Visualization of Tangent Developables on a Volumetric Display

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Abstract. A tangent developable is a developable surface constructed by the union of the tangent lines of a space curve. These surfaces have applications not only in mathematics but also in engineering, such as for designing cars, ships, and apparel. However, since tangent developables typically have complicated and twisted surfaces, it is difficult to understand their structures from their images on a 2D screen. For ease in understanding such complicated structures, a more improved visualization method is required. In this study, we propose and evaluate a software tool for visualizing tangent developables on a volumetric display that draws 3D images directly in real 3D space.

Keywords: Tangent developable, Surface visualization, Volumetric display, 3D display

1 Introduction

A tangent developable is a developable surface that is constructed by the union of the tangent lines of a space curve. As the surface can be developed onto a plane without stretching, it has applications not only in mathematics but also in engineering, such as for designing cars [1], or ships [1], and apparel [2]. Fig. 1 illustrates a tangent developable of a space curve defined by the following parametric equation:

$$c(t) = (\cos^3 t, \sin^3 t, \sqrt{2 - \cos^3 t - \sin^3 t}\ ), \quad -1.0 \leq t < 1.0. \tag{1}$$

As shown in Fig. 1, we can visualize tangent developables on a 2D screen. However, they typically have complicated and twisted surfaces, and therefore it is difficult to understand their structures from 2D images. To ease understanding of their structures, an improved visualization method is required.

A volumetric display is a device that creates 3D images in real 3D space [3]. Most volumetric displays satisfy the physiological factors for stereoscopic viewing while requiring no 3D goggles or head-mounted displays. To date, several types of volumetric displays have been proposed and developed, including a variable-focal type [4], a volume-scanning type [5] [6], an up-conversion type [7] [8], and

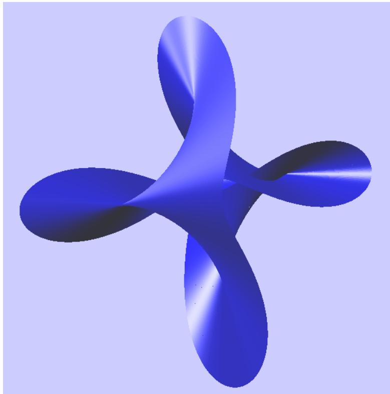

Fig. 1. Example of a tangent developable

a plasma-emission type [9]. Some of these are commercially available; however, they are expensive for researchers who are developing applications for them.

We have developed a volume-scanning type volumetric display that can be easily constructed from commercially available components [10]. Our display creates sequential cross-sectional images of 3D objects on a vacuum fluorescent display (VFD) moving in reciprocating motion; 3D images of the objects appear because of the afterimage effect. Furthermore, we have proposed an additional hardware and a software tool for visualizing isometric deformations of minimal surfaces [11]. As an application of our display, we propose and evaluate, in this study, a software tool for visualizing tangent developables.

2 Tangent Developable

In this section, we quickly review some mathematical aspects of tangent developables. For details, please refer to standard textbooks on the differential geometry of curves and surfaces, *e.g.*, [12] and [13].

A surface in Euclidean 3-space $\mathbb{R}^3$ is said to be *ruled* if it is formed by a one-parameter family of straight lines. It is a classical fact that any ruled surface of constant Gauss curvature zero is locally congruent to either a cylindrical surface, a conical surface, or a tangent surface. By definition, a *tangent surface* is a ruled surface formed by tangent lines of a space curve. Precisely, given a regular curve $c\colon I(\subset \mathbb{R}) \to \mathbb{R}^3$, a map $f(t,u) = c(t) + u\dot{c}(t)$ draws a tangent surface. It is easily verified that $f(t,u) = c(t) + u\dot{c}(t)$ has zero Gaussian curvature on the regular region. Note that a tangent surface $f(t,u)$ has singularities on $u = 0$ or $t = t_0$ where the curvature of c vanishes.

In classical terminology, ruled surfaces of constant Gauss curvature zero have been called *developable* surfaces because they can be isometrically developed to a plane. The fact introduced above asserts that a surface is developable if and only if it is locally congruent to either a cylindrical surface, a conical surface, or a tangent surface. For this reason, a tangent surface is usually called a *tangent developable surface* or a *tangent developable*, in short.

