In Which Sense Is Fuzzy Logic a Logic for Vagueness?

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Abstract. The problem of artificial precision demonstrates the inadequacy of naïve fuzzy semantics for vagueness. This problem is, nevertheless, satisfactorily remedied by fuzzy plurivaluationism; i.e., by taking a class of fuzzy models (a fuzzy plurivaluation), instead of a single fuzzy model, for the semantics of a vague concept. Such a fuzzy plurivaluation in turn represents the class of models of a formal theory, preferably formulated in first- or higher-order fuzzy logic, which formalizes the meaning postulates of the vague concepts involved. The consequence relation of formal fuzzy logic then corresponds to the (super)truth of propositions involving these vague concepts. An adequate formal treatment of vague propositions by means of fuzzy logic thus consists in derivations in the formal calculus of a suitable fuzzy logic, while the particular truth degrees found in engineering applications actually pertain to artificially precisified (so no longer vague) gradual notions.

Keywords: Vagueness, fuzzy plurivaluationism, formal fuzzy logic.

1 Fuzzy Semantics of Vagueness?

Fuzzy sets and fuzzy logic are often claimed (e.g., [5, 15, 6, 18]) to be suitable for capturing the formal semantics of vagueness. Typically, a vague property P is represented by a fuzzy set, i.e., a function $\tilde{P}\colon X \to \mathbf{L}$ from a fixed universe of discourse X to a structure $\mathbf{L}$ of truth degrees, usually the real unit interval $[0, 1]$. Similarly, n-ary vague relations are represented by n-ary fuzzy relations, i.e., functions $\tilde{R}\colon X^n \to \mathbf{L}$. This representation is extensively used for dealing with vague predicates in engineering fuzzy methods such as fuzzy control, decision making, and knowledge representation under vagueness (see, e.g., [14, Part 2] for an overview of applied fuzzy methods). In such applications, particular membership degrees from $\mathbf{L}$ are calculated, and an output is based on the resulting membership degree (see, e.g., [14, Ch. 12–17]).

Fuzzy methods based on this approach are undeniably successful in a broad range of real-life applications. The gradual change in the membership degrees

* This paper is an elaboration of a part of the short comment [1].

from **L** accounts for the graduality of vague notions and thereby solves the sorites paradox for gradable predicates in formal semantics. From the practical point of view, the graduality of **L** provides a feedback mechanism in fuzzy control, which is responsible for its successful applicability.

Nevertheless, fuzzy set theory and fuzzy logic have been criticized by philosophers and linguists as an inadequate model of vagueness (e.g., [11, 20, 12]). Perhaps the most convincing argument against the fuzzy semantics of vagueness is the problem of artificial precision. As discussed below in Section 2, due to this problem, naïve fuzzy models are ruled out as an adequate semantics of vague predicates. Is then the claim of fuzzy logic's suitability for dealing with vagueness to be revoked, or can the claim be salvaged through some refinement? In this paper I argue for the latter, and sketch the way in which fuzzy logic can be viewed as a viable apparatus for dealing with vagueness, albeit in a different manner than employed in the prevailing engineering applications.

In particular, as has been demonstrated in detail by Smith [19], *fuzzy plurivaluationism,* which takes a *set* of fuzzy models (rather than a single fuzzy model) for the fuzzy semantics of a vague predicate, provides a satisfactory solution to the problem of artificial precision; and since most philosophical objections against the fuzzy semantics of vagueness turn out to be reducible to some aspect of this central problem, it actually answers a large portion of the criticisms of the degree-theoretical approach to vagueness. Here I will show that fuzzy plurivaluations can be regarded as the classes of models of theories over fuzzy logic, and consequently that the consequence relation of fuzzy logic captures the notion of truth for vague concepts. An adequate knowledge representation of vagueness should therefore be based on deductions in formal systems of mathematical fuzzy logic, rather than calculations of particular membership degrees. On that account, even if traditional fuzzy methods work well in fuzzy control, strictly speaking they only work with *technical precisifications* of vague notions rather than vague notions themselves—accommodating the graduality, but neglecting the indeterminacy aspect of vagueness.

These theses will be explained and illustrated in the following sections on the example of the vague predicate *tall.* Due to space restrictions, only a sketch of the full argument is given here.

