Formal L-concepts with Rough Intents*

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Abstract. We provide a new approach to synthesis of Formal Concept Analysis and Rough Set Theory. In this approach, the formal concept is considered to be a collection of objects accompanied with two collections of attributes—those which are shared by all the objects and those which are possessed by at least one of the objects. We define concept-forming operators for these concepts and describe their properties. Furthermore, we deal with reduction of the data by rough approximation by given equivalence. The results are elaborated in a fuzzy setting.

1 Introduction

Formal concept analysis (FCA) [12] is a method of relational data analysis identifying interesting clusters (formal concepts) in a collection of objects and their attributes (formal context), and organizing them into a structure called concept lattice. Numerous generalizations of FCA, which allow to work with graded data, were provided; see [19] and references therein.

In a graded (fuzzy) setting, two main kinds of concept forming-operators—antitone and isotone one—were studied [2, 13, 20, 21], compared [7, 8] and even covered under a unifying framework [4, 18]. We describe concept-forming operators combining both isotone and antitone operators in such a way that each formal (fuzzy) concept is given by two sets of attributes. The first one is a *lower intent approximation*, containing attributes shared by all objects of the concept; the second one is an *upper intent approximation*, containing those attributes which are possessed by at least one object of the concept. Thus, one can consider the two intents to be a lower and upper approximation of attributes possessed by an object.

Several authors dealing with synthesis of FCA and Rough Set Theory have noticed that intents formed by isotone and antitone operators (in both, crisp and fuzzy setting) correspond to upper and lower approximations, respectively (see e.g. [15, 16, 24]). To the best of our knowledge, no one has studied concept-forming operators which would provide both approximations being present in one concept lattice.

In this papers we present such concept-forming operators, structure of their concepts, and reduction of the data by means of rough approximations by equivalences. Due to page limitation we omit proofs of some theorems.

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2 Preliminaries

In this section we summarize the basic notions used in the paper.

Residuated Lattices and Fuzzy Sets We use complete residuated lattices as basic structures of truth-degrees. A complete residuated lattice [1, 14, 23] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist; $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$; $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \rightarrow c$. 0 and 1 denote the least and greatest elements. The partial order of $\mathbf{L}$ is denoted by $\leqslant$. Throughout this work, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

Elements a of L are called truth degrees. Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of (truth functions of) "fuzzy conjunction" and "fuzzy implication". Furthermore, we define the complement of $a \in L$ as $\neg a = a \rightarrow 0$.

An $\mathbf{L}$-set (or fuzzy set) A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$. The set of all $\mathbf{L}$-sets in a universe X is denoted $\mathbf{L}^X$, or $\mathbf{L}^X$ if the structure of $\mathbf{L}$ is to be emphasized.

The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^X$ is an $\mathbf{L}$-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$. An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is also denoted $\{{}^{A(x)}\!/x \mid x \in X\}$. If for all $y \in X$ distinct from $x_1, \ldots, x_n$ we have $A(y) = 0$, we also write $\{{}^{A(x_1)}\!/x_1, \ldots, {}^{A(x_n)}\!/x_n\}$.

An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is called normal if there is $x \in X$ such that $A(x) = 1$, and it is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp $\mathbf{L}$-sets can be identified with ordinary sets. For a crisp A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$.

Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$. That is, a binary $\mathbf{L}$-relation $I \in \mathbf{L}^{X\times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I). For $\mathbf{L}$-relation $I \in \mathbf{L}^{X\times Y}$ we define its transpose $I^{\mathrm{T}} \in \mathbf{L}^{Y\times X}$ as $I^{\mathrm{T}}(y, x) = I(x, y)$ for all $x \in X$, $y \in Y$.

The composition operators are defined by

$$(I \circ J)(x, z) = \bigvee_{y\in Y} I(x, y) \otimes J(y, z),$$
$$(I \triangleleft J)(x, z) = \bigwedge_{y\in Y} I(x, y) \rightarrow J(y, z),$$
$$(I \triangleright J)(x, z) = \bigwedge_{y\in Y} J(y, z) \rightarrow I(x, y)$$

for every $I \in \mathbf{L}^{X\times Y}$ and $J \in \mathbf{L}^{Y\times Z}$.

A binary **L**-relation E is called an **L**-equivalence if it satisfies $\mathrm{Id}_X \subseteq E$ (reflexivity), $E = E^{\mathrm{T}}$ (symmetry), $E \circ E \subseteq E$ (transitivity).

An **L**-set $B \in \mathbf{L}^Y$ is compatible w.r.t. **L**-equivalence $E \in \mathbf{L}^{Y \times Y}$ if

$$B(y_1) \otimes E(y_1, y_2) \leqslant B(y_2).$$

for any $y_1, y_2 \in Y$.

