Graph drawing beyond planarity: some results and open problems

Giuseppe Liotta

Dipartimento di Ingegneria Università degli Studi di Perugia, Italy giuseppe.liotta@unipg.it

Abstract. We briefly review some recent findings and outline some emerging research directions about the theory of "nearly planar" graphs, i.e. graphs that have drawings where some crossing configurations are forbidden.

1 Graph drawing beyond planarity

Recent technological advances have generated torrents of relational data that are hard to display and visually analyze due, mainly, to their large size. Application domains where this need is particularly pressing include Systems Biology, Social Network Analysis, Software Engineering, and Networking. What is required is not simply an incremental improvement to scale up known solutions but, rather, a quantum jump in the sophistication of the visualization systems and techniques. New research scenarios for visual analytics, network visualization, and human-computer interaction paradigms must be identified; new combinatorial models must be defined and their corresponding theoretical problems must be computationally investigated; finally, the theoretical solutions must be experimentally evaluated and put into practice. Therefore, a substantial research effort in the graph drawing and network visualization communities started from the following considerations.

- **The Planarity Handicap.** The classical literature on graph drawing and network visualization showcases elegant algorithms and sophisticated data structures under the assumption that the input relational data set can be displayed as a network where no two edges cross (see, e.g., [14,35,36,40]), i.e. as a planar graph. Unfortunately, almost every graph is non-planar in practice and various experimental studies have established that the human ability of understanding a diagram is dramatically affected by the type and number of edge crossings (see, e.g., [42,43,48]).
- **Combinatorial Topology vs. Algorithmics.** A topological graph is a drawing of a graph in the plane such that vertices are drawn as points and edges are drawn as simple arcs between the points. Extremal theory questions such as "how many edges can a certain type of non-planar topological graph have?" have been investigated by mathematicians for decades, typically under the name of Turán-type problems. However, the corresponding computational question: "How efficiently can one compute a drawing Γ of a non-planar graph such that Γ is a topological graph of a certain type?" has been surprisingly disregarded by the algorithmic community until very recent years.

We recall that planar graphs can be expressed in terms of forbidden subgraphs: A graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. Then, a fundamental natural step towards understanding non-planar graphs is to consider network visualizations where some types of crossings are forbidden while some other types are allowed. For example, we recall a sequence of HCI experiments by Huang et al. [32,33,34] proving that crossing edges significantly affect human understanding if they form acute angle, while crossing that form angles from about $\frac{\pi}{3}$ to $\frac{\pi}{2}$ guarantee good readability properties. Hence it makes sense to explore complexity issues related to drawings of graphs where such "sharp angle crossings" are forbidden. As another problem, Purchase et al. [42,43,48]) prove that an edge is difficult to read if it is crossed by many other edges; hence, the current research agenda considers computational issues with graph drawings where every edge is crossed by at most k other edges, for a given constant k.

In addition to requiring that some types of edge crossings must be forbidden, nonplanar drawings must also satisfy a set of geometric optimization goals (often called *aesthetic requirements*) such as, for example, minimizing the area of the drawing for a given resolution rule, maximizing the aspect ratio, minimizing the number of different slopes used to draw the edges, or the number bends along the edges.

In the next section we briefly recall some of the most recent results in the area and propose a few open problems. More formally, a *drawing* of a graph G: (i) injectively maps each vertex u of G to a point p_u in the plane; (ii) maps each edge (u, v) of G to a Jordan arc connecting p_u and p_v that does not pass through any other vertex; (iii) is such that any two edges have at most one point in common. A drawing of a graph is a *straight-line drawing* if every edge is a straight-line segment, it is a *poly-line drawing* if the edges are polygonal chains and may contain bends.

2 Some results and open problems

The "beyond planarity" research area could be briefly described as the (potentially uncountable) collection of problems of the type depicted in Figure 1, where the column "Forbidden" describes a forbidden crossing configuration and the column "Question" describes a corresponding computational question of interest in graph drawing. We remark that both the forbidden configurations and the computational questions of Figure 1 are mere examples within a much larger research framework. In the remainder, we only give some references about the second and the fourth entry of the table. The interested reader is referred, for example, to recent proceedings of the International Symposium on Graph Drawing [49] for more results on the "beyond planarity" topic. (See also http://www.graphdrawing.org/symposia.html.)

2.1 Drawings with large crossing angles

The *crossing angle resolution* of a drawing of a graph measures the smallest angle formed by any pair of crossing edges.

A *RAC drawing* is a drawing of a graph whose edges can cross only orthogonally to one another, i.e. a RAC drawing maximizes the crossing angle resolution. The notion of RAC drawings was first introduced by Didimo et al. in [23], who studied both



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Fig. 1. A table with some forbidden crossing configurations and related computational questions.

straight-line and poly-line drawings. Variants of RAC drawings are drawings in which the minimum crossing angle must be at least a given constat or the drawings where the minimum crossing angle is exactly a given constant. A limited list of recent papers about RAC drawings and their variants includes [4,5,6,7,15,16,17,18,22,25,47]. See also [24] for more references and open problems about drawing graphs with large crossing angles. A sample open problem follows.

Open Problem: Argyriou *et al.* [6] prove that deciding whether a graph has a straightline RAC drawing is NP-hard. Hence, maximizing the crossing angle resolution in a straight-line drawing of a graph is also NP-hard. Is there an efficient approximation algorithm for this problem? Is there a polynomial time solution for special families of graphs (e.g. those having bounded vertex degree)?

Related to the problem above, we recall that there is a polynomial time algorithm to recognize whether a bipartite graph has a straight-line RAC drawing such that the vertices of a same partition set all lie on one of two parallel lines [16].

2.2 Drawings with few crossings per edge

For a fixed non negative integer k, a k-planar drawing is a drawing of a graph where every edge can be crossed by at most k other edges. A k-planar graph is a graph that has a k-planar drawing. Note that the family of 0-planar graphs coincides with the family of planar graphs. The literature about drawings of graphs where every edge can be crossed at most k times has mostly focused on the case k = 1.

Concerning Turán-type problems, Pach and Tóth prove that 1-planar graphs with n vertices have at most 4n - 8 edges, which is a tight upper bound [41]; in the case of straight-line drawings, Didimo [21] proved that a tight bound is 4n - 9. 1-planarity testing is studied by Korzhik and Mohar who prove that recognizing 1-planar graphs is NP-hard [39]; polynomial-time solutions for the recognition problem are known under some additional assumptions and/or for restricted classes of graphs (see, e.g. [8,27,30]).

Straight-line 1-planar drawings have been studied in [3,31,46]. The relation between 1-planar drawings and RAC drawings is considered in [13,28]. A limited list of additional papers on 1-planar graphs includes [1,2,3,9,10,11,26,29,31,37,38,45].

We conclude with a classical open problem about trade-offs of different aesthetic requirements. Assuming that the vertices are points of an integer grid, the *area of a drawing* of a graph is defined as the area of the smallest axis aligned rectangle that includes the drawing.

Open Problem: It is known that every planar graph with n vertices admits a crossingfree straight-line drawing in $\Theta(n^2)$ area [12,44]. On the other hand, every planar graph can be drawn with straight-line edges in O(n) area if one allows O(n) crossings per edge [50]. Does every planar graph with n vertices have a straight-line drawing with $o(n^2)$ area and a o(n) crossings per edge?.

Starting references to study the above problem include [19,20].

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