2 The Problem of Artificial Precision

In the traditional fuzzy approach to vagueness, the semantics of a given set $\mathcal{L} = \{P_i^{(k_i)}\}_{i\in I}$ of (possibly vague) predicates P_i of arities $k_i > 0$, for $i \in I$, is given by a *fuzzy model* $\mathbf{M} = \langle M, \{\|P_i\|_\mathbf{M}\}_{i\in I}\rangle$, where $\|P_i\|_\mathbf{M} \colon M^{k_i} \to \mathbf{L}$ is a fuzzy set (if $k_i = 1$) or a k_i-ary fuzzy relation (if $k_i > 1$) valued in a suitable fixed structure **L** of degrees (usually on the real unit interval $[0, 1]$). The function $\|P_i\|_\mathbf{M}$ is called the *membership function* of the predicate P_i, and the value $\|P_i\|_\mathbf{M}(x)$ for $x \in M$ is called the *membership degree* of the individual x in the fuzzy set $\|P_i\|_\mathbf{M}$ (and similarly for fuzzy relations).

In a fuzzy model, vague predicates are represented by particular membership functions; particular individuals are thus assigned particular degrees of vague properties. For instance, if we choose the membership function $\tilde{T}\colon h \mapsto \min(1, \max(0, (h-150)/50))$, assigning the degree $\tilde{T}(h) \in [0,1]$ to each height $h \in \mathbb{R}^+$ measured in centimeters, to represent the semantics of the predicate *tall,* then a man of height 186 cm is alleged to be tall to degree 0.72.

However, it can be observed that the choice of the membership function $\tilde{T}$ has been completely arbitrary: there is no convincing reason why this particular linear function (rather than, e.g., some S-shaped function) should have been chosen, or why the man of height 186 cm should be assigned the degree 0.72 rather than, e.g., 0.74. Instead of capturing a *vague* property, for which it is *indeterminate* whether it applies to borderline cases or not, a fuzzy model yields something much more precise: a particular real number, such as 0.3428. In view of the arbitrariness of the choice of membership function, this precision, not present in the original vague predicate, is artificial. Consequently, the fuzzy model does not capture the vagueness of the predicate in a satisfactory manner. This is the *problem of artificial precision* of fuzzy semantics of vagueness, and a serious objection to its adequacy.

In engineering applications, the choice of a particular membership function is guided by pragmatic considerations: e.g., a linear function may be preferred for the reason of its efficient computability. The choice is legitimate iff the intended real-world application works well. From the point of view of formal semantics, however, the fuzzy model is not an adequate semantic representation of the original vague notion, but is rather its *technical precisification,* suitable for the particular engineering application in question. If we want to know which propositions are *true* about the original vague notion (e.g., whether all *tall* people from a given sample are *skinny*), the use of the technical precisification may well yield an answer which is an *artifact* of the choice of the membership function. Thus, while fuzzy models are admittedly useful for applied methods in fuzzy control, they are hardly an accurate representation of vague knowledge. Eventually, the meaning of a vague predicate is determined by its usage in natural language; however, one can hardly find any linguistic fact that would determine whether a man of height 186 cm should be considered 0.72- or rather 0.73-tall.

3 Fuzzy Plurivaluationism

A remedy to the problem of artificial precision is offered by *fuzzy plurivaluationism* [19]. Fuzzy plurivaluationism acknowledges that the *meaning-determining facts* of natural language do not uniquely determine the values of membership functions, and so do not narrow the set of admissible fuzzy models for a given predicate down to a singleton set. Therefore, instead of a single fuzzy model, fuzzy plurivaluationism assigns a *set* $\mathcal{P}$ of fuzzy models (as defined in Section 2) as the semantics for a given list $\mathcal{L}$ of (possibly vague) predicates; let us call the set $\mathcal{P}$ of fuzzy models a *fuzzy plurivaluation* for the language $\mathcal{L}$.

The set of admissible fuzzy models for a given language $\mathcal{L}$ does not consist of all fuzzy models: rather, it has to reflect the meanings of the predicates in $\mathcal{L}$. Even though the meaning-determining facts do not pinpoint the exact *values* of membership degrees, they do determine certain *properties* of membership functions and their relationships. For instance, the membership function of the predicate *tall* must in each admissible model be non-decreasing: it is part of the meaning of the word *tall* that people of larger heights are not less *tall* than those of lesser heights; those who do not acknowledge this property do not understand the meaning of the word *tall.* These constraints on admissible fuzzy models can be regarded as the *meaning postulates* of the vague predicates involved, i.e., their semantic properties and relationships that would be acknowledged by competent speakers, and which therefore have to be satisfied in each fuzzy model. In other words, fuzzy plurivaluatistic semantics of a given vague language $\mathcal{L}$ is the set $\mathcal{P}$ of those fuzzy models for the language $\mathcal{L}$ that satisfy the meaning postulates of the predicates in $\mathcal{L}$.[1]

In a *formal* semantics of natural language, these meaning postulates are expected to be expressible in the rigorous language of mathematics and logic, thereby comprising a formal theory $\mathcal{T}$ that constrains the set $\mathcal{P}$ of admissible models. In typical cases, this is indeed so; let us illustrate it on the example of the predicate *tall.*