Formal Concept Analysis in the Fuzzy Setting An **L**-context is a triplet $\langle X, Y, I\rangle$ where X and Y are (ordinary) sets and $I \in \mathbf{L}^{X\times Y}$ is an **L**-relation between X and Y. Elements of X are called objects, elements of Y are called attributes, I is called an incidence relation. $I(x, y) = a$ is read: "The object x has the attribute y to degree a." An **L**-context may be described as a table with the objects corresponding to the rows of the table, the attributes corresponding to the columns of the table and $I(x, y)$ written in cells of the table (for an example see Fig. 1).

	α	β	γ	δ
A	0.5	0	1	0
B	1	0.5	1	0.5
C	0	0.5	0.5	0.5
D	0.5	0.5	1	0.5

Fig. 1. Example of **L**-context with objects A,B,C,D and attributes $\alpha, \beta, \gamma, \delta$.

Consider the following pairs of operators induced by an **L**-context $\langle X, Y, I\rangle$. First, the pair $\langle\uparrow, \downarrow\rangle$ of operators $^{\uparrow} : \mathbf{L}^X \to \mathbf{L}^Y$ and $^{\downarrow} : \mathbf{L}^Y \to \mathbf{L}^X$ is defined by

$$A^{\uparrow}(y) = \bigwedge_{x\in X} A(x) \to I(x, y), \quad B^{\downarrow}(x) = \bigwedge_{y\in Y} B(y) \to I(x, y). \tag{1}$$

Second, the pair $\langle\cap, \cup\rangle$ of operators $^{\cap} : \mathbf{L}^X \to \mathbf{L}^Y$ and $^{\cup} : \mathbf{L}^Y \to \mathbf{L}^X$ is defined by

$$A^{\cap}(y) = \bigvee_{x\in X} A(x) \otimes I(x, y), \quad B^{\cup}(x) = \bigwedge_{y\in Y} I(x, y) \to B(y). \tag{2}$$

To emphasize that the operators are induced by I, we also denote the operators by $\langle\uparrow_I, \downarrow_I\rangle$ and $\langle\cap_I, \cup_I\rangle$. Fixpoints of these operators are called formal concepts. The set of all formal concepts (along with set inclusion) forms a complete lattice, called **L**-concept lattice. We denote the sets of all concepts (as well as the corresponding **L**-concept lattice) by $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ and $\mathcal{B}^{\cap\cup}(X, Y, I)$, i.e.

$$\begin{aligned} \mathcal{B}^{\uparrow\downarrow}(X, Y, I) &= \{\langle A, B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\uparrow} = B,\, B^{\downarrow} = A\}, \\ \mathcal{B}^{\cap\cup}(X, Y, I) &= \{\langle A, B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\cap} = B,\, B^{\cup} = A\}. \end{aligned} \tag{3}$$

For an **L**-concept lattice $\mathcal{B}(X,Y,I)$, where $\mathcal{B}$ is either $\mathcal{B}^{\uparrow\downarrow}$ or $\mathcal{B}^{\cap\cup}$, denote the corresponding sets of extents and intents by $\mathrm{Ext}(X,Y,I)$ and $\mathrm{Int}(X,Y,I)$. That is,

$$\begin{aligned} \mathrm{Ext}(X,Y,I) &= \{A \in \mathbf{L}^X \mid \langle A,B\rangle \in \mathcal{B}(X,Y,I) \text{ for some } B\}, \\ \mathrm{Int}(X,Y,I) &= \{B \in \mathbf{L}^Y \mid \langle A,B\rangle \in \mathcal{B}(X,Y,I) \text{ for some } A\}. \end{aligned} \tag{4}$$

When displaying **L**-concept lattices, we use labeled Hasse diagrams to include all the information on extents and intents. In $\mathcal{B}^{\uparrow\downarrow}(X,Y,I)$, for any $x \in X$, $y \in Y$ and formal **L**-concept $\langle A,B\rangle$ we have $A(x) \geqslant a$ and $B(y) \geqslant b$ if and only if there is a formal concept $\langle A_1,B_1\rangle \leqslant \langle A,B\rangle$, labeled by ${}^{a}\!/x$ and a formal concept $\langle A_2,B_2\rangle \geqslant \langle A,B\rangle$, labeled by ${}^{b}\!/y$. We use labels x resp. y instead of ${}^{1}\!/x$ resp. ${}^{1}\!/y$ and omit redundant labels (i.e., if a concept has both the labels ${}^{a}\!/x$ and ${}^{b}\!/x$ then we keep only that with the greater degree; dually for attributes). The whole structure of $\mathcal{B}^{\uparrow\downarrow}(X,Y,I)$ can be determined from the labeled diagram using the results from [2] (see also [1]).

