Time Process Equivalences for Time Petri Nets*

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1 Introduction

In the core of every theory of systems lies a notion of equivalence between systems: it indicates which particular aspects of systems behaviors are considered to be observable. In concurrency theory, a variety of observational equivalences has been promoted, and the relationships between them have been quite well-understood.

In order to investigate the performance of systems (e.g. the maximal time used for the execution of certain activities and average waiting time for certain requests), many time extensions have been defined for a non-interleaving model of Petri nets. On the other hand, there are few mentions of a fusion of timing and partial order semantics, in the Petri net literature. In [9], processes of timed Petri nets (under the asap hypothesis) have been defined by an algebra of the so-called weighted pomsets. The paper [8] has provided and compared timed step sequence and timed process semantics for timed Petri nets. A method to compute all valid timings for a causal net process of a time Petri net has been put forward in [3]. Branching processes (unfoldings) of time Petri nets have been constructed in [7].

To the best of our knowledge, the incorporation of timing into equivalence notions on Petri nets is even less advanced. In this regard, the paper [4] is a welcome exception, where the testing approach has been extended to Petri nets with associating clocks to tokens and time intervals to arcs from places to transitions. A comparison of different subclasses of time Petri nets has been made in [5], on the base of timed interleaving language and bisimulation equivalences. The papers [1,2] contributed to the classification of the wealth of observational equivalences of linear time – branching time spectrum, based on interleaving, causal tree and partial order semantics, for dense time extensions of event structures with/without internal actions.

The intention of the note is towards developing, studying and comparing trace and bisimulation equivalences based on interleaving, step, partial order, and net-process semantics in the setting of time Petri nets (elementary net systems enriched with the time static intervals on transitions, and with some finiteness

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requirements). This is an extension of the paper [6] to (causal and occurrence) net-process and event structure semantics of the equivalences.

2 Time Petri Nets

In this section, we define some terminology concerning time Petri nets [3].

The domain $\mathbb{T}$ of time values is the set of natural numbers. We denote by $[\tau_1, \tau_2]$ the closed interval between two time values $\tau_1, \tau_2 \in \mathbb{T}$, and by $Interv$ the set of all such intervals. Infinity is allowed at the upper bound. An interval can be of zero length, i.e. $\tau_1 = \tau_2$, containing only a single time value. We use Act to denote an alphabet of actions.

Definition 1. *A* (labeled over Act) time Petri net *is a tuple* $\mathcal{TN} = ((P,\ T,\ F,\ M_0,\ L),\ D)$, *where* (P, T, F, M_0, L) *is a* Petri net *with a set* P *of places, a set* T *of transitions* $(P \cap T = \emptyset)$, *a flow relation* $F \subseteq (P \times T) \cup (T \times P)$, *an initial marking* $M_0 \subseteq P$, *a labeling function* $L : T \to Act$, *and* $D : T \to Interv$ *is a* static timing function *associating with each transition a time interval.*

For $x \in P \cup T$, let ${}^\bullet x = \{y \mid (y, x) \in F\}$ and $x^\bullet = \{y \mid (x, y) \in F\}$ be the *preset* and *postset* of x, respectively. For $X \subseteq P \cup T$, define ${}^\bullet X = \bigcup_{x \in X} {}^\bullet x$ and $X^\bullet = \bigcup_{x \in X} x^\bullet$. For a transition $t \in T$, the boundaries of the interval $D(t) \in Interv$ are called earliest firing time Eft and latest firing time Lft of t.

A *marking* M of $\mathcal{TN}$ is any subset of P. A transition t is *enabled* at a marking M if ${}^\bullet t \subseteq M$ (all its input places have tokens in M), otherwise the transition is *disabled*. Let $En(M)$ be the set of transitions enabled at M.

Consider the *behavior* of a time Petri net $\mathcal{TN}$. A *state* of $\mathcal{TN}$ is a triple (M, I, GT), where M is a marking, $I : En(M) \longrightarrow \mathbb{T}$ is a *dynamic timing function*, and $GT \in \mathbb{T}$ is a *global time moment*. The *initial state* of $\mathcal{TN}$ is a triple $S_0 = (M_0, I_0, GT_0)$, where $I_0(t) = 0$, for all $t \in En(M_0)$, and $GT_0 = 0$.

We call a non-empty subset $U \subseteq T$ a *step enabled at a state* $S = (M, I, GT)$, if $(\forall t \in U \diamond t \in En(M))$ and $(\forall t \neq t' \in U \diamond {}^\bullet t \cap {}^\bullet t' = \emptyset)$. A step $U \subseteq T$ enabled at a state $S = (M, I, GT)$ is *fireable from* S *after a delay time* $\theta \in \mathbb{T}$ if $(\forall t \in U \diamond Eft(t) \leq I(t) + \theta)$ and $(\forall t' \in En(M) \diamond I(t') + \theta \leq Lft(t'))$. Let $Contact(S) = \{t \in U \mid U$ is a step fireable from a state $S = (M, I, GT)$ after some delay time $\theta \in \mathbb{T}$ and $(M \setminus {}^\bullet t) \cap t^\bullet \neq \emptyset)\}$.

The *firing* of a step U fireable from a state $S = (M, I, GT)$ after a delay time θ *leads* to the new state $S' = (M', I', GT')$ given as follows:

(i) $M' = (M \setminus {}^\bullet U) \cup U^\bullet$,

(ii) $\forall t' \in T \diamond I'(t') = \begin{cases} I(t') + \theta, & \text{if } t' \in En(M \setminus {}^\bullet U), \\ 0, & \text{if } t' \in En(M') \setminus En(M \setminus {}^\bullet U), \\ \text{undefined, otherwise,} & \end{cases}$

(iii) $GT' = GT + \theta$.