One may easily imagine that a cylindrical or conical surface can be developed to a plane. Moreover, this is not hard to demonstrate using a sheet of paper. However, it would be rather difficult to directly verify the developability of tangent developables. A typical example is the tangent developable of a helix,

$$h(t) = \left(a\cos\frac{t}{\sqrt{a^2+b^2}}, a\sin\frac{t}{\sqrt{a^2+b^2}}, \frac{bt}{\sqrt{a^2+b^2}} \right) \tag{2}$$

where $a(>0)$ and b are constant real numbers. Note that t is an arc-length parameter. Its tangent developable

$$f(t,u) = h(t) + u\dot{h}(t) \tag{3}$$

has the first fundamental form

$$I = \left(1 + \frac{a^2u^2}{(a^2+b^2)^2}\right) dt^2 + 2dtdu + du^2 \tag{4}$$

so that the choice of a, b preserving $a/(a^2+b^2)$ induces an isometric family of tangent developables. For example, consider the tangent developable $f_\theta(t,u)$ of a helix (2) with the choice

$$a = 1 + \cos\theta, b = \sin\theta, \quad \theta \in (0,\pi). \tag{5}$$

Then $\{f_\theta(t,u) \mid \theta \in (0,\pi)\}$ gives a one-parameter family of isometric tangent developables, which converges to a part of the plane as $\theta \to 0$ (or π). It implies that, in mathematical terms, a tangent developable of a helix can be developed to a plane as non-zero θ continuously goes to zero. As shown in Section 5, our volumetric display visualizes this as an animation in real 3D space without spoiling isometry.

We are more interested in a situation in which the generating curve $c(t)$ can be stretched and contracted. How does the tangent surface of a stretchable curve $c(t)$ behave under $c(t)$ stretching? In such cases, the deformation of the tangent developable is not necessarily isometric and is more complicated. The ease of understanding such deformations is evaluated in Section 5.

3 Our Volumetric Display

Our display creates sequential cross-sectional images of 3D objects on a VFD moving in reciprocating motion; the 3D images of the objects appear because of the afterimage effect. Fig. 2 shows a schematic overview of our display, the reciprocating mechanism, and an example 3D image on our display.

The display has a small display area of $(W \times H \times D)$ $4 \times 2 \times 4[cm]$; however, as shown in Fig 2, it has sufficient resolution to display a single 3D object of $(W \times H \times D)$ $128 \times 64 \times 128$ voxels, and it has a wide view angle (approx. vertical: $170°$ and horizontal: $170°$).

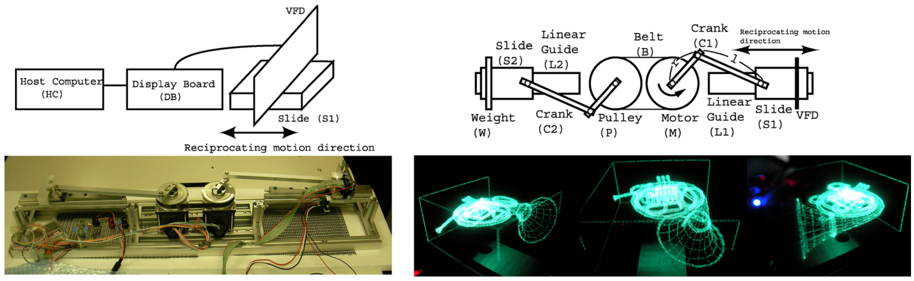

Fig. 2. (Above left) Schematic overview of our display, (Above right, below left) Schematic and photo of the reciprocating mechanism. (Below right) Sample 3D images on the display from three different viewpoints

4 Our Software Tool

Our application provides users with a software tool, an interface called TDsquare (an abbreviation for 3D viewer for tangent developable). After a user provides an equation for a space curve to the tool, it differentiates the equation symbolically, and then generates 3D-voxel data for the union of tangent lines. The user can send the data to our volumetric display via a USB interface and view a true 3D image of the tangent developable. Furthermore, the user can view a 2D preview image of the tangent developable on the PC screen. TDsquare is written in Java and accepts such text input as the following,

```
x = 2*cos(pi*sin(2*pi*t))
y = cos(2*pi*t + pi/6)
z = 2*sin(2*pi*sin(2*pi*t))
t: 0.0 -> 1.0
```

The above input describes the following parametric equation, which defines a space curve,

$$c(t) = (2\cos(\pi\sin(2\pi t)),\ \cos(2\pi t+\pi/6),\ 2\sin(2\pi\sin(2\pi t))),\ 0.0 \leq t < 1.0. \quad (6)$$

As TDsquare is a console application, the user specifies the input text file as a command line argument.