In $\mathcal{B}^{\cap\cup}(X,Y,I)$, for any $x \in X$, $y \in Y$ and formal **L**-concept $\langle A,B\rangle$ we have $A(x) \geqslant a$ and $B(y) \leqslant b$ if and only if there is a formal concept $\langle A_1,B_1\rangle \leqslant \langle A,B\rangle$, labeled by ${}^{a}\!/x$ and a formal concept $\langle A_2,B_2\rangle \geqslant \langle A,B\rangle$, labeled by ${}^{b}\!/y$ (see examples depicted in Fig. 2).

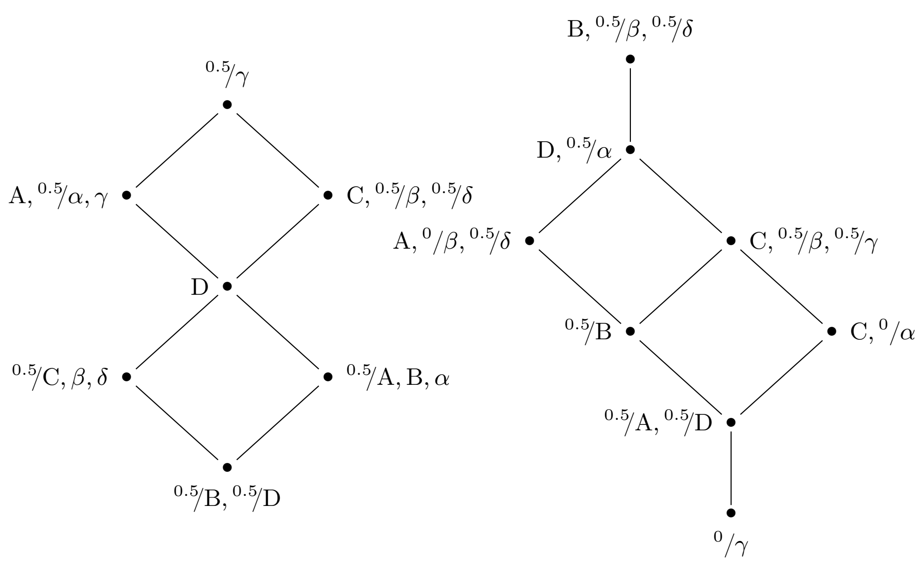

Fig. 2. Concept lattice $\mathcal{B}^{\uparrow\downarrow}(X,Y,I)$ (left) and $\mathcal{B}^{\cap\cup}(X,Y,I)$ (right) of the **L**-context in Fig. 1.

3 L-rough concepts

We consider concept-forming operators induced by **L**-context $\langle X, Y, I\rangle$ defined as follows:

Definition 1. *Let $\langle X, Y, I\rangle$ be an **L**-context. Define* **L**-rough concept-forming operators *as*

$$A^{\vartriangle} = \langle A^{\uparrow}, A^{\cap}\rangle \quad \textit{and} \quad \langle \underline{B}, \overline{B}\rangle^{\triangledown} = \underline{B}^{\downarrow} \cap \overline{B}^{\cup}$$

for $A \in \mathbf{L}^X, \underline{B}, \overline{B} \in \mathbf{L}^Y$. **L**-rough concept *is then a fixed point of $\langle \vartriangle, \triangledown\rangle$, i.e. a pair $\langle A, \langle \underline{B}, \overline{B}\rangle\rangle \in \mathbf{L}^X \times (\mathbf{L} \times \mathbf{L})^Y$ such that $A^{\vartriangle} = \langle \underline{B}, \overline{B}\rangle$ and $\langle \underline{B}, \overline{B}\rangle^{\triangledown} = A$.*[1] *$A^{\uparrow}$ and $A^{\cap}$ are called* lower intent approximation *and* upper intent approximation*, respectively.*

That means, $^{\vartriangle}$ gives intents w.r.t. both $\langle \uparrow, \downarrow\rangle$ and $\langle \cap, \cup\rangle$; $^{\triangledown}$ then gives intersection of extents related to the corresponding intents.

We denote the set of all fixed-points of $\langle \vartriangle, \triangledown\rangle$, in correspondence with (3), as $\mathcal{B}^{\vartriangle\triangledown}(X, Y, I)$ and call it **L**-*rough concept lattice*. Below, we present an analogy of the Main theorem on concept lattices for **L**-rough setting.

