Extension of Batches Petri Nets by Bi-parts batch places

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Abstract. This paper proposes an extension of Batches Petri Nets by a new definition of the batch place called Bi-parts Batch place (BB-place). The flow-density equations that govern the dynamics of controllable batches inside a BB-place is now defined by a triangular relation. To take into account controlled events, the behaviors of batches are discussed according to a variation of speeds and of maximum flows. The switching dynamics of controllable batches is defined on three behaviors: free, congestion and decongestion behaviors. We also propose the computation of the instantaneous firing flow vector associated with continuous and batch transitions thanks to a resolution of a linear programming problem. An example of traffic road illustrates the novel extensions proposed in this paper.

Keywords: Petri Nets, Discrete Event Systems, Hybrid systems, Flow, Batch, Traffic road

1 Introduction

For modeling, analysis and control of discrete event systems like manufacturing systems or traffic road systems, Petri nets are well utilized. Moreover, the discrete Petri net formalism has been extended to also encompass continuous and hybrid models [1], thus offering formal techniques for expressing both fundamental discrete event and continuous time behaviors. Hybrid Petri nets combine the interest of the continuous Petri nets for the representation of the flows and those of the discrete Petri nets for the representation of the controls [4]. In order to integrate variable delays on continuous flows, basic hybrid Petri nets have been extended to Batches Petri Nets (BPNs) [2] [3]. BPNs introduce new kinds of places and transitions: the batch places and the batch transitions. A batch transition acts like a continuous transition while a batch place is defined on the concept of batches. A batch consists in regrouping all elements of the flow with the same behavior. Moreover, a batch is a set of entities (parts, vehicles , etc.) moving through a transfer zone at the same speed. A batch is defined by three characteristics: a length, a density and a position. Inside a batch place, an event hybrid approach allows to describe the evolution of batches. More generally, in the BPN formalism, the dynamics of batches inside a batch place is governed by a flow-density relation representing a switching between free and accumulation behaviors. To represent a more general flow-density relation as the triangular fundamental diagram of traffic road domain [6], the batch place is extended to a Bi-parts Batch place (BB-place) defined by four continuous characteristics: a maximum speed, a maximum density, a length and a maximum flow. The hybrid dynamics is now defined by three behaviors: free, congestion and decongestion behaviors. This dynamics allows batches to switch between free and congested states. The switching of batch states is also discussed according to controlled events, such as the variation of maximum speeds of BB-places or the modification of maximum flows associated to continuous and batch transitions. Finally, to compute the instantaneous firing flow vector we adapt the linear programming problem previously proposed for BPN [12] to BB-places.

This paper is organized as follows. Section 2 recalls some concepts and definitions on batches Petri nets. Section 3 presents the extension of the batch place, called Bi-parts Batch place, and defines the states and the behaviors of controllable batches. The instantaneous firing flow vector is determined thanks to the resolution of a linear programming problem. In Section 4, an example of traffic road system illustrates these contributions.

2 Backgrounds on Batches Petri Nets and controllable batches

This section recalls some concepts and definitions used in this paper. For more details on Batches Petri Nets, the reader is referred to [3], [10], [12] and [13].

2.1 Some definitions

Definition 1 A Generalized Batches Petri Nets (GBPN) *is a 6-tuple* $N = (P, T, Pre, Post, \gamma, Time)$ *where:*

- $P = P^D \cup P^C \cup P^B$ *is a finite set of places partitioned into the three classes of discrete, continuous and batch places.*
- $T = T^D \cup T^C \cup T^B$ *is a finite set of transitions partitioned into the three classes of discrete, continuous and batch transitions.*
- $Pre, Post : (P^D \times T \to \mathbb{N}) \cup ((P^C \cup P^B) \times T \to \mathbb{R}_{\geq 0})$ *are, respectively, the pre-incidence and post-incidence matrices, denoting the weight of the arcs from places to transitions and from transitions to places.*
- $\gamma : P^B \to \mathbb{R}^3_{\geq 0}$ *is the* batch place function. *It associates with each batch place* $p_i \in P^B$ *the triple* $\gamma(p_i) = (V_i, d_i^{max}, S_i)$ *that represents, respectively, a speed, a maximum density and a length.*
- $Time : T \to \mathbb{R}_{\geq 0}$ *associates a non negative number with every transition:*
  - *if* $t_j \in T^D$*, then* $Time(t_j) = d_j$ *denotes the* firing delay *associated with the discrete transition;*
  - *if* $t_j \in T^C \cup T^B$*, then* $Time(t_j) = \Phi_j$ *denotes the* maximal firing flow *associated with the continuous or batch transition.*

□

We denote the number of places and transitions, resp., $m = |P|$ and $n = |T|$ and use the following notations: $m^X = |P^X|$ and $n^X = |T^X|$ for $X \in \{D, C, B\}$. The *preset* and *postset* of transition t_j are: ${}^\bullet t_j = \{p_i \in P \mid Pre(p_i, t_j) > 0\}$ and $t_j^\bullet = \{p_i \in P \mid Post(p_i, t_j) > 0\}$. Similar notations may be used for pre and post transition sets of places and its restriction to discrete, continuous or batch transitions is denoted as ${}^{(d)}p_i = {}^\bullet p_i \cap T^D$, ${}^{(c)}p_i = {}^\bullet p_i \cap T^C$, and ${}^{(b)}p_i = {}^\bullet p_i \cap T^B$. The *incidence matrix* of a GBPN is defined as $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$.

Definition 2 *The* marking *of a GBPN at time τ is defined as $\boldsymbol{m}(\tau) = [m_1(\tau)..m_i(\tau)..m_m(\tau)]^T$ where:*

- *if $p_i \in P^D$ then $m_i \in \mathbb{N}$,i.e, the marking of a discrete place is a non negative integer.*
- *if $p_i \in P^C$ then $m_i \in \mathbb{R}_{\geq 0}$,i.e, the marking of a continuous place is a non negative real.*
- *if $p_i \in P^B$ then $m_i = \{\beta_h, ..., \beta_r\}$,i.e, the marking of a batch place is a series of batches.* □

A batch, i.e., a group of discrete entities characterized by continuous variables, has been defined for Batches Petri Nets. When, three continuous variables are associated with it, it is called a batch. When, four continuous variables are considered [10], it is called a controllable batch, defined as follows.

Definition 3 *A* controllable batch *$C\beta_r(\tau)$ at time τ, is defined by a quadruple, $C\beta_r(\tau) = (l_r(\tau), d_r(\tau), x_r(\tau), v_r(\tau))$ where $l_r(\tau) \in \mathbb{R}_{\geq 0}$ is the* length, *$d_r(\tau) \in \mathbb{R}_{\geq 0}$ is the* density, *$x_r(\tau) \in \mathbb{R}_{\geq 0}$ is the* head position *and $v_r(\tau) \in \mathbb{R}_{\geq 0}$ is the* speed. *An* instantaneous batch flow *of $C\beta_r(\tau)$ is defined by: $\varphi_r(\tau) = v_r(\tau).d_r(\tau)$.* □

Some constraints on batches composing the marking of a batch place, p_i, have to be respected: $0 \leq l_r(\tau) \leq x_r(\tau) \leq S_i$ (position and length constraints), $0 \leq d_r(\tau) \leq d_i^{\max}$ (density constraint) and $0 \leq v_r(\tau) \leq V_i$ (speed constraint).