TDsquare provides the following basic mathematical functions: sin, cos, tan, log, exp, pow (power), arcsin, arccos, and arctan. Additionally, the user can use these functions in composition form. TDsquare analyzes the equation by LL(1) parsing and builds a parse tree. Then, the tree is transformed to a tree corresponding to the symbolically differentiated equation. Finally, the coordinates of the tangent lines are calculated using the tree, then data are generated that are to be sent to the display. Although various ready-made math formula parsers are available today, we implemented an original formula parser including a differentiator for ease in modification and extension in the future. As TDsquare is written in Java, it runs on any PC that supports Java. However, the software that sends generated data to the display runs only on Windows PCs under the present conditions.

Fig. 3 shows sample images of tangent developables on the display and their previews on the software tool. In the figure, TD1 is a tangent developable generated from the above input, and TD2 is a tangent developable of the space curve

$$c(t) = (t^2,\ t^3,\ t^5),\ \ -1.0 \leq t < 1.0. \tag{7}$$

The previews can be rotated both horizontally and vertically by dragging a mouse, and views are shaded in real-time. As the images on our display are drawn directly in real 3D space, the user can view them from any point within the view angle.

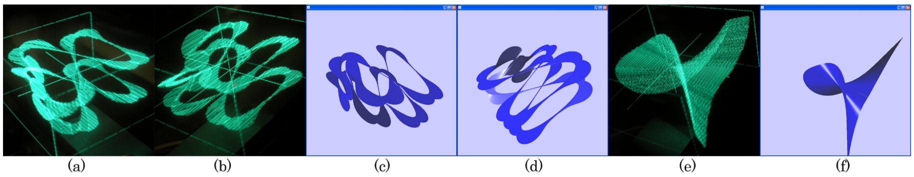

Fig. 3. (a) and (b) 3D images of tangent developable (TD1) on our display (from two different viewpoints). (c) and (d) 2D previews of TD1 on a PC screen. (e) 3D image of tangent developable TD2 on our display and (f) 2D preview of TD2 on a PC screen.

Changes in the constants in the equation that defines a space curve can cause complicated deformations of a tangent developable. Furthermore, our tool can generate data for animating such deformations on our display. As an example, let

$$c(t) = (1.7\cos(4\pi t)\sin(k\pi t),\ 0.5\sin(k\pi t),\ \ -1.7\sin(4\pi t)), 0 \leq t < 1, \tag{8}$$

be a parametric equation of a space curve. When the constant k continuously varies from 0 to 1, its tangent developable (TD3) deforms as shown in Fig. 4 and in the supplementary movie file Movie1. In Movie1[1], most of the flicker is caused by the difference in scanning frequencies of the display and the camera. The preview animation can be rotated and is shaded in real time. The following is the text input that describes the above deformation.

The following is the text input that describes the above deformation.

```
x = 1.7*cos(4*pi*t)*sin(_k*pi*t)
y = 0.5*sin(_k*pi*t)
z = -1.7*sin(4*pi*t)
t: 0.0 -> 1.0
_k: 0.0 -> 1.0
```

[1] http://www.epi.dendai.ac.jp/Yamamoto/MathUI2014/Movie.html

The underscore is a prefix for parameters that cause deformations. The number of animation frames is specified by a command line argument.

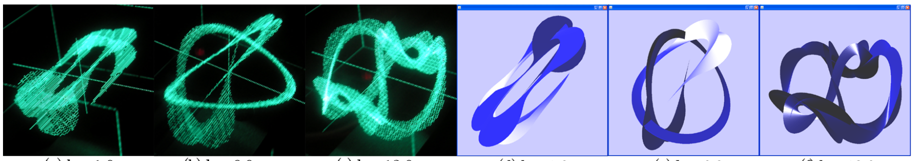

Fig. 4. (a)-(c) Frames from 3D animation of the deformation of TD3 on our display. (d)-(f) 2D previews on a PC screen for the same values of the deformation parameter k as in (a)-(c).