In this case, we write $S \stackrel{(U,\theta)}{\longrightarrow} S'$, and, moreover, $S \stackrel{(A,\theta)}{\longrightarrow} S'$, if $A = L(U) = \sum_{t \in U} L(t)$. A finite or infinite sequence of the form: $S = S^0 \stackrel{(U_1,\theta_1)}{\longrightarrow} S^1 \stackrel{(U_2,\theta_2)}{\longrightarrow} S^2$

$\mathcal{TN}$:

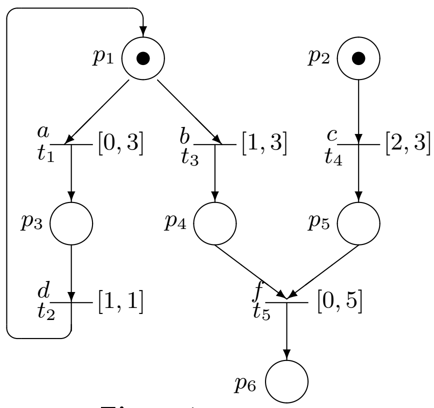

Figure 1.

$\ldots$ ($S = S^0 \stackrel{(\{t_1\},\theta_1)}{\longrightarrow} S^1 \stackrel{(\{t_2\},\theta_2)}{\longrightarrow} S^2 \ldots$,), is a *step (interleaving) firing sequence of $\mathcal{TN}$ from a state S*. Then, $\sigma = (U_1, \theta_1)\ (U_2, \theta_2)\ \ldots$ ($\sigma = (\{t_1\}, \theta_1)\ (\{t_2\}, \theta_2)\ \ldots$) is called a *step (interleaving) firing schedule of $\mathcal{TN}$ from S*. Define the *step (interleaving) language of $\mathcal{TN}$*, $\mathcal{L}_{s(i)}(\mathcal{TN}) = \{(A_1, \theta_1) \ldots (A_k, \theta_k) \mid \sigma = (U_1, \theta_1) \ldots (U_k, \theta_k)$ is a step (interleaving) firing schedule of $\mathcal{TN}$ from the initial state S_0, and $A_k = L(U_k)$ $(k \geq 0)\}$.

A state S of $\mathcal{TN}$ is *reachable* if it appears in some step firing sequence of $\mathcal{TN}$ from the initial state S_0. Let $RS(\mathcal{TN})$ denote the set of all reachable states of $\mathcal{TN}$. We call $\mathcal{TN}$ *T-restricted* if ${}^\bullet t \neq \emptyset \neq t^\bullet$ for all transition $t \in T$; *contact-free* if $Contact(S) = \emptyset$ for all $S \in RS(\mathcal{TN})$; *time-progressive* if for every infinite step firing schedule $(U_1, \theta_1)\ (U_2, \theta_2)\ (U_3, \theta_3)\ \ldots$, the series $\theta_1 + \theta_2 + \theta_3 + \ldots$ diverges. In what follows, we will consider only T-restricted, contact-free and time-progressive time Petri nets.

Example 1. Figure 1 shows a time Petri net $\mathcal{TN}$. Both $\sigma = (\{t_1, t_4\}, 3)$ and $\sigma' = (\{t_1, t_4\}, 3)(\{t_2\}, 1)(\{t_3\}, 1)(\{t_5\}, 2) \ldots$ are step firing schedules of $\mathcal{TN}$ from $S_0 = (M_0, I_0, GT_0)$, where $M_0 = \{p_1, p_2\}$, $I_0(t) = \begin{cases} 0, & \text{if } t \in \{t_1, t_3, t_4\}, \\ undefined, & \text{otherwise}, \end{cases}$ and $GT_0 = 0$. Furthermore, $\widehat{\sigma} = (\{t_2\}, 1)(\{t_3\}, 1)(\{t_5\}, 2) \ldots$ is a step firing schedule of $\mathcal{TN}$ from $S = (M, I, GT)$, where $M = \{p_3, p_5\}$, $I(t) = \begin{cases} 0, & \text{if } t = t_2, \\ undefined, & \text{otherwise}, \end{cases}$ and $GT = 3$. It is easy to see that $\mathcal{TN}$ is really T-restricted, contact-free and time-progressive.

3 Auxiliary Models

First, consider definitions related to time partial orders.

Definition 2. *A* (labeled over *Act*) time partial order *is a tuple $\eta = (X, \prec, \lambda, \tau)$ consisting of a set X; a transitive, irreflexive relation $\prec$; a labeling function $\lambda : X \to Act$; and a timing function $\tau : X \to \mathbb{T}$ such that $e \prec e' \Rightarrow \tau(e) \leq \tau(e')$. As usual, we write $x \preceq y$ for $x \prec y$ or $x = y$. Often $\prec$ is called a strict partial order, while $\preceq$ is a partial order, i.e. a reflexive, antisymmetric and transitive relation.*

Time partial order sets over Act, $\eta = (X, \prec, \lambda, \tau)$ and $\eta' = (X', \prec', \lambda', \tau')$, are *isomorphic* (denoted $\eta \sim \eta'$) iff there is a bijective mapping $\beta : X \to X'$ such that (i) $x \prec \widetilde{x} \iff \beta(x) \prec' \beta(\widetilde{x})$, for all $x, \widetilde{x} \in X$; (ii) $\lambda(x) = \lambda'(\beta(x))$ and $\tau(x) = \tau'(\beta(x))$, for all $x \in X$. The isomorphic class of a time partial order over Act, η, is called a *time pomset over Act* and denoted as $pom(\eta)$.

Second, we aim at defining notions pertaining to time event structures.

Definition 3. *A* (labeled over Act) time event structure *is a tuple* $\xi = (E, \prec, \#, l, \tau)$ *with a set* E *of events; a strict partial order* $\prec \subseteq E \times E$ *such that* $|\downarrow e = \{e' \in E \mid e' \prec e\}| < \infty$*, for all* $e \in E$*; an irreflexive symmetric conflict relation* $\# \subseteq E \times E$ *such that* $(e \# e' \prec e'') \Rightarrow (e \# e'')$*, for all* $e, e', e'' \in E$*; a labeling function* $l : E \to Act$*; a timing function* $\tau : E \to \mathbb{T}$ *such that* $e \prec e' \Rightarrow \tau(e) \leq \tau(e')$.