Definition 4 *A controllable batch $C\beta_r(\tau)$ of batch place p_i, which has its head position equals to the length of the batch place, i.e., $x_r(\tau) = S_i$, is called an* output controllable batch, *denoted $OC\beta_r(\tau)$. The* output density *d_i^{out} of a batch place p_i is defined as follows. If at time τ, batch place p_i has an output controllable batch $OC\beta_r(\tau)$, then $d_i^{out}(\tau) = d_r(\tau)$, else $d_i^{out}(\tau) = 0$.* □

Note that the output density of place p_i at time τ depends on the marking $\boldsymbol{m}(\tau)$ and can also be denoted by $d_i^{\text{out}}(\boldsymbol{m})$.

Definition 5 *The* marking quantity vector *$\boldsymbol{q} \in \mathbb{R}^m$ associated with a marking $\boldsymbol{m}$ is defined as follows:*

$$q_i = \begin{cases} m_i & \text{if } p_i \in P^D \cup P^C \\ \sum_{\beta_r \in m_i} l_r \cdot d_r & \text{if } p_i \in P^B \end{cases},$$

i.e., for a batch place it represents the sum of the quantities of the batches it contains, while it coincides with the marking for other places. □

Definition 6 *The* maximal capacity *of batch place $p_i \in P^B$ is $Q_i = S_i \cdot d_i^{max}$. A place such that $q_i(\tau) = Q_i$ is called a* full batch place. □

2.2 Conditions of enabling

The enabling and firing conditions of timed discrete transitions of a GBPN are those of timed transitions of discrete Petri nets. The enabling conditions of continuous transitions are those of First Order Hybrid Petri Nets [14] and Hybrid Petri Nets [4] i.e., one distinguishes weakly and strongly enabled transitions. Similar conditions for batch transitions have been defined [12].

Condition 7 *A discrete transition $t_j \in T^D$ is enabled at $\boldsymbol{m}$ if for all $p_i \in {}^{\bullet}t_j$, $m_i \geq Pre(p_i, t_j)$.*

A discrete transition $t_j \in T^D$ that is enabled at a marking $\boldsymbol{m}$ and has also been continuously enabled for a time equal to its firing delay, fires yielding a new marking $\mathbf{m'} = \mathbf{m} + \boldsymbol{C}(\cdot, t_j)$. □

Condition 8 *A continuous transition $t_j \in T^C$ is enabled at $\boldsymbol{m}$ if for all $p_i \in {}^{(d)}t_j$, $m_i \geq Pre(p_i, t_j)$. We say that the continuous transition is:*

- strongly enabled *if $\forall p_k \in {}^{(c)}t_j$, $m_k > 0$.*
- weakly enabled *if $\exists p_r \in {}^{(c)}t_j$, $m_r = 0$.* □

Condition 9 *A batch transition $t_j \in T^B$ is enabled at $\boldsymbol{m}$ if:*

- $\forall p_i \in {}^{(d)}t_j$, $m_i \geq Pre(p_i, t_j)$.
- $\forall p_s \in {}^{(b)}t_j$, $d_s^{\mathrm{out}} > 0$.

We say that the batch transition is:

- strongly enabled *if $\forall p_k \in {}^{(c)}t_j$, $m_k > 0$.*
- weakly enabled *if $\exists p_r \in {}^{(c)}t_j$, $m_r = 0$.* □

2.3 Batch dynamics

A flow-density relation is intrinsically associated with a batch place p_i that governs the dynamics of batches. In a GBPN, this relation is a linear function when the density is strictly inferior to the maximal density of p_i. Moreover, a batch place describes the transfer of batches according to a switching dynamics between two behaviors: the free behavior and the accumulation behavior [3]. According to this relation, described in Fig. 1, every accumulated batch has the same density, i.e. $d_r = d_i^{max}$ and its batch flow verifies $0 \leq \varphi_r \leq \Phi_i^{max}$, while every free batch respects $\varphi_r = V_i \cdot d_r$.

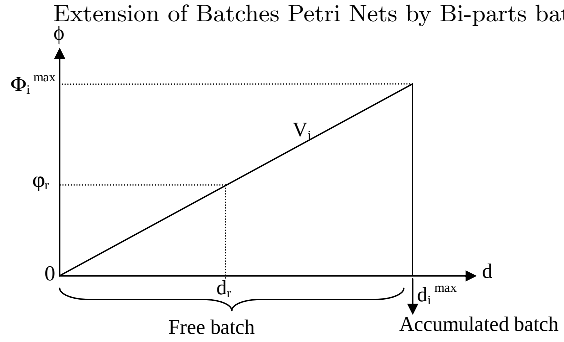


Fig. 1. Relation flow-density of a batch place

2.4 Instantaneous firing flows

The *instantaneous firing flow* (IFF) $\varphi_j(\tau) \leq \Phi_j$, associated with a continuous or a batch transition $t_j \in T^C \cup T^B$, represents the quantity of firing of transition t_j by time unit. The IFF vector at time τ is denoted by $\boldsymbol{\varphi}(\tau) \in \mathbb{R}^{n^C+n^B}$. In [12], a method for computing the IFF of enabled continuous and batch transitions has been introduced. It is based on the resolution of a linear programming problem that takes the net structure and the current state into account.

Definition 10 *Given a marked GBPN $(N, \boldsymbol{m})$ with incidence matrix $\boldsymbol{C}$, let:*

- $T_{\mathcal{N}}(\boldsymbol{m}) \subset T^C \cup T^B$ *be the subset of continuous and batch transitions that are not enabled at $\boldsymbol{m}$;*
- $P_{\emptyset}(\boldsymbol{m}) = \{p_i \in P^C \mid m_i = 0\}$ *be the subset of empty continuous places;*
- $P_F(\boldsymbol{m}) = \{p_i \in P^B \mid q_i = Q_i\}$ *be the subset of full bach places.*

Any admissible IFF vector $\boldsymbol{\varphi}$, at $\boldsymbol{m}$, is a feasible solution of the following linear set:

$$\begin{cases} (a)\ 0 \leq \varphi_j \leq \Phi_j & \forall t_j \in T^C \cup T^B \\ (b)\ \varphi_j = 0 & \forall t_j \in T_{\mathcal{N}}(\boldsymbol{m}) \\ (c)\ C(p_i, \cdot) \cdot \boldsymbol{\varphi} \geq 0 & \forall p_i \in P_{\emptyset}(\boldsymbol{m}) \\ (d)\ C(p_i, \cdot) \cdot \boldsymbol{\varphi} \leq 0 & \forall p_i \in P_F(\boldsymbol{m}) \\ (e)\ Post(p_i, \cdot) \cdot \boldsymbol{\varphi} \leq V_i \cdot d_i^{\max} & \forall p_i \in P^B \\ (f)\ Pre(p_i, \cdot) \cdot \boldsymbol{\varphi} \leq V_i \cdot d_i^{\mathrm{out}}(\boldsymbol{m}) & \forall p_i \in P^B \end{cases} \quad (1)$$

The set of all feasible solutions is denoted $S(N, \boldsymbol{m})$. □

3 Triangular Batches Petri Nets

Triangular Batches Petri Nets (TrBPN) extends the GBPN formalism by associated a new continuous characteristic to the batch place. This new place, called Bi-parts Batch place (BB-place) integrates a triangular flow-density relation, the propagation speed of congestion and the critical density, concepts that are observed in traffic systems.