Theorem 1 (Main theorem on L-rough concept lattices).

(a) **L**-*rough concept lattice $\mathcal{B}^{\vartriangle\triangledown}(X, Y, I)$ is a complete lattice with suprema and infima defined as follows*

$$\bigwedge_i \langle A_i, \underline{B}_i, \overline{B}_i\rangle = \langle \bigcap_i A_i, \langle \bigcup_i \underline{B}_i, \bigcap_i \overline{B}_i\rangle^{\triangledown\vartriangle}\rangle,$$
$$\bigvee_i \langle A_i, \underline{B}_i, \overline{B}_i\rangle = \langle (\bigcup_i A_i)^{\vartriangle\triangledown}, \bigcap_i \underline{B}_i, \bigcup_i \overline{B}_i\rangle.$$

(b) Moreover, a complete lattice $\mathbf{V} = \langle V, \leqslant\rangle$ is isomorphic to $\mathcal{B}^{\vartriangle\triangledown}(X, Y, I)$ iff there are mappings

$$\gamma : X \times L \to V \quad \textit{and} \quad \mu : Y \times L \times L \to V$$

such that $\gamma(X \times L)$ is supremally dense in $\mathbf{V}$, $\mu(Y \times L \times L)$ is infimally dense in $\mathbf{V}$, and

$$a \otimes \underline{b} \leqslant I(x, y) \textit{ and } I(x, y) \leqslant a \to \overline{b} \quad \textit{is equivalent to} \quad \gamma(x, a) \leqslant \mu(y, \underline{b}, \overline{b})$$

for all $x \in X, y \in Y, a, \underline{b}, \overline{b} \in L$.

When drawing a concept lattice we label nodes as in $\mathcal{B}^{\uparrow\downarrow}$ for lower intent approximations and $\mathcal{B}^{\cap\cup}$ for upper intent approximations. We write $\underline{a}/y$ or $\overline{a}/y$ instead of just a/y to distinguish them. Fig. 3 (middle) shows an **L**-rough concept lattice for the **L**-context from Fig. 1.

The following theorem explains that normal extents have natural intent approximations; that is $\underline{B} \subseteq \overline{B}$.

[1] In what follows, we naturally identify $\langle A, \langle \underline{B}, \overline{B}\rangle\rangle$ with $\langle A, \underline{B}, \overline{B}\rangle$.

Theorem 2. *For normal $A \in \mathbf{L}^X$, we have $A^{\uparrow} \subseteq A^{\cap}$, for crisp singleton $A \in \mathbf{L}^X$, we have $A^{\uparrow} = A^{\cap}$.*

Proof. Since A is normal, there is $x_1 \in X$ such that $A(x_1) = 1$. Then we have

$$\begin{aligned} A^{\uparrow}(y) &= \bigwedge_{x \in X} A(x) \to I(x,y) \leqslant A(x_1) \to I(x_1,y) = I(x_1,y) = \\ &= A(x_1) \otimes I(x_1,y) \leqslant \bigvee_{x \in X} A(x) \otimes I(x,y) = A^{\cap}(y) \end{aligned} \tag{5}$$

for each $y \in Y$.

For A being a crisp singleton, one can show $A^{\uparrow} = A^{\cap}$ by changing all inequalities in (5) to equalities. ◻

Since $\langle \vartriangle, \triangledown \rangle$ is defined via $\langle \uparrow, \downarrow \rangle$ and $\langle \cap, \cup \rangle$, one can expect that there is a strong relationship between the associated concept lattices. In the rest of this section, we summarize them.

Theorem 3. *For $\mathcal{S} \subseteq \mathbf{L}^X$, let $[\mathcal{S}]$ denote an $\mathbf{L}$-closure span of $\mathcal{S}$, i.e. the smallest $\mathbf{L}$-closure system containing $\mathcal{S}$. We have*

$$[\mathrm{Ext}^{\uparrow\downarrow}(X,Y,I) \cup \mathrm{Ext}^{\cap\cup}(X,Y,I)] = \mathrm{Ext}^{\vartriangle\triangledown}(X,Y,I).$$

Proof. "$\subseteq$": Let $A \in \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$. Then $A = A \cap X = \langle A^{\uparrow}, Y \rangle^{\triangledown} \in \mathrm{Ext}^{\vartriangle\triangledown}(X,Y,I)$. Similarly for $A \in \mathrm{Ext}^{\cap\cup}(X,Y,I)$.