Time event structures over Act, $\xi = (E, \prec, \#, l, \tau)$ and $\xi' = (E', \prec', \#', l', \tau')$, are *isomorphic* (denoted $\xi \sim \xi'$) iff there is a bijective mapping $\beta : E \to E'$ such that (i) $e \prec e' \Leftrightarrow \beta(e) \prec' \beta(e')$ and $e \# e' \Leftrightarrow \beta(e) \#' \beta(e')$, for all $e, e' \in E$; (ii) $l(e) = l'(\beta(e))$ and $\tau(e) = \tau'(\beta(e))$, for all $e \in E$. The isomorphic class of a time event structure over Act, ξ, is denoted as $les(\xi)$.

Third, consider definitions associated with (labeled) time nets.

Definition 4. *A* (labeled over Act) time net *is a finitary, acyclic net* $TN = (B, E, G, l, \tau)$ *with a set* B *of conditions, a set* E *of events, a flow relation* $G \subseteq (B \times E) \cup (E \times B)$ *such that* $\{e \mid (e, b) \in G\} = \{e \mid (b, e) \in G\} = E$*, a labeling function* $l : E \to Act$*, and a time function* $\tau : E \to \mathbb{T}$ *such that* $e\ G^+\ e' \Rightarrow \tau(e) \leq \tau(e')$.

Time nets over Act, $TN = (B, E, G, l, \tau)$ and $TN' = (B', E', G', l', \tau')$, are *isomorphic* (denoted $TN \simeq TN'$) iff there exists a bijective mapping $\beta : B \cup E \to B' \cup E'$ such that (i) $\beta(B) = B'$ and $\beta(E) = E'$; (ii) $x\ G\ y \iff \beta(x)\ G'\ \beta(y)$, for all $x, y \in B \cup E$; (iii) $l(e) = l'(\beta(e))$ and $\tau(e) = \tau'(\beta(e))$, for all $e \in E$.

Consider additional notions and notations for a time net TN. Let $\prec = G^+$, $\preceq = G^*$, and $\tau(TN) = \sup\{\tau(e) \mid e \in E\}$. Specify ${}^\bullet x = \{y \mid (y, x) \in G\}$ and $x^\bullet = \{y \mid (x, y) \in G\}$, for $x \in B \cup E$, and, moreover, ${}^\bullet X = \bigcup_{x \in X} {}^\bullet x$ and $X^\bullet = \bigcup_{x \in X} x^\bullet$, for $X \subseteq B \cup E$. Furthermore, define the sets ${}^\bullet TN = \{b \in B \mid {}^\bullet b = \emptyset\}$, $TN^\bullet = \{b \in B \mid b^\bullet = \emptyset\}$. Given $e, e' \in E$, $x, x' \in (B \cup E)$, and $E' \subseteq E$,

- $\downarrow e = \{x \mid x \preceq e\}$ (predecessors),
- E' is a *downward-closed subset of* E if $\downarrow e' \cap (E \times E) \subseteq E'$, for all $e' \in E'$. In this case, E' is called *timely sound* if $\tau(e') \leq \tau(e)$, for all $e' \in E'$ and $e \in E \setminus E'$, and define the set $Cut(E') = (E'^\bullet \cup {}^\bullet TN) \setminus {}^\bullet E'$,
- $x \# x' \iff \exists e \neq e' \diamond e \preceq x, \wedge e' \preceq x' \wedge {}^\bullet e \cap {}^\bullet e' \neq \emptyset$ (conflict),
- E' is a *conflict-free subset of* E, if $\neg(e' \# e'')$, for all $e', e'' \in E'$,
- E' is a *configuration* of TN if E' is a finite, downward-closed, conflict-free subset of E,
- $x \smile x' \iff \neg((x \prec x') \vee (x' \prec x) \vee (x \# x'))$ (concurrency).

– $\emptyset \neq E'$ is a *step of* TN iff $e \smile e'$ and $\tau(e) = \tau(e')$, for all $e, e' \in E'$. In this case, let $\tau(E') = \tau(e)$ for some $e \in E'$.

Given time nets $TN = (B, E, G, l, \tau)$, $\widehat{TN} = (\widehat{B}, \widehat{E}, \widehat{G}, \widehat{l}, \widehat{\tau})$ and $TN' = (B', E', G', l', \tau')$, TN is a *prefix of* TN' (denoted $TN \longrightarrow TN'$) if $B' \subseteq B$, E is a finite, downward-closed and timely sound subset of E', $G = G' \cap (B \times E \cup E \times B)$, $l = l' \mid_E$, and $\tau = \tau' \mid_E$; $\widehat{TN}$ is a *suffix of* TN' *w.r.t.* TN if $\widehat{E} = E' \setminus E$, $\widehat{B} = B' \setminus B \cup TN^{\bullet}$, $\widehat{G} = G' \cap (\widehat{B} \times \widehat{E} \cup \widehat{E} \times \widehat{B})$, $\widehat{l} = l' \mid_{\widehat{E}}$, and $\widehat{\tau} = \tau' \mid_{\widehat{E}}$. We write $TN \xrightarrow{\widehat{TN}} TN'$ iff $TN \longrightarrow TN'$ and $\widehat{TN}$ is a suffix of TN' w.r.t. TN.

Lemma 1. *Given* $TN \xrightarrow{\widehat{TN}} TN'$ *and* $\widehat{e} \in \widehat{E}$, *the following holds:*

(i) ${}^{\bullet}TN = {}^{\bullet}TN'$ *and* ${}^{\bullet}\widehat{TN} = TN^{\bullet}$,
(ii) $({}^{\bullet}\widehat{e} \setminus {}^{\bullet}\widehat{TN}) \subseteq ({}^{\bullet}\widehat{e} \setminus {}^{\bullet}TN')$,
(iii) *if* ${}^{\bullet}\widehat{e} \subseteq \widetilde{B}' \subseteq \widetilde{B}$, *then* $\{b \in \widetilde{B}' \mid \widetilde{\varphi}(b) \in {}^{\bullet}\widehat{\varphi}(\widehat{e})\} = {}^{\bullet}\widehat{e}$ *in* $\widetilde{TN} \in \{TN, \widehat{TN}, TN'\}$.