3.1 Definitions and notations

Firstly we extend the definition of a GBPN, by enriching the characteristic function γ of a batch place p_i by a new parameter corresponding to a maximum flow Φ_i^{max}. Nodes of a Triangular Batches Petri Nets are represented in Fig. 2.

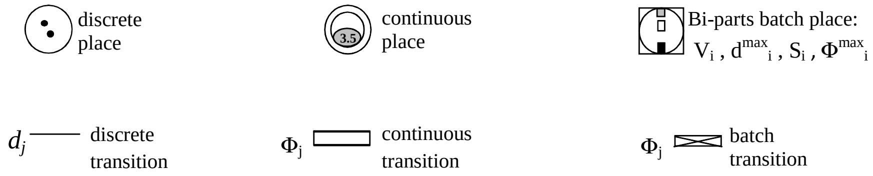


Fig. 2. Nodes of Triangular Batches Petri Nets

Definition 11 *A Triangular Batches Petri Nets (TrBPN) is a GBPN with a new batch place called* Bi-parts Batch place *(BB-place). The set of BB-places of a TrBPN is denoted as P^{BB}. The batch place function for a BB-place is $\gamma : P^{BB} \rightarrow \mathbb{R}^4_{>0}$. Its associates with BB-place $p_i \in P^{BB}$, the quadruple $\gamma(p_i) = (V_i, d_i^{max}, S_i, \Phi_i^{max})$ that represents, respectively, a maximum speed, a maximum density, a length and a maximum flow.* □

Definition 12 *The* marking of a BB-place *at time τ is a series of controllable batches. If $p_i \in P^{BB}$ then $m_i = \{C\beta_h, \cdots, C\beta_r\}$.* □

From these definitions, we associate to a BB-place p_i a new flow-density relation with a triangular form that must respected by controllable batches. This relation can adequately represent the different situations and states of the flow circulating inside a BB-place. In fact it has a propagation speed of congestion and a critical density that are defined as follows:

Definition 13 *For a BB-place p_i with $\gamma(p_i) = (V_i, d_i^{max}, S_i, \Phi_i^{max})$, the propagation speed of congestion, denoted W_i and, the critical density d_i^{cri} are respectively defined by:*

$$W_i = \frac{\Phi_i^{max} \cdot V_i}{d_i^{max} \cdot V_i - \Phi_i^{max}} \tag{2}$$

$$d_i^{cri} = \frac{\Phi_i^{max}}{V_i} \tag{3}$$

□

Definition 14 *The* flow-density relation *that governs the dynamics of controllable batches inside BB-places with $\gamma(p_i) = (V_i, d_i^{max}, S_i, \Phi_i^{max})$ is defined as follows:*

$$\varphi_r = \begin{cases} d_r.V_i & if \quad 0 \leq d_r \leq d_i^{cri} \\ W_i.(d_i^{max} - d_r) & if \, d_i^{cri} < d_r \leq d_i^{max} \end{cases} \quad (4)$$

□

To allow a dynamic reconfiguration of flow systems with accumulation behavior by manual control we propose here a variation of a BB-place speed. Indeed the variation of the speed of BB-place imposes a variation of the critical density and the maximum flow of the BB-place. The critical density and the maximum flow will be named respectively instantaneous critical density $d_i^{cri}(\tau)$ and instantaneous maximum flow $\phi_i^{max}(\tau)$ when the speed of a BB-place is time-varying, $0 \leq v_i(\tau) \leq V_i$, but the propagation speed of congestion is assumed to be constant, W_i.

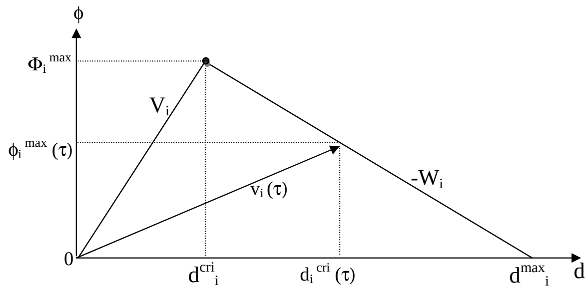


Fig. 3. Relation flow-density of a BB-place

Proposition 15 *Let a BB-place* p_i *with* $\gamma(p_i) = (V_i, d_i^{max}, S_i, \Phi_i^{max})$*, and a speed* $v_i(\tau)$*. At time* τ *an* instantaneous critical density $d_i^{cri}(\tau)$ *and an* instantaneous maximum flow $\phi_i^{max}(\tau)$ *are defined as follow:*

$$d_i^{cri}(\tau) = \frac{W_i.d_i^{max}}{v_i(\tau) + W_i}, \quad (5)$$

$$\phi_i^{max}(\tau) = v_i(\tau).d_i^{cri}(\tau) \quad (6)$$

with $0 \leq \phi_i^{max}(\tau) \leq \Phi_i^{max}$ *and* $\frac{\Phi_i^{max}}{V_i} \leq d_i^{cri}(\tau) \leq d_i^{max}$. □

Proof. For a density $d_r = d_i^{cri}(\tau) > d_i^{cri}$, we have from (4):

$$\phi_i^{max}(\tau) = W_i.(d_i^{max} - d_i^{cri}(\tau)) \quad (7)$$

From both the expression of the instantaneous maximum flow in (6) and (7), we deduce:

$$\begin{aligned} v_i(\tau).d_i^{cri}(\tau) &= W_i.(d_i^{max} - d_i^{cri}(\tau)) \\ &\Rightarrow W_i.d_i^{max} \\ &= v_i(\tau).d_i^{cri}(\tau) + W_i.d_i^{cri}(\tau) \\ &= d_i^{cri}(\tau).(v_i(\tau) + W_i) \\ &\Rightarrow d_i^{cri}(\tau) = \frac{W_i.d_i^{max}}{v_i(\tau) + W_i} \end{aligned}$$

□

Note that the instantaneous critical density of BB-place p_i at time τ depends on the instantaneous speed $v_i(\tau)$ and can also be denoted by $d_i^{cri}(v_i)$. By the same, for the instantaneous maximum flow, can also be denoted by $\phi_i^{max}(v_i)$.

Definition 16 *States of batches. Let $C\beta_r(\tau) = (l_r(\tau), d_r(\tau), x_r(\tau), v_r(\tau))$ be a controllable batch of BB-place p_i, with $v_i(\tau)$ the instantaneous speed of p_i.*

- *$C\beta_r$ is called a* free controllable batch *if its density is lower than the critical density of p_i: $d_r(\tau) \leq d_i^{cri}(\tau)$;*
- *$C\beta_r$ is called a* congested controllable batch *if its density is greater than the critical density of p_i: $d_r(\tau) > d_i^{cri}(\tau)$.* □

3.2 Variation of speeds and flows in BPN

We assume that the dynamic evolution of batches inside a BB-place p_i takes into account the variation the maximum flow of continuous and batch transitions and the maximum speed of p_i. For this we present two controlled events as follow:

- A controlled speed event is a triplet (p_i, v_i, τ), where $p_i \in P^{BB}$ is a BB-place, $v_i \in [0, V_i]$ is an instantaneous speed of BB-place and τ is the date of occurrence of this event.
- A controlled flows event is a triplet (t_j, ϕ_j, τ); where $t_j \in T^C \cup T^B$ is a continuous or batch transitions, $\phi_j \in [0, \Phi_j]$ is a instantaneous flow of continuous or batch transitions and τ is the date of occurrence of this event.