"$\supseteq$": Let $A \in \mathrm{Ext}^{\vartriangle\triangledown}(X,Y,I)$ and let $\langle B_1, B_2 \rangle = A^{\vartriangle}$. Then we have $A = B^{\downarrow} \cap B^{\cup} \in [\mathrm{Ext}^{\uparrow\downarrow}(X,Y,I) \cup \mathrm{Ext}^{\cap\cup}(X,Y,I)]$ since $B^{\downarrow} \in \mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$ and $B^{\cup} \in \mathrm{Ext}^{\cap\cup}(X,Y,I)$.

From Theorem 3 one can observe that no extent from $\mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$ and $\mathrm{Ext}^{\cap\cup}(X,Y,I)$ is lost.

Corollary 1. $\mathrm{Ext}^{\uparrow\downarrow}(X,Y,I) \subseteq \mathrm{Ext}^{\vartriangle\triangledown}(X,Y,I)$ *and* $\mathrm{Ext}^{\cap\cup}(X,Y,I) \subseteq \mathrm{Ext}^{\vartriangle\triangledown}(X,Y,I)$.

In addition, no concept is lost.

Corollary 2. *For each $\langle A, \underline{B} \rangle \in \mathcal{B}^{\uparrow\downarrow}(X,Y,I)$ there is $\langle A, \underline{B}, A^{\cap} \rangle \in \mathcal{B}^{\vartriangle\triangledown}(X,Y,I)$.*

For each $\langle A, \overline{B} \rangle \in \mathcal{B}^{\cap\cup}(X,Y,I)$ there is $\langle A, A^{\uparrow}, \overline{B} \rangle \in \mathcal{B}^{\vartriangle\triangledown}(X,Y,I)$.

Remark 1. One can observe from Fig. 3 that in $\mathrm{Ext}^{\vartriangle\triangledown}(X,Y,I)$ there exist extents which are present neither in $\mathrm{Ext}^{\uparrow\downarrow}(X,Y,I)$ nor in $\mathrm{Ext}^{\cap\cup}(X,Y,I)$. On the other hand, lower intent approximations are exactly those from $\mathrm{Int}^{\uparrow\downarrow}(X,Y,I)$ and upper intent approximations are exactly those from $\mathrm{Int}^{\cap\cup}(X,Y,I)$.

With results on mutual reducibility from [8] we can state the following theorem on representation of $\mathcal{B}^{\vartriangle\triangledown}$ by $\mathcal{B}^{\uparrow\downarrow}$.