An *s-linearization of a time net* TN is a finite or infinite sequence $\rho = V_1 V_2 \ldots$ of steps of TN, such that every event of TN is included in the sequence exactly once, and both causal and time orders are preserved: $(e_i \prec e_j \vee \tau(e_i) < \tau(e_j)) \Rightarrow i < j$, for all $e_i \in V_i$ and $e_j \in V_j$ $(i, j \geq 1)$. An s-linearization of TN of the form: $\rho = \{e_1\}\{e_2\}\ldots$, is called an *i-linearization* of TN. For an s-linearization $\rho = V_1 V_2 \ldots$ of TN, define $E_\rho^k = \bigcup_{1 \leq i \leq k} V_i$ $(k \geq 0)$. Clearly, E_ρ^k is a downward-closed subset of E.

A *(labeled over Act) time net* $TN = (B, E, G, l, \tau)$ is called a *time causal net*, if $|{}^{\bullet}b| \leq 1 \ \wedge \ |b^{\bullet}| \leq 1$, for all $b \in B$; a *time occurrence net*, if $|{}^{\bullet}b| \leq 1$, and $\neg(x \ \#_{TN} x)$, for all $x \in B \cup E$. Clearly, $\eta(TN) = (E, \prec \cap (E \times E), l, \tau)$ is a time partial order, if TN is a time causal net, and $\xi(TN) = (E_{TN}, \prec_{TN} \cap (E_{TN} \times E_{TN}), \#_{TN} \cap (E_{TN} \times E_{TN}), l_{TN}, \tau_{TN})$ is a (labeled over Act) time event structure, if TN is a time occurrence net.

Lemma 2. *Every time causal net* TN *has an s-linearization* $\rho = V_1 V_2 \ldots$. *Moreover, it holds:* $Cut(E_\rho^{k+1}) = \big(Cut(E_\rho^k) \setminus {}^{\bullet}V_{k+1}\big) \cup V^{\bullet}{}_{k+1}$, *and* $\big(Cut(E_\rho^k) \setminus {}^{\bullet}e\big) \cap e^{\bullet} = \emptyset$, *for all* $e \in V_{k+1}$ $(k \geq 0)$.

Example 2. The time causal net $TN' = (B',\ E',\ G',\ l',\ \tau')$ is depicted in Figure 2(a), where the net elements are accompanied by their names, and the values of the functions l' and τ' are indicated nearby the events. Define the time causal nets $TN = (B,\ E,\ G,\ l,\ \tau)$, with $B = \{b_1, b_2, b_3, b_4\}$, $E = \{e_1, e_4\}$, $G = G' \cap (B \times E \cup E \times B)\}$, $l = l' \mid_E$, $\tau = \tau' \mid_E$, and $\widehat{TN} = (\widehat{B},\ \widehat{E},\ \widehat{G},\ \widehat{l},\ \widehat{\tau})$, with $\widehat{B} = B' \setminus B \cup \{b_3, b_4\}$, $\widehat{E} = E' \setminus E$, $\widehat{G} = G' \cap (\widehat{B} \times \widehat{E} \cup \widehat{E} \times \widehat{B})$, $\widehat{l} = l' \mid_{\widehat{E}}$, $\widehat{\tau} = \tau' \mid_{\widehat{E}}$. It is easy to see that TN is a prefix of TN', $\widehat{TN}$ is a suffix of TN' w.r.t. TN, and, moreover, $TN \xrightarrow{\widehat{TN}} TN'$. Notice that $\rho_{TN'} = \{e_1, e_4\}\{e_2\}\{e_3\}\{e_5\}\ldots$ is an s-linearization of TN'. The time occurrence net $\widetilde{TN}$ is depicted in Figure 2(b).

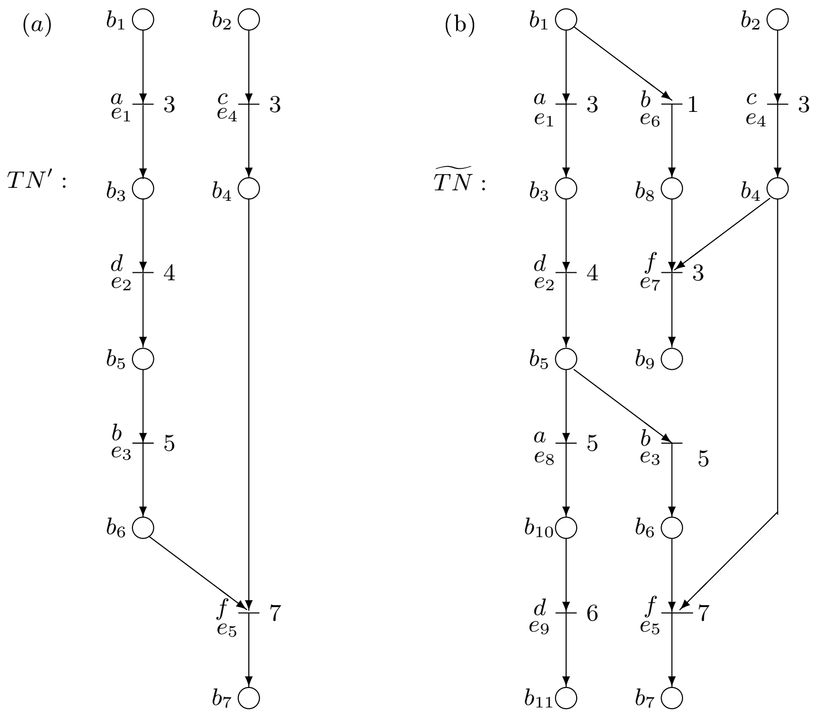

Figure 2.