4 Example: the Predicate *Tall*

Let us present a (simplified) analysis of the meaning of the vague predicate *tall.* One condition that the intended usage of the term *tall* (in any given context) has to satisfy is the one mentioned earlier:

If a person of height x is tall and $y \geq x$,
then a person of height y is tall as well. (1)

Further conditions on the predicate *tall* that are part of its meaning are, for instance, those related to prototypical cases, e.g.:

A person of Michael J. Fox's height is not tall, while (2)
a person of Christopher Lee's height is tall. (3)

In the degree-theoretical framework of fuzzy plurivaluationism, these meaning postulates for the predicate *tall* can be reformulated as conditions on its membership function. Let the membership function of the vague unary predicate *tall,* assigning degrees from $[0, 1]$ to heights in centimeters, be denoted by

[1] Since typical meaning postulates of vague concepts do not refer to truth degrees (which are just part of our semantic modeling rather than linguistic usage), it is no wonder that the membership functions are underdetermined by the meaning postulates and that $\mathcal{P}$ is usually a non-singleton class of models.

$\tilde{T}\colon \mathbb{R}^+ \to [0,1]$. The meaning postulates (1)–(3) then correspond to the following conditions:

$$(y \geq x) \to (\tilde{T}(y) \geq \tilde{T}(x)) \tag{4}$$

$$\tilde{T}(a) = 0 \tag{5}$$

$$\tilde{T}(b) = 1, \tag{6}$$

where a and b are constants denoting the respective heights of Michael J. Fox and Christopher Lee in cm (163 and 196, according to Google Search).

The fuzzy plurivaluation representing the meaning of the predicate *tall* according to our analysis would therefore be the set $\mathcal{P}$ of all fuzzy models in which the membership function $\tilde{T}$ of the predicate *tall* satisfies the constraints (4)–(6), i.e., is non-decreasing in heights and assigning the full and zero degrees to prototypical cases.[2]

The conditions (4)–(6), together with appropriate axioms for the ordering of real numbers,[3] can be regarded as a mathematical theory $\mathcal{T}$, formalizable in classical first-order logic. The set $\mathcal{P}$ of admissible fuzzy models for *tall* is then the set of the models of the theory $\mathcal{T}$; i.e., $\mathcal{P} = \{\mathbf{M} \mid \mathbf{M} \models \mathcal{T}\}$.

Membership functions of predicates from $\mathcal{L}$ in a particular model $\mathbf{M} \in \mathcal{P}$ are (fuzzy) *precisifications* of the vague concepts from $\mathcal{L}$. The condition that $\mathbf{M} \models \mathcal{T}$ for all $\mathbf{M} \in \mathcal{P}$ ensures that the meaning postulates of all vague concepts $P_i \in \mathcal{L}$ are jointly satisfied by their fuzzy precisifications in each model $\mathbf{M} \in \mathcal{P}$.

5 Supertruth

As mentioned in Section 2, a proposition φ about the predicates in the language $\mathcal{L}$ that is true in only some, but not all, of the models $\mathbf{M} \in \mathcal{P}$, can be an artifact of the choice of membership functions in these particular models. Only such propositions that are true in *all* models $\mathbf{M} \in \mathcal{P}$, i.e., in all models satisfying the meaning postulates of the predicates in $\mathcal{L}$, are true independently of particular choices of membership functions in these models, and so can be regarded as truths about the vague concepts from $\mathcal{L}$. Since fuzzy plurivaluationism is a

[2] In reality, the meaning postulates for the predicate *tall* are more complex than the simplified version (1)–(3) discussed here. A more detailed analysis of the predicate would have to include, inter alia, the meaning postulate that "imperceptible changes in height correspond to negligible changes in the degree of tallness", formalized as the congruence of admissible membership functions of *tall* w.r.t. fuzzy indistinguishability relations on heights and degrees of truth (i.e., a certain generalization of Lipschitz continuity). The avoidance of the sorites paradox for prototypical cases might require that rather than postulating the degrees 0 and 1 for prototypical cases, only a fuzzy indistinguishability from the full truth and full falsity be required. For the sake of simplicity, we leave these refinements aside, as they are not central to the thesis presented here.

[3] Notice that the crisp predicate $\leq$ on heights is part of our language $\mathcal{L}$, as it appears in the meaning postulates of *tall*.