Variation of a speed of BB-place: States and characteristics of batches change when the maximum speed of the BB-place is changed, we suppose that the speed of BB-place p_i changes at time τ from $v_i(\tau)$ to $v_i'(\tau)$. Two situations must be considered: easier the speed decreases or increases: i.e., $v_i' < v_i$, or increases, i.e., $v_i' > v_i$.

A) $v_i'(\tau) < v_i(\tau)$: three cases have to be considered (see Fig.4).

- case 1: $C\beta_1 = (l_1, d_1, x_1, v_1)$ is a free controllable batch. When the speed of BB-place changes, batch $C\beta_1$ reduces its speed but keeps its density. It stays a free batch $C\beta_1 = (l_1, d_1, x_1, v_i')$.

– case 2: $C\beta_2 = (l_2, d_2, x_2, v_2)$ is a congested controllable batch with a higher speed than v'_i ($v_2 > v'_i$). When the speed of BB-place changes, batch $C\beta_2$ reduces its speed to v'_i but keeps its density. It becomes a free batch with $C\beta_2 = (l_2, d_2, x_2, v'_i)$.

– case 3: $C\beta_3 = (l_3, d_3, x_3, v_3)$ is a congested controllable batch with a lower speed than v'_i ($v_3 < v'_i$). When the speed of BB-place changes, $C\beta_3$ keeps its speed and its density while it stays a congested batch.

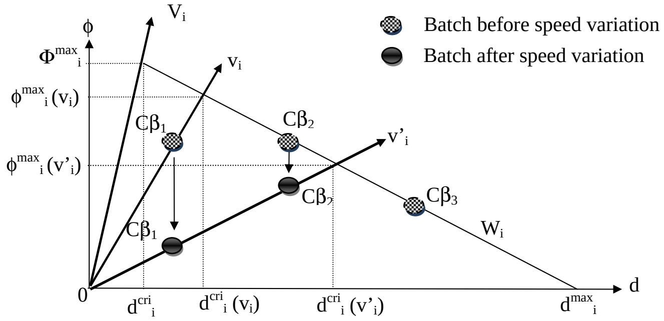


Fig. 4. Batch's changes after a decreasing of BB-place speed

B) $v'_i(\tau) > v_i(\tau)$: three cases have to be considered (see Fig.5)

– case 1: $C\beta_1 = (l_1, d_1, x_1, v_1)$ is a free controllable batch and its density is lower than $d_i^{cri}(\tau)$ at speed v'_i (i.e., $d_1 < d_i^{cri}(\tau)$). When the BB-place speed changes, batch $C\beta_1$ increases its speed to v'_i and keeps its density. It stays a free batch with $C\beta_1 = (l_1, d_1, x_1, v'_i)$ (see case 1 in Fig.5).

– case 2: $C\beta_2 = (l_2, d_2, x_2, v_2)$ is a free controllable batch and its density is greater than $d_i^{cri}(\tau)$ at speed v'_i (i.e., $d_2 > d_i^{cri}(\tau)$). When the BB-place speed changes, batch $C\beta_2$ keeps its density but increases its speed to a speed v'_2 that respects $v_i < v'_2 < v'_i$. It becomes a congested batch with $C\beta_2 = (l_2, d_2, x_2, v'_2)$ (see case 2 in Fig.5).

– case 3: $C\beta_3 = (l_3, d_3, x_3, v_3)$ is a congested controllable batch. When the speed of BB-place changes, this batch does not changed and stays a congested batch.

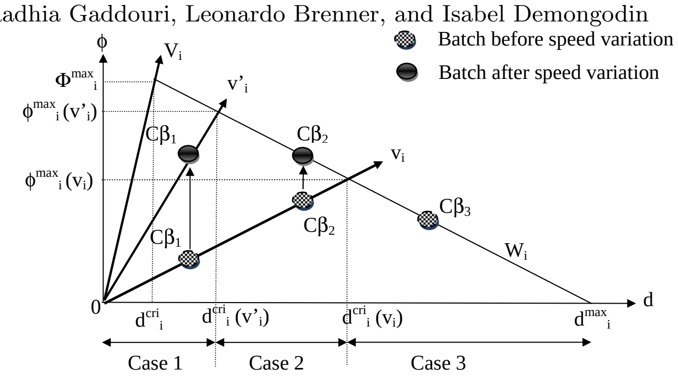


Fig. 5. Batch's changes after a increasing of BB-place speed

Variation of continuous or batch transition flow: States and characteristics of batches change when the maximum flow of continuous or batch transition is changed, we suppose that the flow ϕ_j of continuous or batch transition place t_j changes at time τ from $\phi_j(\tau)$ to $\phi'_j(\tau)$. Two situations must be considered: easier the flow decreases i.e., $\phi'_j(\tau) < \phi_j(\tau)$, or increases, i.e., $\phi_j(\tau') > \phi_j(\tau)$.

A) $\phi'_j(\tau) < \phi_j(\tau)$: two cases have to be considered (see Fig.6 a))

- case 1: $C\beta_1 = (l_1, d_1, x_1, v_1)$ is a free controllable batch. When the flow of continuous or batch transition changes, batch $C\beta_1$ becomes a congested batch with new density and new speed $C\beta'_1(\tau) = (l_1, d'_1, x_1, v'_1)$.
- case 2: $C\beta_2 = (l_2, d_2, x_2, v_2)$ is a congested controllable batch. When the flow of continuous or batch transition changes, batch $C\beta_2$ becomes completely congested, its density increases and its speed decreases $C\beta'_2(\tau) = (l_2, d'_2, x_2, v'_2)$.

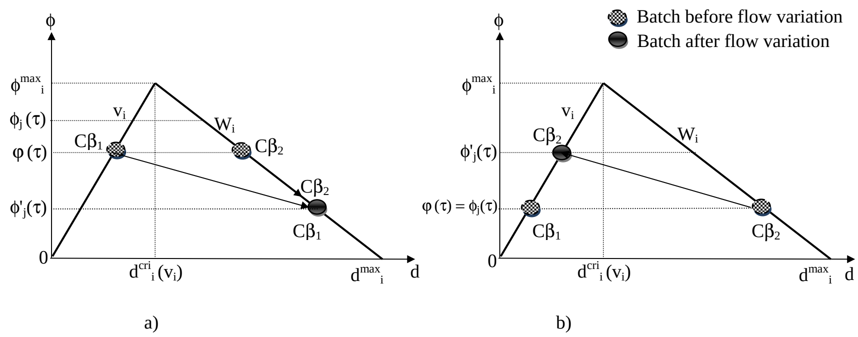


Fig. 6. Variation of batch transition flow: decreasing and increasing of flow

B) $\phi'_j(\tau) > \phi_j(\tau)$: two cases have to be considered (see Fig.6 b))

– case 1: $C\beta_1 = (l_1, d_1, x_1, v_1)$ is a free controllable batch. When the flow of continuous or batch transition changes, batch $C\beta_1$ stays a free batch.
– case 2: $C\beta_2 = (l_2, d_2, x_2, v_2)$ is a congested controllable batch. When the flow of continuous or batch transition changes, batch $C\beta_2$ becomes a free batch with $C\beta_2 = (l_2, d'_2, x_2, v_i)$: its speed becomes equal to the speed of batch place and its density is reduced.