4 Time Process Semantics

We start with defining a special mapping from a time net TN to a time Petri net $\mathcal{TN}$ w.r.t. its marking. Given a time Petri net $\mathcal{TN} = ((P,\ T,\ F,\ M_0,\ L),\ D)$ with a marking M and a time net $TN = (B, E, G, l, \tau)$, a mapping $\varphi : B \cup E \to P \cup T$ is a *homomorphism* from TN to $\mathcal{TN}$ w.r.t. M iff the following conditions hold:

- $\varphi(B) \subseteq P$, $\varphi(E) \subseteq T$,
- the restriction of φ to ${}^\bullet e$ is a bijection between ${}^\bullet e$ and ${}^\bullet\varphi(e)$ and the restriction of φ to $e^\bullet$ is a bijection between $e^\bullet$ and $\varphi(e)^\bullet$, for all $e \in E$,
- $\left({}^\bullet e = {}^\bullet e' \wedge \varphi(e) = \varphi(e')\right) \ \Rightarrow\ e = e'$,
- the restriction of φ to ${}^\bullet TN$ is a bijection between ${}^\bullet TN$ and M,
- $l(e) = L(\varphi(e))$, for all $e \in E$.

4.1 Time C-Processes

First, introduce the notion of a time C-process of $\mathcal{TN}$ w.r.t. its marking.

Definition 5. *Given a time Petri net $\mathcal{TN}$ with its marking M, a* time C-process *of $\mathcal{TN}$* w.r.t. *M is a pair $\pi = (TN, \varphi)$ with a time causal net TN and a homomorphism φ from TN to $\mathcal{TN}$ w.r.t. M. Let $\tau(\pi) = \tau(TN)$.*

We use $\mathcal{CP}(\mathcal{TN}, M_0)$ ($\mathcal{CP}(\mathcal{TN}, M)$) to denote the set of time C-processes of $\mathcal{TN}$ w.r.t. the initial marking M_0 (a marking M). Let $\pi = (TN, \varphi)$, $\pi' = (TN', \varphi') \in \mathcal{CP}(\mathcal{TN}, M_0)$. Then, $\pi \xrightarrow{\widehat{\pi}=(\widehat{TN},\widehat{\varphi})} \pi'$ iff $TN \xrightarrow{\widehat{TN}} TN'$, $\varphi = \varphi'|_{B\cup E}$, and $\widehat{\varphi} = \varphi'|_{\widehat{B}\cup\widehat{E}}$. From now on, whenever $\pi \xrightarrow{\widehat{\pi}} \pi'$, we shall write $\pi \xrightarrow{(a,\theta)} \pi'$ if $\widehat{E} = \{e\}$, $\widehat{\tau}(e) = \tau(\pi) + \theta$, $\widehat{l}(e) = a$; and $\pi \xrightarrow{(A,\theta)} \pi'$ if $\widehat{\precsim} \cap (\widehat{E} \times \widehat{E}) = \emptyset$, $\widehat{l}(\widehat{E}) = \sum_{e\in\widehat{E}} \widehat{l}(e) = A$, $\widehat{\tau}(e) = \tau(\pi) + \theta$, for all $e \in \widehat{E}$.

Given $\pi = (TN, \varphi) \in \mathcal{CP}(\mathcal{TN}, M)$, a state $S = (M, I, GT)$ of $\mathcal{TN}$, and $B' \subseteq B_{TN}$, the latest global time moment when tokens appear in all input places of the transition $t \in En(\varphi(B'))$ is defined as follows:

$$\mathbf{TOE}_{\pi,S}(B', t) = \max\Big(\{\tau_{TN}({}^\bullet b) \mid b \in B'_{[t]} \setminus {}^\bullet TN\} \cup \{\overline{GT}\}\Big),$$

where $B'_{[t]} = \{b \in B' \mid \varphi_{TN}(b) \in {}^\bullet t\}$, $\overline{GT} = GT - I(t)$, if $B'_{[t]} \subseteq {}^\bullet TN$, and $\overline{GT} = GT$, otherwise. Notice that the above is an extension of the definition of $\mathbf{TOE}(\cdot,\cdot)$ from [3] to the case of the time C-processes of $\mathcal{TN}$ w.r.t. an arbitrary one and not only the initial marking.

Definition 6. *A time C-process $\pi = (TN, \varphi)$ of $\mathcal{TN}$ w.r.t. M is* a time C-process of $\mathcal{TN}$ w.r.t. $S = (M, I, GT) \in RS(\mathcal{TN})$ *iff for all $e \in E$ it holds:*

(i) $\tau(e) \geq GT$,
(ii) $\tau(e) \geq \boldsymbol{TOE}_{\pi,S}({}^\bullet e, \varphi(e)) + Eft(\varphi(e))$,
(iii) $\forall t \in En(\varphi(C_e)) \diamond \tau(e) \leq \boldsymbol{TOE}_{\pi,S}(C_e, t) + Lft(t)$,
where $C_e = Cut(Earlier(e))$ *with* $Earlier(e) = \{e' \in E \mid \tau(e') < \tau(e)\}$.

The time C-process $\pi_0 = (TN_0 = (B_0, \emptyset, \emptyset, \emptyset, \emptyset),\ \varphi_0)$ of $\mathcal{TN}$ w.r.t. the initial state is called the *initial time C-process of $\mathcal{TN}$*. We use $\mathcal{CP}(\mathcal{TN}, S_0)$ ($\mathcal{CP}(\mathcal{TN}, S)$) to denote the set of time C-processes of $\mathcal{TN}$ w.r.t. the initial state S_0 (a state $S \in RS(\mathcal{TN})$).

Theorem 1. *Given $\pi = (TN, \varphi)$, $\pi' = (TN', \varphi') \in \mathcal{CP}(\mathcal{TN}, S_0)$ such that $\pi \xrightarrow{\widehat{\pi}} \pi'$, $\widehat{\pi} = (\widehat{TN}, \widehat{\varphi}) \in \mathcal{CP}(\mathcal{TN}, \widehat{S} = (\widehat{M}, \widehat{I}, \widehat{GT}))$, where $\widehat{M} = \varphi(TN^\bullet)$,* $\widehat{I}(t) = \begin{cases} \tau(TN) - \boldsymbol{TOE}_{\pi,S_0}(TN^\bullet, t), & \textit{if } t \in En(\widehat{M}), \\ \textit{undefined}, & \textit{otherwise}, \end{cases}$ *and $\widehat{GT} = \tau(TN)$.*

Finally, we intend to realize for a time Petri net the relationships between its firing schedules from reachable states and its time C-processes w.r.t. the states.