(fuzzified) variant of supervaluationism (see, e.g., [4, 20, 13, 19]), let us call such propositions φ that are true in all admissible fuzzy models $\mathbf{M} \in \mathcal{P}$ (i.e., true in all admissible fuzzy precisifications) *supertrue.*

Thus, those and only those propositions φ which are supertrue represent truths about the vague concepts from $\mathcal{L}$ regardless of a chosen precisification. Like in supervaluationism, it is only supertruth that matters, rather than truth in a particular model.

Mathematically, φ is supertrue iff $\mathbf{M} \models \varphi$ for all models $\mathbf{M} \in \mathcal{P}$, i.e., for all models $\mathbf{M}$ such that $\mathbf{M} \models \mathcal{T}$; that is, iff $\mathcal{T} \models \varphi$. Thus, supertruths about vague predicates are exactly the *logical consequences* of their meaning postulates.[4] If $\mathcal{T}$ is formulated in classical first-order logic, then due to the strong completeness theorem, supertruths are also exactly those φ that can be formally derived from $\mathcal{T}$, i.e., φ is supertrue iff $\mathcal{T} \vdash \varphi$.

It can be observed, however, that the formalization of the meaning postulates in first-order classical logic is not completely straightforward. In particular, vague *predicates* from $\mathcal{L}$ are represented by *functions,* which is a type mismatch that has to be handled by the formalization: consequently, logical *connectives* occurring in the meaning postulates have to be represented by certain *operations* on truth degrees. For instance, in our toy example of Section 4, compare the meaning postulate (1), whose logical form is $\varphi \,\&\, \psi \to \xi$, with its formalization (4) of a rather different logical form; or the postulate (2) with (5), where the negation ("is not tall") changed into equality with 0. It would certainly be preferable if the theory $\mathcal{T}$ could be formalized in a more straightforward way which preserves the logical form of the meaning postulates, rendering predicates as predicates (rather than functions) and connectives as connectives (rather than operations, inequalities, or equalities). This is exactly what mathematical fuzzy logic aims for, and where it enters the picture.

6 Mathematical Fuzzy Logic

Fuzzy logics form a well-developed family of many-valued and substructural logics [3]. For a detailed information on mathematical fuzzy logic we refer the reader to [2]. Here we shall just briefly introduce the perhaps best known system of formal fuzzy logic, namely (infinite-valued) Łukasiewicz logic.

The syntax of Łukasiewicz first-order logic is the same as that of classical first-order logic. Let us take $\neg$ and $\to$ for the primitive connectives. The *standard* semantics of Łukasiewicz logic employs the real unit interval $[0, 1]$ as the set of truth degrees. Let $\mathcal{L}$ be a non-empty set of predicate symbols and individual constants.[5] The value $\|t\|_{\mathbf{M},v}$ of a term (i.e., a variable or constant) t under

[4] Let us note that the meaning postulates correspond to what in supervaluationism is called the *penumbral connections;* see, e.g., [4, 20, 13, 19].

[5] Further on we expand the definition of fuzzy models (see Section 2) by interpretations $\|c\|_{\mathbf{M}} \in M$ of individual constants $c \in \mathcal{L}$. Function symbols can be added to Łukasiewicz logic as well (see [2]); we leave them aside here, as they are not needed in the present paper.

a valuation v of individual variables in a fuzzy model $\mathbf{M}$ for $\mathcal{L}$ is the element $\|c\|_{\mathbf{M}} \in M$ if t is a constant $c \in \mathcal{L}$ or the element $v(x) \in M$ if t is a variable x. The truth degree $\|\varphi\|_{\mathbf{M},v}$ of a formula φ in the language $\mathcal{L}$ under a valuation v in a fuzzy model $\mathbf{M}$ for $\mathcal{L}$ is defined by the following recursive Tarski conditions:

$$\|P(t_1,\dots,t_n)\|_{\mathbf{M},v} = \|P\|_{\mathbf{M}}(\|t_1\|_{\mathbf{M},v},\dots,\|t_n\|_{\mathbf{M},v}) \tag{7}$$
$$\|\neg\varphi\|_{\mathbf{M},v} = 1 - \|\varphi\|_{\mathbf{M},v} \tag{8}$$
$$\|\varphi \to \psi\|_{\mathbf{M},v} = \min(1 - \|\varphi\|_{\mathbf{M},v} + \|\psi\|_{\mathbf{M},v}, 1) \tag{9}$$
$$\|(\forall x)\varphi\|_{\mathbf{M},v} = \inf_{a\in M} \|\varphi\|_{\mathbf{M},v_{x:a}} \tag{10}$$
$$\|(\exists x)\varphi\|_{\mathbf{M},v} = \sup_{a\in M} \|\varphi\|_{\mathbf{M},v_{x:a}}, \tag{11}$$

where $v_{x:a}(x) = a$ and $v_{x:a}(y) = v(y)$ for each variable y different from x. Further propositional connectives of Łukasiewicz logic are defined as follows:[6]

$$\varphi \mathbin{\&} \psi \ \equiv_{\mathrm{df}}\ \neg(\varphi \to \neg\psi) \tag{12}$$
$$\varphi \vee \psi \ \equiv_{\mathrm{df}}\ (\varphi \to \psi) \to \psi \tag{13}$$
$$\varphi \leftrightarrow \psi \ \equiv_{\mathrm{df}}\ (\varphi \to \psi) \mathbin{\&} (\psi \to \varphi) \tag{14}$$