3.3 Dynamics of controllable batch

Let us first recall some concepts necessary to the understanding of the evolution of controllable batches, dedicated to a bacth place that can be also applied to a BB-place.

Definition 17 *The* input (*resp.,* output) flow *of a batch place or continuous place p_i at time τ is the sum of all flows entering (resp., leaving) the place and can be written, respectively, as:*

– $\phi_i^{\mathrm{in}}(\tau) = \sum_{t_j \in {}^\bullet p_i} Post(p_i, t_j) \cdot \varphi_j(\tau) = Post(p_i, \cdot) \cdot \boldsymbol{\varphi}(\tau)$.
– $\phi_i^{\mathrm{out}}(\tau) = \sum_{t_j \in p_i^\bullet} Pre(p_i, t_j) \cdot \varphi_j(\tau) = Pre(p_i, \cdot) \cdot \boldsymbol{\varphi}(\tau)$.

Definition 18 *At time τ, various static functions can be applied on batches composing the marking of batch place p_i:*

– *Create. If the input flow of p_i is not null, i.e., $\phi_i^{in}(\tau) \neq 0$, a controllable batch $C\beta_r(\tau) = (0, d_r(\tau), 0, v_r(\tau))$ with $d_r(\tau) = \phi_i^{in}(\tau)/v_i(\tau)$ and $v_r(\tau) = v_i(\tau)$, is created and added to the marking of p_i, i.e., $m_i(\tau) = m_i(\tau) \cup \{\beta_r(\tau)\}$.*
– *Destroy. If the length of a batch, $C\beta_r(\tau)$, is null, $l_r(\tau) = 0$, and if it is not a created batch, $x_r(\tau) \neq 0$, batch $C\beta_r(\tau)$ is destroyed, noted $C\beta_r(\tau) = \mathbf{0}$, and removed from the marking of p_i, i.e., $m_i(\tau) = m_i(\tau) \setminus \{\beta_r(\tau)\}$.*
– *Merge. If two batches with the same density and the same speed are in contact, they can be merged. Let batches $C\beta_r(\tau) = (l_r(\tau), d_r(\tau), x_r(\tau), v_r(\tau))$ and $C\beta_h(\tau) = (l_h(\tau), d_h(\tau), x_h(\tau), v_h(\tau))$, such that $x_r(\tau) = x_h(\tau) + l_r(\tau)$, $d_r(\tau) = d_h(\tau)$ and $v_r(\tau) = v_h(\tau)$. In this case, batch $C\beta_r(\tau)$ becomes $C\beta_r(\tau)$ $= (l_r(\tau) + l_h(\tau), d_r(\tau), x_r(\tau), v_r(\tau))$, batch $\beta_h(\tau)$ is destroyed, $\beta_h(\tau) = \mathbf{0}$, and $m_i(\tau) = m_i(\tau) \setminus \{\beta_h(\tau)\}$*
– *Split. It is always possible to split a batch into two batches in contact with the same density and the same speed.* □

The density and the speed of batches cannot varied in time while their value can change when an event occurs. In other words, these both characteristics are piecewise constants while the length and the position are linear in time. Consequently, for any batch $C\beta_r(\tau) = (l_r(\tau), d_r(\tau), x_r(\tau), v_r(\tau))$, it holds: $\dot{d_r} = \dot{v_r} = 0$.

Inside a BB-place, various equations govern the dynamics of batches : inputting, moving and existing. Between two events, a batch can move in three different behaviors: free behavior, congestion behavior and decongestion behavior.

(*Free behavior*) Controllable batch $C\beta_r(\tau) = (l_r(\tau), d_r(\tau), x_r(\tau), v_r(\tau))$ of BB-place p_i

is in a *free behavior*, if it moves freely at its transfer speed $v_r(\tau)$. Three different dynamics can occur.

- *Input.* A created controllable batch, $C\beta_r(\tau) = (0, d_r(\tau), 0, v_r(\tau))$, without contact with another batch or in contact with a downstream batch $C\beta_h(\tau)$ that has a greater speed (i.e., $v_h(\tau) \geq v_r(\tau)$), freely enters in place p_i according to $\dot{x_r} = \dot{l_r} = v_r(\tau)$.
- *Move.* A controllable batch, $C\beta_r(\tau) = (l_r(\tau), d_r(\tau), x_r(\tau), v_r(\tau))$, which is a free batch, freely moves inside BB-place p_i according to $\dot{x_r} = v_r(\tau)$; $\dot{l_r} = 0$.
- *Exit.* An output controllable batch $C\beta_r(\tau) = (l_r(\tau), d_r(\tau), S_i, v_r(\tau))$, which has its flow equals to the output flow of p_i, or which is free with a lower batch flow than the output flow, freely exits from place p_i according to $\dot{x_r} = 0$; $\dot{l_r} = -v_r(\tau)$.

(*Congestion behavior*) Controllable batch $C\beta_r(\tau) = (l_r(\tau), d_r(\tau), x_r(\tau), v_r(\tau))$ of BB-place p_i is in a *congestion behavior*, if it cannot move at its speed but must reduces it, i.e., it starts a congestion.

An output controllable batch $OC\beta_r(\tau)$ of BB-place p_i, which is in a congested behavior at time τ, is split into two batches in contact (Def. 18) as follow:

- $C\beta_r(\tau) = (l_r(\tau), d_r(\tau), S_i, v_r(\tau))$ and $OC\beta_{r'}(\tau) = (0, d_{r'}(\tau), x_{r'}(\tau), v_{r'}(\tau))$

  with: $d_{r'}(\tau) = d_i^{max} - \frac{\phi_i^{out}}{W_i}$; $v_{r'}(\tau) = \frac{\phi_i^{out}}{d_{r'}(\tau)}$ and $x_{r'}(\tau) = S_i$

From time τ on, the evolution of both batches $C\beta_r$ and $OC\beta_{r'}$ is governed by (Eq.21) and (Eq.22) in [9]

A complete and general description of the equations that govern the congestion behavior can be found in [9]. Note that in the dynamics of congestion, we assume that the density of a batch in a congestion behavior is equal to $d_i^{max} - \frac{\phi_i^{out}}{W_i}$ that can easily deduced from Eq. 4. When a batch starts a congestion, it is split into two batches in contact where the downstream batch is congested.

(*Decongestion behavior*) Congested controllable batch $C\beta_r(\tau) = (l_r(\tau), d_r(\tau), x_r(\tau), v_r(\tau))$ of BB-place p_i is in a *decongestion behavior*, if it can move with a higher speed.

