D**				
bd				

Fig. 10

Contrary to Venn who abandoned unhappily the continuity of his diagrams, Marquand did not seem to be bothered with this constraint, as long as the diagrams are easy to extend for further terms. All one has to do is to divide again the square dichotomically to introduce an additional term. After Marquand, several tabular schemes have been introduced and continued to be used in subsequent years [7; 8]. Much later, interest in such diagrams has been renewed in the 1950s when computer scientists had to simplify logical forms in order to get better and cheaper electronic circuits. Such methods have been notably introduced by Edward W. Veitch in 1952 [9] and Maurice Karnaugh in 1953 [10].

IV. KARNAUGH MAPS FOR N TERMS

In order to simplify a logical form F, one first divides each term of F into its simplest components, then one collects together the components to get a simpler expression of F. Let us consider the logical form: F = A'B'C + AD + A'BD + A' B'C' (where A' stands for the negation of A, etc.). There are four variables A, B, C, D. Karnaugh uses a square divided into 16 subdivisions; each subdivision corresponds to one combination of the variables [Fig. 11]. For each combination where F is true, one puts 1 in the appropriate subdivision. Each red curve in [Fig. 11, left] highlights the subdivisions that correspond to one term of F. For instance, the eastern circle encloses the cases where term AD is true.

Any equivalent form of F would still have the same truth value for any given combination of the variables. Hence, looking for a simpler (equivalent) form of F does not involve changing the content of the subdivisions. It rather requires looking for a different assemblage of the subdivisions, with fewer and larger curves yielding to fewer and more general terms. In present case, [Fig. 11, right] shows how we get a simpler expression of F. For instance, the vertical blue curve encloses all affirmed subdivisions where A'B' is fixed. Similarly, the horizontal curve encloses eight affirmed subdivisions with one fixed variable D. Hence, we obtain simple form: F = A'B' + D.

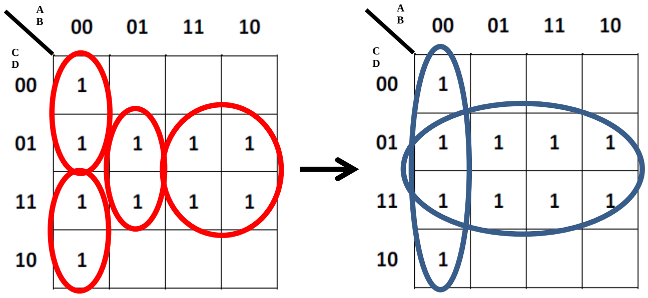

Fig. 11

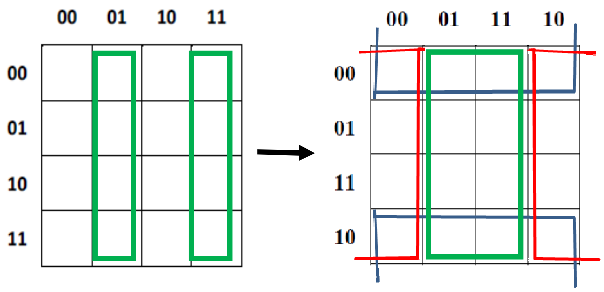

Fig. 12

Karnaugh considered that finding such assemblages of connected squares could be done by "direct inspection" [10, p. 594]. This is made possible by the fact that the variables are ordered is such a way as to always have one variable unchanged between adjacent squares. Indeed, the appeal to Gray's sequence: 00, 01, 11, 10 (see the horizontal sequence at the top of the map) makes simplification easier. It is noteworthy that Veitch first used the same sequence as Marquand's: 00, 01, 10, 11 [Fig. 12, left]. Thus, several variables were represented with discontinuous spaces. For instance the two green curves represent together a single variable. In Karnaugh's map [Fig. 12, right], that variable is represented with one continuous space, as shown by the green curve. Here we see how Karnaugh changed the sequence in order to restore the continuity of classes that were abandoned by Marquand and Veitch. Interestingly, Veitch himself adopted later Karnaugh's sequence [11].

It might be objected that some variables in Karnaugh maps are also discontinuous (see the red and blue curves in [Fig. 12, right]). However, Karnaugh considered those opposite ends of columns and rows to be adjacent, as if the map was inscribed on a torus or a cylinder. As such, their connectivity was saved.

V. Conclusion: Toward Regularity

The discussion of Venn, Marquand and Karnaugh diagrams shows the crucial role of continuity and connectivity in the making of those diagrams for more than 3 terms. These topological properties have been differently handled by these authors. Venn knew how to draw continuous classes but sacrificed that continuity in favour of regularity in his 5-term diagrams. On the contrary, Marquand was not bothered as to whether the classes were continuous or the subdivisions connected. Finally, Karnaugh made his best to save the connectivity of spaces by imagining a three-dimensional construction (cylinder) even when the diagrams were drawn on a two-dimensional surface.

It is obvious that when N is superior to 5 or 6 terms, using continuous figures becomes tedious as it makes it difficult to get regular diagrams. Regularity here is meant as the possession of some features (symmetry, congruence, familiarity, recurrence) that simplify the identification of the terms involved in each sub-division of the diagram. Mathematicians tackled this problem for more than a century in order to construct 'nice' Venn diagrams for N terms. From a mathematical viewpoint, the continuity of the diagrams is essential as is rightly explained by Anthony W. F. Edwards: "Both Venn and Carroll gave up at four sets and offered five-set diagrams whose fifth set did not consist of a closed curve, so that some regions became disjoint. In our terminology, they were not really Venn diagrams at all: once one admits the possibility of sets being bounded by more than one closed curve, one might as well just list all the binary numbers between 0 and 2^n-1 and put a little ring round each!" [12, p. 32]. From a logical viewpoint, the matter is different however. Making diagrams continuous and spaces connected is not a challenge in itself. The logician rather expects such diagrams to provide a visual aid for solving logic problems. As such, continuity and connectivity are pursued as long as they contribute to making the diagrams helpful. When the number of terms increases, discontinuous diagrams loose the advantages of having every class within a single space but provide regular schemes where it is easier to locate every subdivision.

For instance, Venn abandoned his diagrams in favour of Marquand's tables for more than 6 terms. Venn, referring to the 8x8 Marquand diagram for 6 terms argued that: "The scheme is very compendious: thus one adapted for 10 terms, and involving 1024 combinations, can be conveniently printed on one of these pages. Of course there is not the help to the eye here, afforded by keeping all the subdivisions of a single class within one boundary […] But this is almost inevitable where we deal with many class terms" [5, p.140]. In a way, Karnaugh maps might be perceived as a response to Venn by suggesting that methods exist to deal with the simplification on subdivisions of areas into contiguous parts observable by 'direct inspection'. In all the cases of the above, only 0 and 1 are the values in the diagrams and maps, but generalisations to other than binary Boolean algebras should pose no problems. These generalisations retain the desired contiguity of maps and the directly observable properties of simplification.

This paper shows how continuity, connectivity and regularity acted as major constraints for diagram designers (Venn, Marquand and Karnaugh). Their opposed solutions show the difficulties they faced and the choices they made. It provides a nice illustration of the uneasy balance between visual aid and logical efficiency that was constantly pursued by logicians [13].

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