Consequently we obtain $\|\varphi\&\psi\|_{\mathbf{M},v} = \max(\|\varphi\|_{\mathbf{M},v}+\|\psi\|_{\mathbf{M},v}-1, 0)$, and similarly for other defined connectives.

We say that a formula φ is true in $\mathbf{M}$, written $\mathbf{M} \models \varphi$, if $\|\varphi\|_{\mathbf{M},v} = 1$ for all valuations v in $\mathbf{M}$. A fuzzy model $\mathbf{M}$ is a model of a theory (i.e., a set of formulae) $\mathcal{T}$, written $\mathbf{M} \models \mathcal{T}$, if $\mathbf{M} \models \varphi$ for each $\varphi \in \mathcal{T}$. Let the class $\{\mathbf{M} \mid \mathbf{M} \models \mathcal{T}\}$ of models of $\mathcal{T}$ be denoted by $\mathrm{Mod}(\mathcal{T})$. A formula φ is a logical consequence of $\mathcal{T}$, written $\mathcal{T} \models \varphi$, if $\mathbf{M} \models \varphi$ for all $\mathbf{M} \in \mathrm{Mod}(\mathcal{T})$.

The axiomatic system for first-order Łukasiewicz logic consists of the following axiom schemata, where the term t is substitutable for x in φ and ν does not contain free x:

$$\begin{aligned}
&(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))\\
&\varphi \to (\psi \to \varphi)\\
&(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)\\
&((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)\\
&(\forall x)\varphi(x) \to \varphi(t)\\
&(\forall x)(\nu \to \varphi) \to (\nu \to (\forall x)\varphi),
\end{aligned}$$

plus the derivation rules of modus ponens (from φ and $\varphi \to \psi$ derive ψ) and generalization (from φ derive $(\forall x)\varphi$). The notions of proof and provability in a theory (written $\mathcal{T} \vdash \varphi$) are defined as in classical logic.

[6] Usually (see [2]), an additional conjunctive connective $\varphi \wedge \psi \equiv_{\mathrm{df}} \varphi \mathbin{\&} (\varphi \to \psi)$, an additional disjunctive connective $\varphi \oplus \psi \equiv_{\mathrm{df}} \neg(\varphi \to \neg\psi)$, and optionally, the unary connective $\triangle\varphi$ ("determinately φ", with the standard semantics $\|\triangle\varphi\|_{\mathbf{M},v} = 1$ if $\|\varphi\|_{\mathbf{M},v} = 1$, otherwise $\|\triangle\varphi\|_{\mathbf{M},v} = 0$) are introduced in Łukasiewicz logic. However, we will not need these connectives in the present exposition.

The completeness of the above axiomatic system with respect to Łukasiewicz first-order logic is somewhat tricky. To obtain the completeness theorem ($\mathcal{T} \models \varphi$ iff $\mathcal{T} \vdash \varphi$) for the standard $[0,1]$-valued semantics described above, an additional infinitary derivation rule has to be added [10]. Nevertheless, the axiomatic system is sound and complete if more general algebras of truth degrees (besides the standard algebra on $[0,1]$) are admitted in fuzzy models.[7]

A similar situation is found in other systems of formal fuzzy logic, which differ from Łukasiewicz logic mainly in the choice of algebraic operations in the Tarski conditions (8)–(9) for propositional connectives and the ensuing modification of the axiomatic system.[8]