At time τ, the congested output batch $OC\beta_r(\tau)$ is split into two batches in contact (see Def. 18) as follow:

- $OC\beta_r(\tau) = (l_r(\tau), d_r(\tau), S_i, v_r(\tau))$ and $OC\beta_{r'}(\tau) = (0, d_{r'}(\tau), x_{r'}(\tau), v_{r'}(\tau))$

with $d_r(\tau) = d_i^{max} - \frac{\phi_i^{out}}{W_i}$, $v_r(\tau) = \frac{\phi_i^{out}}{d_r(\tau)}$ and $x_r(\tau) = S_i$

and $v_{r'}(\tau) = v_i$, $d_{r'}(\tau) = \frac{\phi_i^{out}}{v_i}$ and $x_{r'}(\tau) = S_i$

3.4 IB-state and events

In the dynamics of Triangular Batches Petri Nets are based on a discrete event approach with linear or constant continuous evolutions between two timed events. Then between two events or two dates, the state of the hybrid model has an invariant state defined as follow:

Definition 19 *The* invariant behavior state *(IB-state) of a TrBPN corresponds to a period of time such that:*

- *the marking in the discrete places is constant;*
- *the IFF of the continuous and batch transitions is constant;*
- *the reserved marking of discrete and continuous places is constant.*

The IB-state changes if and only if one (or possibly several at the same time) of the following kind of events occurs:

- Internal events (timed events inside a BB-place)
  - i.1 a batch becomes an output batch $C\beta_r = OC\beta_r$;
  - i.2 two batches meet;
  - i.3 an output batch is destroyed $OC\beta_r = \mathbf{0}$.
- External events
  - e.1 a discrete transition is fired: t_j;
  - e.2 a continuous place becomes empty: $m_i^n = 0$;
  - e.3 a discrete transition becomes enabled $m_i^n = a$;
  - e.4 a batch becomes an output batch (i.e. event i.1 above);
  - e.5 an output batch is destroyed (i.e. event i.3 above);
- Controlled events
  - c.1 the flow of a batch transitions is modified: $\phi_j(\tau) = \phi'_j(\tau)$;
  - c.2 the speed of a BB-place is modified: $v_i(\tau) = v'_i(\tau)$.

As in Batches Petri Nets, the behavior of a TrBPN can be represented by an evolution graph where a node represents an IB-state of the dynamic model. The nodes are linked by arcs with a labeled transition that determines the occurred events and the past delay between two consecutive IB-states (see [3]). An example of evolution graph will be given in Section 4.

3.5 Computation of instantaneous firing flows

To computate the instantaneous firing flows (IFF) φ, the speed v_i of batch place p_i and the flow of continuous and batch transitions ϕ_j are considered as variables that can be suitably chosen by a supervisor to drive the evolution of the net. The computation of IFF for a TrBPN is deduced from Def. 10.

Proposition 20 *Given a TrBPN (N, m) with incidence matrix C. For a given speed of BB-place $v_i(\tau)$ and a flow of continuous and batch transitions ϕ_j.*

Any admissible IFF vector $\boldsymbol{\varphi}$, at $\boldsymbol{m}$, is a feasible solution of the following linear set:

– $T_{\mathcal{N}}(\boldsymbol{m}) \subset T^C \cup T^B$ *be the subset of continuous and batch transitions that are not enabled at* $\boldsymbol{m}$*;*
– $P_{\emptyset}(\boldsymbol{m}) = \{p_i \in P^C \mid m_i = 0\}$ *be the subset of empty continuous places;*
– $P_F(\boldsymbol{m}) = \{p_i \in P^{BB} \mid q_i = Q_i\}$ *be the subset of full BB-places.*

$$\begin{cases} (a) & 0 \leq \varphi_j(\tau) \leq \phi_j & \forall t_j \in T^C \cup T^B \\ (b) & \varphi_j(\tau) = 0 & \forall t_j \in T_N(m) \\ (c) & C(p_i,.).\varphi(\tau) \geq 0 & \forall p_i \in P_{\emptyset}(m) \\ (d) & C(p_i,.).\varphi(\tau) \leq 0 & \forall p_i \in P_F(m) \\ (e) & Post(p_i,.).\varphi(\tau) \leq v_i(\tau).d_i^{cri}(\tau) & \forall p_i \in P^{BB} \\ (f) & Pre(p_i,.).\varphi(\tau) \leq v_i(\tau).d_i^{out}(\tau) & \forall p_i \in P^{BB} \\ (g) & Pre(p_i,.).\varphi(\tau) \leq v_i(\tau).d_i^{cri}(\tau) & \forall p_i \in P^{BB} \end{cases} \tag{8}$$

□

Proof. Constraints (a)-(d) are not changed compared to the linear program (1) in Def. 10. The constraint (e) in (1) becomes the constraint (e), it means that the flow arriving in a BB-place $Post(p_i,.).\varphi$ should not exceed the maximum flow of BB-place p_i which is now defined by $\phi_i^{max} = v_i.d_i^{cri}$.

Then, the constraint (f) in (1) is now duplicated in two constraints (f) and (g) as follows: the constraint (f), implies that the total flow exiting BB-place p_i $Pre(p_i,.).\varphi$ should not be greater than the output flow $v_i.d_i^{out}$ generated by the output batch exiting the place. The constraint (g) implies that the total flow leaving the BB-place $Pre(p_i,.).\varphi$ should not exceed the maximum flow $\phi_i^{max} = v_i.d_i^{cri}$ of BB-place p_i. In the case of a free batch, the flow exiting $Pre(p_i,.).\varphi$ will be limited by the constraint (f), and in the opposite case, i.e a congested batch, the $Pre(p_i,.).\varphi$ is limited in constraint (f) by a flow $v_i.d_i^{out}$ which is greater than the maximum flow of BB-place $v_i.d_i^{cri}$. Therefore in this case it is the constraint (g) will prevail, as it limits the flow exiting $Pre(p_i,.).\varphi$ by the maximum flow.

3.6 Dynamics of a TrBPN

The dynamics of Triangular Batches Petri Nets is based on a discrete event approach with linear or constant continuous evolutions between timed events. Between two timed events, the state of the net has an invariant behavior state (IB-state) (see Def. 19).The state of the system is calculated only when it undergoes discontinuity. This dynamic tests on the existence of controlled event at current date (see Section. 3.2), determines the state of batches (see Def. 16) and the enabled transitions (see Def. 3.5) to calculate the instantaneous firing flow of continuous and batch transition.Then all timed events that change the global state of system are computed, the date of the nearest event, which is the nearest in time, becomes the current date and at this instant, new markings are computed. This dynamics is stopped when there is no event or when there is an invariant state (IB-state) already defined previously. Fig. 7 shows the most important steps of this dynamics.