5 Evaluation

For examples TD1, TD2, and TD3, we evaluated the differences in ease of understanding the structures (TD1 and TD2), and the deformation (TD3) between the 2D preview and 3D images. For each tangent developable, each of thirteen participants observed the preview image for 1 min, and then each observed the 3D image on our display for 1 min. During observations of the 2D preview images, participants could rotate the image freely by dragging the mouse. After observing the 2D and 3D images of each tangent developable, participants were asked to state whether the 2D or 3D image was more helpful in understanding the structure of TD1 and TD2 and the deformation of TD3. All participants are students at the faculty of electronic engineering of our university and have no background knowledge of tangent developables. For TD1, ten participants chose the 3D image ($p = 0.092$, two-sided binomial test). For TD2, seven participants chose the 3D image ($p = 1.0$, two-sided binomial test). For TD3, twelve participants chose the 3D image ($p = 0.0034$, two-sided binomial test). Sample TD2 has a simpler shape than the other two, and in this case, the results from participants show no statistically significant difference between the preview and the display. However, the results for TD1 and TD3 imply an advantage in using our display to help understand the structures of complicated tangent developables and their deformations.

Another advantage to our display is preservation of isometry. As described in Section 2, some deformations of tangent developables are isometric. Fig. 5 and the supplementary movie file Movie2[2] illustrate an isometric deformation of a tangent developable for the helix defined by (3).

In those illustrations, the following parameter θ was used for the deformation,

$$a = 0.25 \times (1 + \cos\theta), b = 0.25 \times \sin\theta, \quad \theta \in (0, \pi). \tag{9}$$

[2] http://www.epi.dendai.ac.jp/Yamamoto/MathUI2014/Movie.html

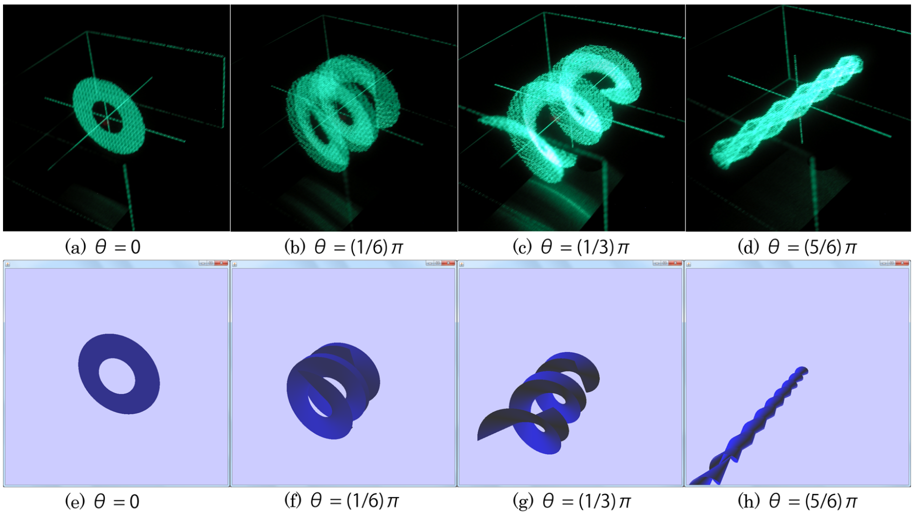

Fig. 5. (a)-(d) Isometric deformation in (9) of a tangent developable on our 3D display. (e)-(h) 2D previews on a PC screen for the same values of the deformation parameter θ as in (a)-(d).

As images on 2D displays are mappings from 3D space to 2D space, 2D images do not preserve isometry. However, the images on our display are drawn in real 3D space, thereby preserving isometry.

6 Conclusions

We have proposed and implemented a tool for visualizing tangent developables on a volumetric display. We further evaluated the ease in understanding structures of tangent developables using our tool as compared to using images on a 2D display. Future tests should include evaluations that compare visualizations on our display with those produced on other 3D displays. Nevertheless, the present results show some advantages of using our tool rather than using images on a 2D display; this is particularly true for visualizing elastic deformations of tangent developables.

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