7 Formalization of Vague Predicates in Fuzzy Logic

Let us return to our toy example of Section 4. The straightforward formalization of the meaning postulates (1)–(3) of *tall* in Łukasiewicz fuzzy logic is given by the following axioms:[9]

$$Tx \mathbin{\&} (y \geq x) \rightarrow Ty \tag{15}$$

$$\neg Ta \tag{16}$$

$$Tb. \tag{17}$$

Using a few simple observations on Tarski conditions (8)–(9) for Łukasiewicz logic, such as that $\|\varphi \rightarrow \psi\|_{\mathbf{M},v} = 1$ iff $\|\varphi\|_{\mathbf{M},v} \leq \|\psi\|_{\mathbf{M},v}$, and that $\|\neg\varphi\|_{\mathbf{M},v} = 0$ iff $\|\varphi\|_{\mathbf{M},v} = 1$, which are actually true in all t-norm-based fuzzy logics, one can easily verify that the formulae (15)–(17) are true in a fuzzy model $\mathbf{M}$ iff the membership function $\tilde{T} = \|T\|_{\mathbf{M}}$ satisfies the conditions (4)–(6). In other words, the class $\mathrm{Mod}(\mathcal{T})$ of models of the theory $\mathcal{T} = \{(15), (16), (17)\}$ in Łukasiewicz logic is exactly the class $\mathcal{P}$ of fuzzy models identified in Section 4 as the plurivaluationistic meaning of the vague predicate *tall.*

The semantics of Łukasiewicz fuzzy logic thus captures the plurivaluationistic meaning of *tall* if its meaning postulates (1)–(3) are straightforwardly formalized as (15)–(17) in Łukasiewicz logic. Moreover, it can be observed that the formalization in Łukasiewicz logic does not suffer from the type mismatch pointed out

[7] In particular, the axiomatic system is sound and complete with respect to *safe* (i.e., such that all suprema and infima required by the Tarski conditions exist) fuzzy models over the algebras of truth degrees for which the propositional fragment of Łukasiewicz logic is sound (these are called *MV-algebras;* it is sufficient to consider linearly ordered ones). See [2] for more details.

[8] Incidentally, the plurivaluationistically perhaps best justified fuzzy logic MTL (see Section 9) does enjoy the completeness with respect to $[0,1]$-valued fuzzy models; thus in MTL it is sufficient to consider just $[0,1]$ as the system of truth degrees.

[9] Recall that our predicate *tall* takes heights in cm for arguments, so Tx is interpreted as "a person of height x cm is tall". Again, a and b are the constants denoting the heights of the two actors appearing in (2)–(3), i.e., the numbers 163 and 196. Note that the axioms (15)–(17) are required to be *true* in admissible fuzzy models: i.e., we only claim the equivalence of (4)–(6) to the *full* truth of (15)–(17).

at the end of Section 5: in (15)–(17), the vague predicate *tall* is indeed formalized as a *predicate* symbol T of Łukasiewicz logic, while its interpretation by the membership *function* $\tilde{T}$ in admissible fuzzy models $\mathbf{M} \in \mathcal{P}$ is hidden in the semantics of Łukasiewicz first-order logic.

A similar situation is found in the plurivaluationistic semantics for other vague predicates besides our example of *tall:* typically, the class of admissible fuzzy models $\mathcal{P}$ can be defined as the class $\mathrm{Mod}(\mathcal{T})$ of models of a theory $\mathcal{T}$ in Łukasiewicz (or another suitable) first-order fuzzy logic, where $\mathcal{T}$ is a straightforward formalization of the meaning postulates in this fuzzy logic. This is, of course, no coincidence: Łukasiewicz and other fuzzy logics have been *designed* to straightforwardly formalize typical (i.e., relational and quantificational) conditions between truth degrees of gradual notions.

8 Supertruth as Entailment in Fuzzy Logic

The utility of formal fuzzy logic in fuzzy plurivaluationistic semantic of vagueness goes beyond the delimitation of fuzzy plurivaluations by means of theories over fuzzy logic. Recall from Section 5 that genuine truths about vague predicates (i.e., truths which are not artifacts of particular choices of membership functions) are those propositions which are true in all admissible fuzzy models for the vague language $\mathcal{L}$; i.e., those propositions which are supertrue in the fuzzy plurivaluation $\mathcal{P} = \mathrm{Mod}(\mathcal{T})$, where $\mathcal{T}$ is the theory over fuzzy logic formalizing the meaning postulates for $\mathcal{L}$. Now observe that by definition, a formula φ of Łukasiewicz fuzzy logic in the language $\mathcal{L}$ is true in all models $\mathbf{M} \in \mathcal{P} = \mathrm{Mod}(\mathcal{T})$ iff $\mathcal{T} \models \varphi$ in Łukasiewicz fuzzy logic; in other words, if φ is entailed by the meaning postulates of $\mathcal{L}$ in Łukasiewicz logic (or another fuzzy logic chosen for formalization of the meaning postulates for $\mathcal{L}$).

Supertruths about vague predicates thus coincide with fuzzy-logical consequences of their meaning postulates. The completeness theorems for fuzzy logics further translate[10] the latter into the *provability* from $\mathcal{T}$ in fuzzy logic. Supertruths thus can be found by axiomatic derivation in fuzzy logic.