Lemma 3. *Given $\pi = (TN, \varphi) \in \mathcal{CP}(\mathcal{TN}, S)$, an s-linearization $\rho = V_1V_2\ldots$ of TN, $e \in V_{k+1}$, $t \in En\big(\varphi(Cut(E^k_\rho))\big)$, $t' \in En\big(\varphi(C_e)\big)$, and $t'' \in En\big(\varphi(Cut(E^{k+1}_\rho))\big)$ ($k \geq 0$), the following holds:*

(i) $\boldsymbol{TOE}_{\pi,S}(Cut(E^k_\rho), \varphi(e)) = \boldsymbol{TOE}_{\pi,S}({}^\bullet e, \varphi(e))$,
(ii) $\boldsymbol{TOE}_{\pi,S}(Cut(E^k_\rho), t) = \boldsymbol{TOE}_{\pi,S}(C_e, t)$, *if* $t \in En\big(\varphi(C_e)\big)$,
(iii) $\boldsymbol{TOE}_{\pi,S}(Cut(E^k_\rho), t) = \tau(V_{k+1})$, *if* $t \notin En\big(\varphi(C_e)\big)$,

(iv) $\boldsymbol{TOE}_{\pi,S}(Cut(E_\rho^k), t) = \boldsymbol{TOE}_{\pi,S}(Cut(E_\rho^{k+1}), t)$, *if* $t \in En\big(\varphi(Cut(E_\rho^{k+1}))\big)$,
(vi) $\boldsymbol{TOE}_{\pi,S}(Cut(E_\rho^{k+1}), t'') = \tau(V_{k+1})$, *if* $t'' \notin En\big((\varphi(Cut(E_\rho^k))) \setminus {}^\bullet V_{k+1}\big)$.

For $\pi = (TN, \varphi) \in \mathcal{CP}(\mathcal{TN}, S)$, define the function $FS_{\pi,S}$ which maps any s-linearization $\rho = V_1 V_2 \ldots$ of TN to the sequence of the form: $FS_{\pi,S}(\rho) = (\varphi(V_1), \tau(V_1) - GT)\ (\varphi(V_2), \tau(V_2) - \tau(V_1)) \ldots$.

Proposition 1. – *Given* $\pi = (TN, \varphi) \in \mathcal{CP}(\mathcal{TN},\ S = (M, I, GT))$ *and an* $s(i)$*-linearization* $\rho = V_1 V_2 \ldots$ *of* TN, $FS_{\pi,S}(\rho)$ *is a step (interleaving) firing schedule of* $\mathcal{TN}$ *from the state* S, *with intermediate states* $S^k = (M^k, I^k, GT^k)$ $(k \geq 0)$, *where* $M^k = \varphi\big(Cut(E_\rho^k)\big)$, $GT^k = \tau(V_k)$, *and* $I^k(t) = \begin{cases} \tau(V_k) - \boldsymbol{TOE}_{\pi,S}\big(Cut(E_\rho^k), t\big), & \textit{if } t \in En(M^k), \\ \textit{undefined}, & \textit{otherwise}, \end{cases}$ *Here,* $\tau(V_0) = GT$.

– *For any step (interleaving) firing schedule* σ *of* $\mathcal{TN}$ *from a state* $S \in RS(\mathcal{TN})$, *there is a unique (up to an isomorphism) time process* $\pi \in \mathcal{CP}(\mathcal{TN}, S)$ *such that* $FS_{\pi,S}(\rho) = \sigma$, *where* ρ *is an* $s(i)$*-linearization of* TN.

Notice that the above Proposition is an extension of Theorems 19 and 21 from [3] to the cases of s-linearizations of time C-processes of $\mathcal{TN}$ w.r.t. arbitrary reachable states and step firing schedules of $\mathcal{TN}$ from the states.

Example 3. Define a mapping φ' from the time causal net TN' (see Fig. 2(a)) to the time Petri net $\mathcal{TN}$ (see Fig. 1), as follows: $\varphi'(b_i) = p_i$ $(1 \leq i \leq 3)$, $\varphi'(b_4) = p_5$, $\varphi'(b_5) = p_1$, $\varphi'(b_6) = p_4$, $\varphi'(b_7) = p_6$, and $\varphi'(e_i) = t_i$ $(1 \leq i \leq 5)$. Next, for the time causal nets TN and $\widehat{TN}$ specified in Example 1, set $\varphi = \varphi' \mid_{E \cup B}$ and $\widehat{\varphi} = \varphi' \mid_{\widehat{E} \cup \widehat{B}}$, respectively. Clearly, $\pi' = (TN', \varphi')$ and $\pi = (TN, \varphi)$ are time C-process of $\mathcal{TN}$ w.r.t. M_0. As $TN \xrightarrow{\widehat{TN}} TN'$, we get $\pi \xrightarrow{\widehat{\pi} = (\widehat{TN}, \widehat{\varphi})} \pi'$. Further, take $\widetilde{B} = \{b_1, b_2\}$, $S' = (M', I', GT')$, where $M' = \{p_1, p_2\}$, $I'(t) = \begin{cases} 0, & \text{if } t \in \{t_1, t_4\}, \\ undefined, & \text{otherwise}, \end{cases}$ and $GT' = 3$, and $t_1 \in En(\varphi'(\widetilde{B}))$. Calculate $\mathbf{TOE}_{\pi',S'}(\widetilde{B}, t_1) = \max\Big(\{\tau_{TN'}({}^\bullet b) \mid b \in \widetilde{B}_{[t_1]} \setminus {}^\bullet TN'\} \cup \{\overline{GT}\}\Big) = \max\Big(\emptyset \cup \{3 - 0\}\Big) = 3$. It is not difficult to check that $\pi' = (TN', \varphi')$, $\pi = (TN, \varphi) \in \mathcal{CP}(\mathcal{TN}, S_0)$. Then, $\widehat{\pi} \in \mathcal{CP}(\mathcal{TN}, S)$, where $M = \{p_3, p_5\}$, $I(t) = \begin{cases} 0, & \text{if } t = t_2, \\ undefined, & \text{otherwise}, \end{cases}$ and $GT = 3$, due to Theorem 1. For the s-linearization $\rho_{TN'} = \{e_1, e_4\}\{e_2\}\{e_3\}\{e_5\} \ldots$ of TN' from Example 2, we can get $FS_{\pi',S_0}(\rho_{TN'}) = \sigma'$, by using Proposition 1.