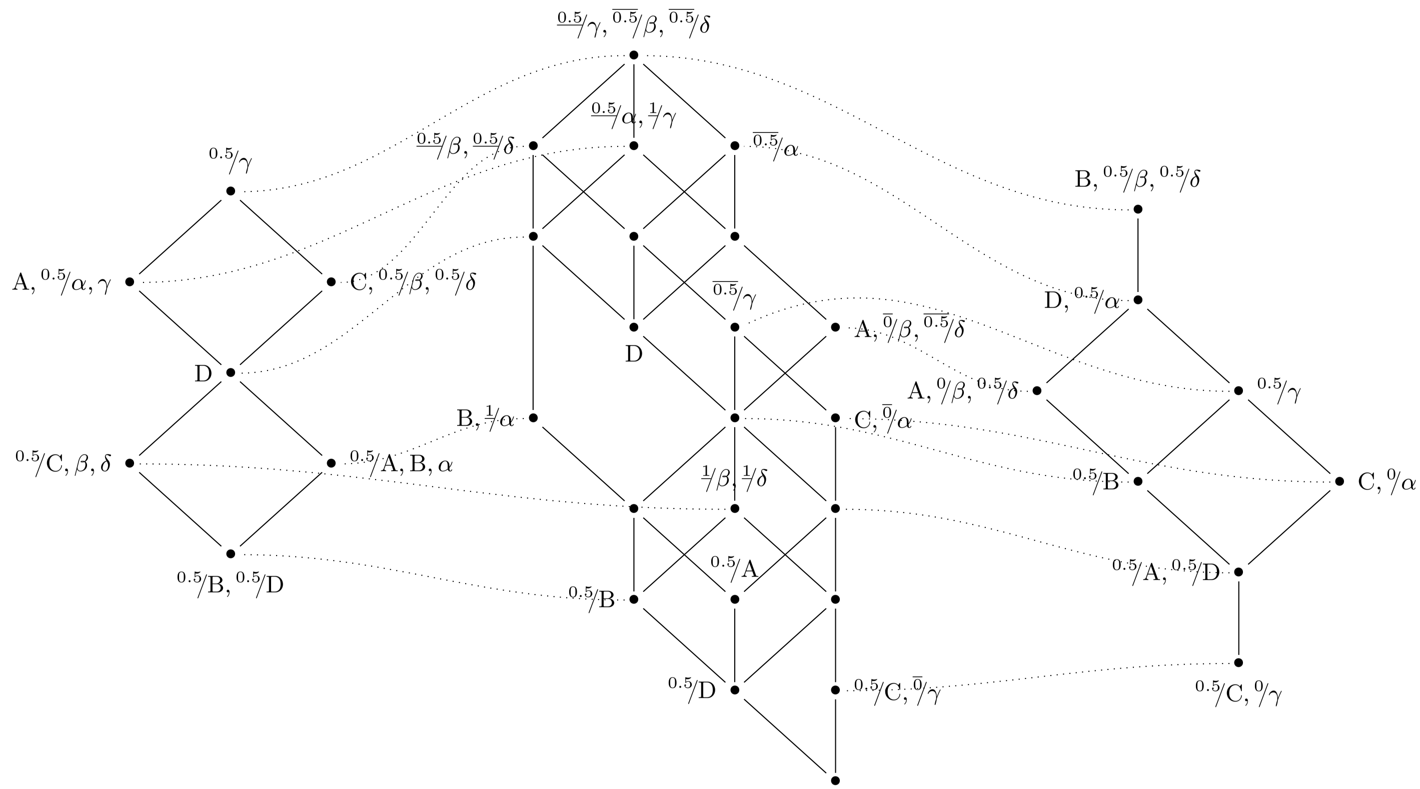

Fig. 3. $\mathcal{B}^{\vartriangle\triangledown}(X, Y, I)$ (middle) and positions of original concepts in $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ (left) and $\mathcal{B}^{\cap\cup}(X, Y, I)$ (right) with $\mathbf{L}$ being a three-element Łukasiewicz chain

Theorem 4. *For a* **L***-context* $\langle X, Y, I\rangle$*, consider the* **L***-context* $\langle X, Y \times L, J\rangle$ *where* J *is defined by*

$$J(x, \langle y, a\rangle) = \begin{cases} I(x, y) & \text{if } a = 1, \\ I(x, y) \to a & \text{otherwise.} \end{cases}$$

Then we have that $\mathcal{B}^{\uparrow\downarrow}(X, Y \times L, J)$ *is isomorphic to* $\mathcal{B}^{\vartriangle\triangledown}(X, Y, I)$ *as a lattice. In addition,*

$$\mathrm{Ext}^{\uparrow\downarrow}(X, Y \times L, J) = \mathrm{Ext}^{\vartriangle\triangledown}(X, Y, I).$$

Proof (sketch). In [8] we show that for **L**-contexts $\langle X, Y, I\rangle$ and $\langle X, Y \times L\backslash\{1\}, J\rangle$ such that

$$J(x, \langle y, a\rangle) = I(x, y) \to a$$

it holds that $\mathrm{Ext}^{\cap\cup}(X, Y, I) = \mathrm{Ext}^{\uparrow\downarrow}(X, Y \times L\backslash\{1\}, J)$. Using this fact, one can check that mapping i defined as

$$i(\langle A, \underline{B}, \overline{B}\rangle) \mapsto \langle A, \underline{B}' \cup \overline{B}'\rangle,$$

where $\underline{B}' \in L^{Y\times\{1\}}, \overline{B}' \in L^{Y\times L\backslash\{1\}}$ with

$$\underline{B}'(\langle y, 1\rangle) = \underline{B}(y),$$
$$\overline{B}'(\langle y, a\rangle) = \overline{B}(y) \to a,$$

is the desired isomorphism from $\mathcal{B}^{\vartriangle\triangledown}(X, Y, I)$ to $\mathcal{B}^{\uparrow\downarrow}(X, Y \times L, J)$.

Theorem 4 shows how we can obtain a concept lattice formed by $\langle\uparrow, \downarrow\rangle$ which is isomorphic to **L**-rough concept lattice of given **L**-context.

4 Rough approximation of an L-context and L-concept lattice

In [17] Pawlak introduced Rough Set Theory where uncertain elements are approximated with respect to an equivalence relation representing indiscernibility.

Formally, given *Pawlak approximation space* $\langle U, E\rangle$, where U is a non-empty set of *objects* (universe) and E is an equivalence relation on U, the *rough approximation* of a crisp set $A \subseteq U$ by E is the pair $\langle A^{\Downarrow_E}, A^{\Uparrow_E}\rangle$ of sets in U defined by

$x \in A^{\Downarrow_E}$ iff for all $y \in U$, $\langle x, y\rangle \in E$ implies $y \in A$,

$x \in A^{\Uparrow_E}$ iff there exists $y \in U$ such that $\langle x, y\rangle \in E$ and $y \in A$.