This is in fact the way which formal fuzzy logicians (unlike most designers of applied fuzzy methods, who tend to employ single fuzzy models) have implicitly used for more than a decade in modeling vagueness: cf., e.g., the way in which the vague predicate *small* is treated in [9]—namely by first axiomatizing its basic properties and, subsequently, deriving theorems from these axioms in formal fuzzy logic. The same approach has been employed in the formal modeling of vague linguistic hedges [7, 16], vague evaluative linguistic expressions [16], and vague quantifiers [17].

[10] Modulo the provisos of Section 6 on the preconditions of the completeness theorems—namely, adding infinitary derivation rules or admitting more general systems of truth degrees than $[0, 1]$—for some fuzzy logics (including Łukasiewicz logic; for the fundamental fuzzy logic MTL mentioned below, however, these preconditions are not needed).

9 The Plurality of Fuzzy Logics

On top of that, mathematical fuzzy logic applies the same fuzzy plurivaluationistic approach to propositional connectives, too. In Section 6 the reader might rightly have wondered what justified the particular truth functions in the Tarski conditions (8)–(9) for the connectives of Łukasiewicz logic (namely, $1 - x$ for negation and $\min(1-x+y, 1)$ for implication): why not, for instance, $1-x^2$ and $\max(1-x, y)$, or any other choice? Which linguistic facts justify our particular choice of truth functions for these connectives? Is it not an arbitrary choice, too, similar to the choice of particular membership functions for vague predicates?

Indeed it is. The standard account of Łukasiewicz logic is actually somewhat simplified: mathematical fuzzy logic in fact takes an axiomatic approach even to propositional connectives, constraining them only by certain properties which can be viewed as embodying their meaning postulates. The most common approach (originating in [6]) posits that conjunction is commutative, associative, monotone, neutral w.r.t. determinate truth, and continuous in degrees. These constraints represent the meaning postulates on *and,* namely that φ *and* ψ is as true as ψ *and* φ, etc. Similar constraints are imposed on other connectives and their mutual interplay. Within these constraints, the truth functions of connectives are allowed to vary across admissible models, analogously to the indeterminacy of membership functions in fuzzy plurivaluations—precisely because their (accepted, reasonable) meaning postulates fail to narrow them down to single truth functions.

Particular systems of fuzzy logic (of which there are many—see [2]) differ from each other by making different choices regarding the requisite properties of propositional connectives, i.e., their stipulated meaning postulates. E.g., the above constraints on conjunction lead to the fuzzy logic BL, while slightly weaker requirements produce the (arguably more fundamental) fuzzy logic MTL. In these logics, the set of admissible truth functions for conjunction is only narrowed down to the (still rather broad) class of so-called (left-)continuous t-norms.

In some contexts, additional constraints on connectives are appropriate, leading to stronger fuzzy logics: for instance, adding the law of double negation (which formalizes a meaning postulate for *not,* namely: "*not not* φ is as true as φ") to BL yields Łukasiewicz logic. *Some* of such more specialized fuzzy logic (including Łukasiewicz) can be shown to be sound and complete with respect to fuzzy models with a fixed standard choice of truth functions for connectives; e.g., in Łukasiewicz logic, the functions $1 - x$ for negation and $\min(1 - x + y, 1)$ for implication. In such logics, even if the constraints on connectives admit truth functions other than the single standard ones, fuzzy models with non-standard truth functions happen to be irrelevant for the consequence relation. In such logics, then, logical consequence is fully determined by evaluating by means of the single standard truth functions, and so we can pretend that the standard truth functions are the only admissible ones (as we did in (8)–(9)). In general, though, all fuzzy logics admit variable truth functions of propositional connectives across their general models (within the constraints based on their meaning postulates chosen for the given logic).[11] Thus, in effect, mathematical fuzzy logic treats both vague predicates and fuzzy propositional connectives plurivaluationistically, regarding them as underdetermined by their defining meaning postulates.

10 Fuzzy Logic as a Model of Vagueness

Fuzzy plurivaluationism de facto characterizes vagueness as a combination of two phenomena: the *graduality* of vague concepts, by which we are forced to use fuzzy, rather than classical, models (to avoid the sorites paradox), and the *indeterminacy* of meaning, by which we are forced to use sets of fuzzy models rather than single fuzzy models (to avoid the problem of artificial precision).

The graduality component of fuzzy plurivaluationism acknowledges that most vague predicates apply to some individuals more than to others; this is modeled by means of truth degrees (the "fuzzy" aspect). The indeterminacy component acknowledges that the meaning-determining facts do not narrow down the meanings of vague concepts to single fuzzy models, and leave the membership functions of vague predicates (as well as the truth functions of fuzzy connectives) underdetermined; this is modeled by letting the membership and truth functions vary across admissible fuzzy models, constrained only by the meaning postulates of the vague concepts involved (the "plurivaluationistic" aspect).