4.2 Time O-Processes

Define the notion of a time O-process of $\mathcal{TN}$ w.r.t. its marking.

Definition 7. *Given a time Petri net* $\mathcal{TN}$ *with its marking* M, *a* time O-process *of* $\mathcal{TN}$ w.r.t. M *is a pair* $\nu = (TN, \psi)$ *with a time occurrence net* TN *and a homomorphism* ψ *from* TN *to* $\mathcal{TN}$ *w.r.t.* M.

A *computation of a time O-process* $\nu = (TN = (B,\ E,\ G,\ l,\ \tau), \psi)$ of $\mathcal{TN}$ w.r.t. M is a finite time C-process $\pi = (TN' = (B',\ E',\ G',\ l',\ \tau'), \psi|_{B' \cup E'})$ of $\mathcal{TN}$ w.r.t. M such that $E' \subseteq E$ is a configuration of TN. A time O-process $\nu = (TN, \psi)$ of $\mathcal{TN}$ w.r.t. M is a time O-process of $\mathcal{TN}$ w.r.t. $S = (M, I, GT) \in RS(\mathcal{TN})$ iff all computations of ν belong to the set $\mathcal{CP}(\mathcal{TN}, S)$. We use $\mathcal{OP}(\mathcal{TN}, S_0)$ to denote the set of all time O-processes of $\mathcal{TN}$ w.r.t. S_0.

Example 4. To illustrate the notions above, first define a mapping ψ from the time O-net $\widetilde{TN}$ (see Fig. 2(b)) to the time Petri net $\mathcal{TN}$ (see Fig. 1) as follows: $\psi(b_1) = \psi(b_5) = \psi(b_{11}) = p_1$, $\psi(b_2) = p_2$, $\psi(b_3) = \psi(b_{10}) = p_3$, $\psi(b_4) = p_5$, $\psi(b_6) = \psi(b_8) = p_4$, $\psi(b_7) = \psi(b_9) = p_6$ and $\psi(e_1) = \psi(e_8) = t_1$, $\psi(e_2) = \psi(e_9) = t_2$, $\psi(e_3) = \psi(e_6) = t_3$, $\psi(e_4) = t_4$, $\psi(e_5) = \psi(e_7) = t_5$. Clearly, both π and π', specified in Example 3, are time C-processes of of $\mathcal{TN}$ w.r.t. S_0, and, moreover, are computations of ν. It is easy to see that all computations of ν belong to the set $\mathcal{CP}(\mathcal{TN}, S')$. Then, ν is a time O-process of $\mathcal{TN}$ w.r.t. S_0.

5 Hierarchy of Behavioral Equivalences

First, consider equivalence notions rested on classical state-based behaviors of time Petri nets.

Definition 8. *Time Petri nets $\mathcal{TN}$ and $\mathcal{TN}'$ labeled over Act are:*

- step (interleaving) trace equivalent *(denoted $\mathcal{TN} \equiv_{s(i)} \mathcal{TN}'$) iff $\mathcal{L}_{s(i)}(\mathcal{TN}) = \mathcal{L}_{s(i)}(\mathcal{TN}')$,*
- step (interleaving) bisimilar *(denoted $\mathcal{TN} \leftrightarrows_{s(i)} \mathcal{TN}'$) iff there is a relation $R \subseteq RS(\mathcal{TN}) \times RS(\mathcal{TN}')$ such that $(S_0, S_0') \in R$ (S_0 and S_0' are the initial states of $\mathcal{TN}$ and $\mathcal{TN}'$, respectively) and for all $(S, S') \in R$ it holds:*
  - *if $S \stackrel{(A,\theta)}{\longrightarrow} S_1$ ($S \stackrel{(\{a\},\theta)}{\longrightarrow} S_1$) in $\mathcal{TN}$, then $S' \stackrel{(A,\theta)}{\longrightarrow} S_1'$ ($S' \stackrel{(\{a\},\theta)}{\longrightarrow} S_1'$) in $\mathcal{TN}'$ and $(S_1, S_1') \in R$,*
  - *and vice versa.*

Before defining behavioral equivalences on time processes of time Petri nets, we need auxiliary notions. Given a time Petri net $\mathcal{TN}$, define the following sets:

- $Trace_{i\text{–}pr}(\mathcal{TN}) = \{(\{a_1\}, \theta_1) \ldots (\{a_n\}, \theta_n) \in (2^{Act} \times \mathbb{T})^* \mid \pi_0 \stackrel{(a_1,\theta_1)}{\longrightarrow} \pi_1 \ldots \pi_{n-1} \stackrel{(a_n,\theta_n)}{\longrightarrow} \pi_n\ (n \geq 0)$ in $\mathcal{TN}\}$,
- $Trace_{s\text{–}pr}(\mathcal{TN}) = \{(A_1, \theta_1) \ldots (A_n, \theta_n) \in (\mathbb{N}^{Act} \times \mathbb{T})^* \mid \pi_0 \stackrel{(A_1,\theta_1)}{\longrightarrow} \pi_1 \ldots \pi_{n-1} \stackrel{(A_n,\theta_n)}{\longrightarrow} \pi_n\ (n \geq 0)$ in $\mathcal{TN}\}$,
- $Trace_{pom\text{–}pr}(\mathcal{TN}) = \{pom(\eta(TN)) \mid \pi = (TN, \varphi) \in \mathcal{CP}(\mathcal{TN}, S_0)\}$,
- $Trace_{c\text{–}pr}(\mathcal{TN}) = \{[TN]_{\simeq} \mid \pi = (TN, \varphi) \in \mathcal{CP}(\mathcal{TN}, S_0)\}$,
- $Trace_{les\text{–}pr}(\mathcal{TN}) = \{les(\xi(TN)) \mid \nu = (TN, \psi) \in \mathcal{OP}(\mathcal{TN}, S_0)\}$,
- $Trace_{o\text{–}pr}(\mathcal{TN}) = \{[TN]_{\simeq} \mid \nu = (TN, \psi) \in \mathcal{OP}(\mathcal{TN}, S_0)\}$.