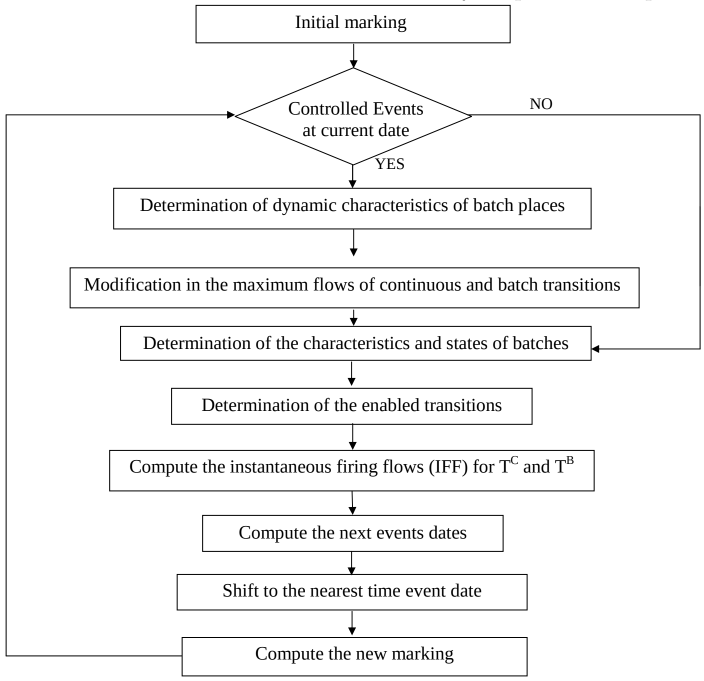
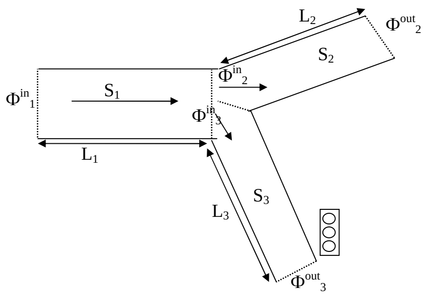


Fig. 7. Dynamics of a triangular Batches Petri Nets

4 Example

To illustrate the main contributions of this paper, we consider a traffic road intersection composed of three sections and a traffic light as shown by Figure 8.



Section	1(1lane)	2(1lane)	3(1lane)
L_i (*km*)	12	3.6	9
V^{max}_i (*km/h*)	120	120	60
d^{max}_i (*veh/km*)	320	320	320
Φ^{max}_i (*veh/h*)	7000	4080	4080
Φ^{in}_i (*veh/h*)	0	3060	1040
Φ^{out}_i (*veh/h*)	4100	4080	4080

Fig. 8. Traffic road intersection

Section S_1 has an output flow equal to 4100 veh/h. This output flow of S_1 is divided into an input flow of S_2 equal to 3060 veh/h and an input flow of S_3

equal to 1040 veh/h. In other words, a part of outgoing vehicles of S_1 goes to the second section S_2 and the other one goes to the third section S_3. The input flow of section S_1 is supposed null meaning that no vehicles enters the intersection. Some traffic events appear during the evolution of the traffic:

1- 10.8 minutes after the beginning, an accident appears at the end of section S_2 reducing its output flow to 2040 veh/h;
2- 13.8 minutes after the beginning, there is a reduction in the maximum speed of section S_3 that becomes equal to 20 km/h;
3- 13.2 minutes after the reduction in the speed of S_3, there is an increase in the speed of S_3 that becomes equal to 60 km/h.

The initial state of this intersection is: in section S_1, there are 409 vehicles having a speed of 120 km/h. They are supposed uniformly distributed from the entrance to the end of section S_1 with a density of 34.1 veh/km. Both sections S_2 and S_3 are empty.

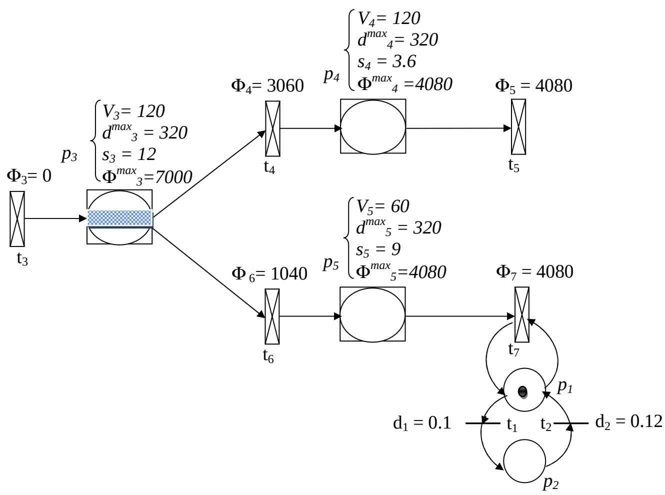


Fig. 9. TrBPN of the traffic road intersection

The TrBPN model for this traffic road intersection is shown in Figure 9. Discrete places p_1, p_2 represent, respectively, the green and red traffic light. Delays $d_1 = 0.1h$ and $d_2 = 0.12h$ of discrete transitions t_1 and t_2 represent, respectively, the durations of green and red traffic light. BB-places p_3, p_4 and p_5 represent, respectively, sections S_1, S_2 and S_3. Maximum flows Φ_3, Φ_4, Φ_5, Φ_6 and Φ_7 of batch transitions t_3, t_4, t_5, t_6 and t_7 represent the maximum input/output flow of each section.

The initial state of intersection, 409 vehicles distributed in section S_1, are represented in the TrBPN model by the output controllable batch $OC\beta_3(\tau_0) = (12, 34.1, 12, 120)$ that composes the initial marking of place p_3. As the traffic light is supposed to be green, place p_1 contains one token while place p_2 is emptied. The initial marking of the model is $m(\tau_0) = (1, 0, \{OC\beta_3(\tau_0)\}, \mathbf{0}, \mathbf{0})$.

The three traffic events cited above, correspond to controlled events in the TrBPN formalism that are defined, according to Section 3.2, by:

– traffic event 1 is related to transition t_5: $(t_5, 2040, 10.8)$;
– traffic event 2 is related to BB-place p_5: $(p_5, 20, 13.8)$;
– traffic event 3 is also related to BB-place p_5: $(p_5, 60, 27)$.

To be able to compute the instantaneous firing flow using linear programming, we define as objective function to maximize the output flows of the sections, i.e., the IFF of batch transitions, $J = \max\{\boldsymbol{\varphi}\}$.

At the initial state ($\tau_0 = 0\ h$), BB-place p_3 contains an output batch $OC\beta_3(\tau_0) = (12, 34.1, 12, 120)$ and there are no controlled events at this date. Following the dynamics of TrBPN (Figure 7), we determine the enabled transitions (t_1, t_3, t_4 and t_6) and we compute the instantaneous firing flow vector solving of the linear program (9) below. Results obtained are $\varphi_3(\tau_0) = 0$, $\varphi_4(\tau_0) = 3060$, $\varphi_5(\tau_0) = 0$, $\varphi_6(\tau_0) = 1040$, and $\varphi_7(\tau_0) = 0$.

$$\begin{cases} \text{(a)} & 0 \leq \varphi_4 \leq 3060 \\ \text{(a')} & 0 \leq \varphi_6 \leq 1040 \\ \text{(a")} & 0 \leq \varphi_3 \leq 0 \\ \text{(b)} & \varphi_5 = \varphi_7 = 0 \\ \text{(e)} & \varphi_3 \leq 7000 \\ \text{(e')} & \varphi_4 \leq 4080 \\ \text{(e")} & \varphi_6 \leq 4080 \\ \text{(f)} & \varphi_4 + \varphi_6 \leq 4100 \\ \text{(g)} & \varphi_4 + \varphi_6 \leq 7000 \end{cases} \tag{9}$$

At this initial state, there is a creation of two batches in BB-places p_4 and p_5, respectively $C\beta_4(\tau_0)$ and $C\beta_5(\tau_0)$. The state of the output batch $OC\beta_3(\tau_0)$ is free (see Def.16).