Application-oriented fuzzy methods do reflect the graduality component of vagueness, by using fuzzy models; however, they neglect the semantic indeterminacy of vague predicates and rather work with their fuzzy precisifications in a single fuzzy model. As a result, they are able to compute particular membership degrees of these gradual precisifications (and act accordingly to the degree, e.g., in fuzzy control). However, the results can be artifacts of the choice of the particular fuzzy model; their correctness is assessed on pragmatic grounds (namely, whether the application works well or not).

On the other hand, supervaluationism—a rival theory of vagueness to degree-theoretical ones—can be cast as a bivalent version of fuzzy plurivaluationism, representing vague predicates by sets of classical, bivalent models.[12] This reflects the indeterminacy aspect of vagueness, but neglects the graduality of most vague notions. The supervaluationistic representation of vagueness is therefore subject

[11] The admittance of different sets of truth degrees across fuzzy models (cf. footnote 7) is an effect of the same principle: the system of truth degrees is only determined by the postulates on connectives, rather than given in advance, and the postulates do not narrow it down to algebras with isomorphic universes. In only a few fuzzy logics (such as MTL or Gödel logic, see [2]), the real-valued completeness theorem holds, entailing that we can ignore all the non-$[0,1]$-valued algebras.

[12] Smith [19] makes a subtle distinction between two variants of supervaluationism that occur in the literature, using the terms 'supervaluationism' and 'plurivaluationism'. The distinction is not important in our context, therefore we use the more established term 'supervaluationism' for both variants.

to several problems exposing its insufficiency (most prominently, the 'jolt problem', instantiated, e.g., by the fact that it is supertrue that there is the least *large* natural number; see [19]).

Fuzzy plurivaluationism combines both detected aspects of vagueness; and as we have seen in Section 8, the consequence relation of fuzzy logic describes its notion of supertruth, identifying it with fuzzy-logical consequences of the meaning postulates for the vague notions. In this sense, fuzzy logic is the logic underlying the fuzzy plurivaluationistic semantics of vagueness. Consequently, fuzzy logic *as the consequence relation* can be regarded as the logic of vagueness (fuzzy-plurivaluationistically modeled).

On the other hand, *particular semantic models* of fuzzy logic (with fixed membership functions) are insufficient for capturing vagueness, as exposed by the problem of artificial precision described in Section 2, as well as various further objections raised by philosophers of vagueness (summarized, e.g., in [20, 12, 19]).[13] This answers the question of whether or not, or in which sense, fuzzy logic can be regarded as a logic suitable for the formal semantics of vagueness: the answer is *yes* for fuzzy logic in the sense of the consequence relation (provided we accept the fuzzy plurivaluationistic model of vagueness), and *no* for fuzzy logic in the sense of particular fuzzy models.

The practical problem with this answer is that fuzzy plurivaluationistic semantics, or fuzzy logic qua consequence relation, does not make it possible to calculate any truth degrees or membership functions, as they vary across the admissible fuzzy models. Thus it is hardly imaginable that we could construct, for instance, a fuzzy controller based on the fuzzy-logical consequence: for such applications, the method of artificial precisification (retaining graduality to ensure a feedback mechanism for the controlled process, while eliminating indeterminacy in order to be able to calculate values for controlling) is obviously superior. Nevertheless, the more adequate representation of vagueness by fuzzy plurivaluations and consequence-based fuzzy logic may open the way to another kind of applications in which the indeterminacy aspect of vagueness matters, such as in linguistic modeling, knowledge representation, or logical inference (e.g., in answering database queries) under vagueness. Imaginably, for instance, logical programming aimed at accommodating genuinely vague notions (as opposed to the current practice of fuzzy logical programming employing particular truth degrees) might be achieved by replacing the inference rules of classical logic by suitable derivation rules of formal fuzzy logic. An area with practical applications where steps towards employing consequence-based fuzzy logic has already been made (starting with [8]) is fuzzy description logic. However, the applicability of consequence-based formal fuzzy logic to practical problems (rather than just as

[13] It is worth noting that most philosophical criticisms of fuzzy logic as the logic of vagueness assume fixed membership functions, and thus do not apply to fuzzy plurivaluationism nor fuzzy logic as consequence relation. The fuzzy-plurivaluationistic analysis agrees with these objections against the naïve fuzzy semantics of fixed membership functions, and explains the problems as stemming from its failure to address the indeterminacy aspect of vagueness.

a theoretical background of fuzzy methods with fixed membership functions) is still a matter of future research.

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