$A^{\Downarrow_E}$ and $A^{\Uparrow_E}$ are called *lower and upper approximation* of the set A by E, respectively.

In the fuzzy setting, one can generalize $\langle A^{\Downarrow_E}, A^{\Uparrow_E}\rangle$ as in [10, 11, 22],

$$A^{\Downarrow_E}(x) = \bigwedge_{y\in U}(E(x,y) \to A(y)),$$
$$A^{\Uparrow_E}(x) = \bigvee_{y\in U}(A(y) \otimes E(x,y))$$

for $\mathbf{L}$-equivalence $E \in \mathbf{L}^{U\times U}$ and $\mathbf{L}$-set $A \in \mathbf{L}^U$.

Considering $\mathbf{L}$-context $\langle U, U, E\rangle$, we can easily see that ${}^{\Downarrow_E}$ is equivalent to ${}^{\cup_E}$; and ${}^{\Uparrow_E}$ is equivalent to ${}^{\cap_{E^{\mathrm{T}}}}$. Since E is symmetric, we can also write

$$\langle \Downarrow_E, \Uparrow_E\rangle = \langle \cup_E, \cap_E\rangle. \tag{6}$$

Note that for $\mathbf{L}$-set A, $A^{\Downarrow_E}$ is its largest subset compatible with E and $A^{\Uparrow_E}$ is its smallest superset compatible with E.

Below, we deal with situation where lower and upper intent approximations are further approximated using Pawlak's approach. In other words, instead of lower intent approximation $A^{\uparrow}$ we consider the largest subset of $A^{\uparrow}$ compatible with a given indiscernibility relation E, and similarly, instead of upper intent approximation $A^{\cap}$ we consider its smallest superset compatible with E. In Theorem 5 we show how to express this setting using $\mathbf{L}$-rough concept forming operators.

Definition 2. *Let $\langle X, Y, I\rangle$ be an $\mathbf{L}$-context, E be an $\mathbf{L}$-equivalence on Y. Define* $\mathbf{L}$-rough concept-forming operators *as follows:*

$$A^{\vartriangle_E} = \langle A^{\uparrow\Downarrow_E}, A^{\cap\Uparrow_E}\rangle,$$
$$\langle \underline{B}, \overline{B}\rangle^{\triangledown_E} = \underline{B}^{\Uparrow_E\downarrow} \cap \overline{B}^{\Downarrow_E\cup}.$$

Directly from (6) and results in [5] we have:

Theorem 5. *Let $\langle X, Y, I\rangle$ be an $\mathbf{L}$-context, E be an $\mathbf{L}$-equivalence on Y. We have*

$$A^{\vartriangle_E} = \langle A^{\uparrow_{I\triangleright E}}, A^{\cap_{I\circ E}}\rangle \quad \textit{and} \quad \langle \underline{B}, \overline{B}\rangle^{\triangledown_E} = \underline{B}^{\downarrow_{I\triangleright E}} \cap \overline{B}^{\cup_{I\circ E}}.$$

Again, for normal extents we obtain natural upper and lower intent approximations.

Theorem 6. *For normal $A \in \mathbf{L}^X$ we have $A^{\uparrow_{I\triangleright E}} \subseteq A^{\cap_{I\circ E}}$.*

In correspondence with (3) and (4), we denote set of the set of fixpoints of $\langle \vartriangle_E, \triangledown_E\rangle$ in $\mathbf{L}$-context $\langle X, Y, I\rangle$ by $\mathcal{B}^{\vartriangle_E\triangledown_E}(X, Y, I)$ and set of its extents and intents by $\mathrm{Ext}^{\vartriangle_E\triangledown_E}(X, Y, I)$ and $\mathrm{Int}^{\vartriangle_E\triangledown_E}(X, Y, I)$, respectively.

The following theorem shows that a use of a rougher $\mathbf{L}$-equivalence relation leads to a reduction of size of the $\mathbf{L}$-rough concept lattices. Furthermore, this reduction is natural, i.e. it preserves extents.

Theorem 7. *Let $\langle X, Y, I\rangle$ be an* **L***-context, and E_1, E_2 be* **L***-equivalences on Y, such that $E_1 \subseteq E_2$. Then*

$$\mathrm{Ext}^{\vartriangle_{E_2}\triangledown_{E_2}}(X, Y, I) \subseteq \mathrm{Ext}^{\vartriangle_{E_1}\triangledown_{E_1}}(X, Y, I).$$

Example 1. Fig. 4 shows **L**-rough concept lattice of the **L**-context in Fig. 1 and rough **L**-concept lattice approximated using the following **L**-equivalence relation on Y.