From the initial IB-state, nine IB-states have been reached as it is shown in the evolution graph in Figure 10. The set of timed events are:

(Ev1, delay: 0.1 h) Discrete transition (t_1) is enabled; destruction of the output batch of BB-place p_3, $OC\beta_3 = \mathbf{0}$;
(Ev2, delay: 0.03 h) Batch $C\beta_4$ of BB-place p_4 becomes an output batch, $OC\beta_4$;
(Ev3, delay: 0.03 h) Maximum flow of transition t_5 is modified and becomes $\phi_5 = 2040veh/h$;
(Ev4, delay: 0.04 h) Output batch of p_4 is destroyed, $OC\beta_{4'} = \mathbf{0}$;
(Ev5, delay: 0.02 h) Discrete transition t_2 is enabled; batch $C\beta_5$ of place p_5 becomes an output batch $OC\beta_5$;
(Ev6, delay: 0.01 h) Speed of BB-place p_5 is reduced, $v_5 = 20km/h$;
(Ev7, delay: 0.1 h) Discrete transition t_1 is enabled;
(Ev8, delay: 0.12 h) Discrete transition t_2 is enabled and speed of p_5 is increased, $v_5 = 60km/h$;

(Ev9, delay: 0.01 h) Destruction of the output batch of BB-place p_5, $OC\beta_{5''} = \mathbf{0}$.

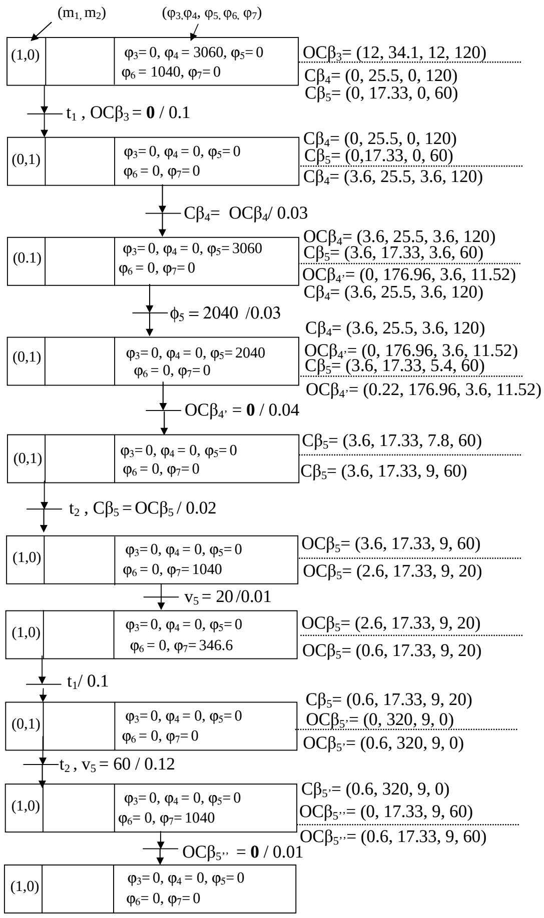


Fig. 10. Evolution graph of the TrBPN model

We detail now two events of the set of timed events that illustrate our contributions in TrBPN: a modification of maximum flow of batch transition t_5, and a variation of the maximum speed of BB-place p_5. These both events are controlled events.

At Ev3, there is a controlled event due to an accident and the maximum flow of batch transition t_5 is reduced to 2040 veh/h. The output flow of the output batch $OC\beta_4$ is now higher then the maximum flow of batch transition t_5 and the output batch adopt a congestion behavior. In this case, the output batch $OC\beta_4$ is split into two batches $C\beta_4(\tau_3) = (3.6, 25.5, 3.6, 120)$ and $OC\beta_{4'}(\tau_3) = (0, 176.96, 3.6, 11.52)$ according to Def.18. The batch $C\beta_4$ is a free batch while

$OC\beta_{4'}$ is congested batch. The batch $C\beta_5(\tau_3) = (3.6, 17.33, 5.4, 60)$ of BB-place p_5 is also a free batch and continues to move freely inside BB-place keeping its length, only its position change. This case corresponds to the case 1 of A) in the section of variation of flow (see Section 3.2).

To compute the IFF of all enabled continuous and batch transitions, we solve the linear program of the equation 10 and we obtain the follow results ($\varphi_3(\tau_3) = 0$,$\varphi_4(\tau_3) = 0$, $\varphi_5(\tau_3) = 2040$, $\varphi_6(\tau_3) = 0$ and $\varphi_7(\tau_3) = 0$).

$$\begin{cases} (a) & 0 \leq \varphi_5 \leq 2040 \\ (a') & 0 \leq \varphi_3 \leq 0 \\ (b) & \varphi_4 = \varphi_6 = \varphi_7 = 0 \\ (f) & \varphi_5 \leq 3060 \\ (g) & \varphi_5 \leq 4080 \end{cases} \tag{10}$$

At Ev6, there is a controlled event that decreases the batch place speed ($v_5 = 20\ km/h$). The output batch $OC\beta_5(\tau_6) = (2.6, 17.33, 9, 20)$ is a free batch and has its speed reduce to 20 km/h. In this case, the speed reduction do not change. The state of the output batch $OC\beta_5(\tau_6)$ remains in a free batch. However, this speed reduction implies in a reduction of the flow of the batch to 346.6 veh/h. This case corresponds to the case 3 of A) in the section of decrease in the maximum speed of BB-place (see Section 3.2).

We recompute the IFF of all enabled continuous and batch transitions and we obtain ($\varphi_3(\tau_6) = 0$,$\varphi_4(\tau_6) = 0$, $\varphi_5(\tau_6) = 0$, $\varphi_6(\tau_6) = 0$ and $\varphi_7(\tau_6) = 346.6$). We remark that the reduction of the flow of the batch $OC\beta_5(\tau_6)$ is observed in the IFF of $\varphi_7(\tau_6)$.

5 Conclusion

We proposed in this paper an extension of Batches Petri Nets by a new definition of batch place called Bi-parts Batche place (BB-place). To model a variable delay inspired in the triangular fundamental diagram, we improve the γ function of a batch place with a maximal flow parameter. This maximal flow, the speed and the density allow us to define a new dynamic equations of flow-density. These equations represent more accurately the bi-part behavior observed in the systems flow based on the triangular diagram. To include this bi-part behavior in this new extension of BPN, we redefined some concepts like, the states of batches, conditions of enabling and firing transitions. Additionally, a definition of controlled events was proposed in this paper and it allows a manual control of the speed of BB-place and the maximal flow of batch and continuous transitions. Another contribution of this paper is the proposition of a method to compute the IFF that take into account the bi-parts behavior of the BB-place proposed in this paper. An application of all these contributions is shown in the example in the section 4. The Triangular Batches Petri Nets formalism proposed in this paper allows us to represent more accurately the vehicle traffic in transportation systems than BPN. The next step is to use the control laws already defined in the literature to test and analyze the performance of these laws (such as the

Ramp Metering System (RMS), and Variable Speed Limit (VSL)) in order to reduce congestion.

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