	α	β	γ	δ
α	1	0.5	0	0
β	0.5	1	0	0
γ	0	0	1	0.5
δ	0	0	0.5	1

To demonstrate Theorem 7, the concepts with the same extents in the two lattices are connected.

5 Conclusions and further research

We proposed a novel approach to synthesis of RTS na FCA. It provides a lot of directions to be further explored. Our future research includes:

Study of attribute implications using whose semantics is related to the present setting. That will combine results on fuzzy attribute implications [9] and attribute containment formulas [6].

Generalization of the current setting. Note that the operators $^{\uparrow}$ and $^{\cap}$ which compute the universal and the existential intent, need not be induced by the same relation to keep most of the described properties. Actually, this feature is used in Section 4. In our future research, we want to elaborate more on this. For instance, it can provide interesting solution of problem of missing values in a formal fuzzy context—the idea is to use $^{\uparrow}$ induced by the context with missing values substituted by 0, and $^{\cap}$ induced by the context with missing values substituted by 1.

Reduction of **L***-rough concept lattice via linguistic hedges* As two intents are considered in each **L**-rough concept, the size of concept lattice can grow very large. The RST approach to reduction of data, i.e. use of rougher **L**-relation, directly leads to reduction of **L**-rough concept lattice as we showed in Theorem 7. FFCA provides other ways to reduce the size, one of them is parametrization of concept-forming operators using hedges.

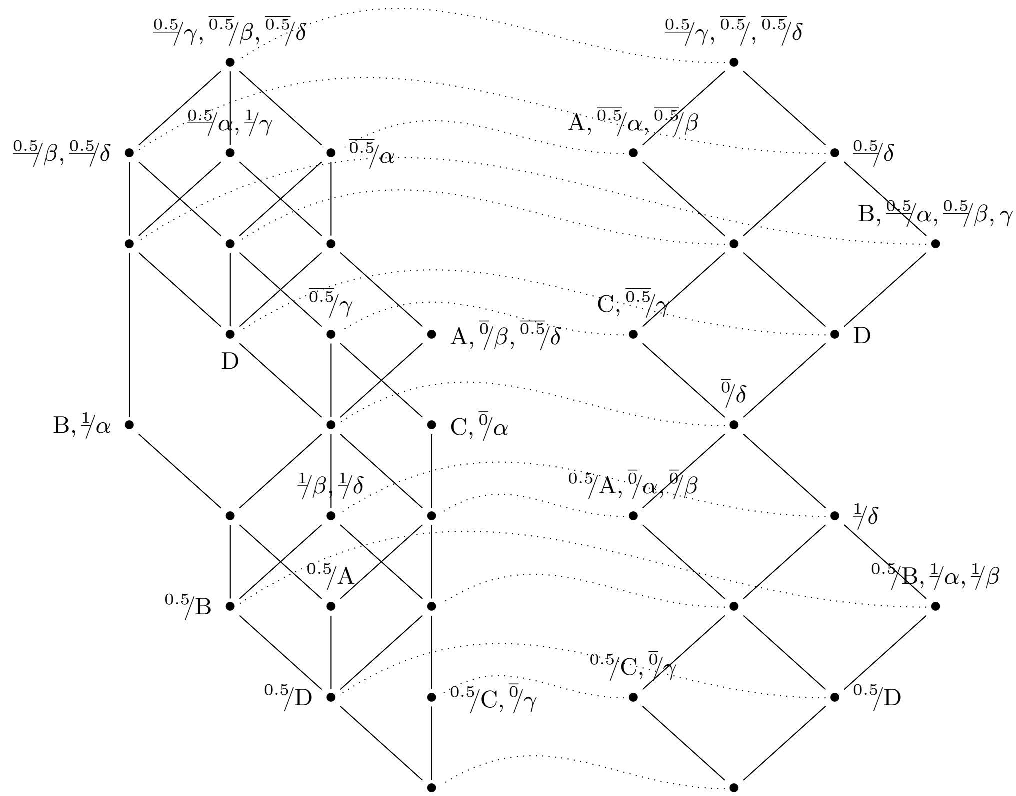

Fig. 4. Rough **L**-concept lattices $\mathcal{B}^{\triangle\triangledown}(X, Y, I)$ (left) and $\mathcal{B}^{\triangle_E \triangledown_E}(X, Y, I)$ (right) with **L** being three-element Łukasiewicz chain. The corresponding extents are connected.

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