Definition 9. *Let $* \in \{i-pr, s-pr, pom-pr, c-pr, les-pr, o-pr\}$ and $\star \in \{i-pr, s-pr, pom-pr, c-pr\}$. Then,*

– $\mathcal{TN}$ *and* $\mathcal{TN}'$ $*$-trace equivalent *(denoted* $\mathcal{TN} \equiv_* \mathcal{TN}'$*) iff* $Trace_*(\mathcal{TN}) = Trace_*(\mathcal{TN}')$,
– *a relation* $R \subseteq \mathcal{CP}(\mathcal{TN}, S_0) \times \mathcal{CP}(\mathcal{TN}', S_0')$ *is* $\star$-bisimulation between $\mathcal{TN}$ and $\mathcal{TN}'$ *(denoted* $R : \mathcal{TN} \Leftrightarrow_\star \mathcal{TN}'$*) iff* $(\pi_0, \pi_0') \in R$*, and for all* $(\pi, \pi) \in R$*, the following holds:*
  1. *whenever* $\pi \xrightarrow{\widehat{\pi}} \widetilde{\pi}$ *in* $\mathcal{TN}$ *and*
     * $|\widehat{E}| = 1$*, if* $\star = i{-}pr$,
     * $\widehat{\preceq} \cap (\widehat{E} \times \widehat{E}) = \emptyset$*, if* $\star = s{-}pr$,

     then $\pi' \xrightarrow{\widehat{\pi}'} \widetilde{\pi}'$ *in* $\mathcal{TN}'$, $(\widetilde{\pi}, \widetilde{\pi}') \in R$*, and*
     * $\eta(\widehat{TN}) \simeq \eta(\widehat{TN}')$*, if* $\star \in \{i{-}pr, s{-}pr, pom{-}pr\}$,
     * $\widehat{TN} \simeq \widehat{TN}'$*, if* $\star = c{-}pr$,
  2. *Symmetric to item 1.*

– $\mathcal{TN}$ *and* $\mathcal{TN}'$ *are* $\star$-bisimilar *(denoted* $\mathcal{TN} \Leftrightarrow_\star \mathcal{TN}'$*) iff there is* $\star$*-bisimulation* $R : \mathcal{TN} \Leftrightarrow_\star \mathcal{TN}'$.

Proposition 2. *Let* $\leftrightarrow \in \{\equiv, \Leftrightarrow\}$ *and* $* \in \{i, s\}$*. Then,* $\mathcal{TN} \leftrightarrow_* \mathcal{TN}' \iff \mathcal{TN} \leftrightarrow_{*-pr} \mathcal{TN}'$.

Finally, we state the relationships between the time process equivalences of time Petri nets.

Theorem 2. *Let* $\leftrightarrow, \rightleftharpoons \in \{\equiv, \Leftrightarrow\}$ *and* $\star, * \in \{i{-}pr, s{-}pr, pom{-}pr, c{-}pr, les{-}pr, o{-}pr\}$*. Then,*

$$\mathcal{TN} \leftrightarrow_\star \mathcal{TN}' \;\Rightarrow\; \mathcal{TN} \rightleftharpoons_* \mathcal{TN}'$$

iff there is a directed path from $\leftrightarrow_\star$ *to* $\rightleftharpoons_*$ *in Fig. 3.*

$$\begin{tikzcd}
 & & \equiv_{les-pr} \arrow[d] & \equiv_{o-pr} \arrow[l] \arrow[d] \\
\Leftrightarrow_{i-pr} \arrow[d] & \Leftrightarrow_{s-pr} \arrow[l] \arrow[d] & \Leftrightarrow_{pom-pr} \arrow[l] \arrow[d] & \Leftrightarrow_{c-pr} \arrow[l] \arrow[d] \\
\equiv_{i-pr} & \equiv_{s-pr} \arrow[l] & \equiv_{pom-pr} \arrow[l] & \equiv_{c-pr} \arrow[l]
\end{tikzcd}$$

Figure 3.

Proof. ($\Leftarrow$) All the implications in Fig. 1 follow from the Definitions, Theorems and Lemmas considered prior to that.
($\Rightarrow$) We now demonstrate that it is impossible to draw any arrow from one equivalence to the other such that there is no directed path from the first equivalence to the second one in the graph in Fig. 1.

For this purpose, we consider the time Petri nets depicted in Fig. 2. It is easy to see that $\mathcal{TN}_1$ and $\mathcal{TN}_2$ are $\equiv_{c-pr}$–equivalent but not $\Leftrightarrow_{i-pr}$–equivalent because, for example, any time C-process π of $\mathcal{TN}_2$ w.r.t. the initial state, containing one event (with its input and output conditions) labeled by an action b and time moment 0, can be extended up to a time C-process π' of $\mathcal{TN}_2$ w.r.t. the initial state, containing two events (with their input and output conditions) labeled by actions b and a and time moments 0 and 5, respectively, by the time C-process $\widehat{\pi}$ of $\mathcal{TN}_2$ w.r.t. its state corresponding to the finishing of π, but it is not the case in $\mathcal{TN}_1$.

Second, $\mathcal{TN}_2$ and $\mathcal{TN}_3$ are $\leftrightarrows_{i\text{–}pr}$–equivalent but not $\equiv_{s\text{–}pr}$–equivalent because, for example, there is a time C-process of $\mathcal{TN}_3$ w.r.t. the initial state, containing two concurrent events (with their input and output conditions) labeled by actions a and b and time moments 0, but it is not the case in $\mathcal{TN}_2$.

Third, $\mathcal{TN}_3$ and $\mathcal{TN}_4$ are $\leftrightarrows_{s\text{–}pr}$–equivalent but not $\equiv_{pom\text{–}pr}$–equivalent because, for example, there is a time C-process of $\mathcal{TN}_3$ w.r.t. the initial state, containing two events (with their input and output conditions) labeled by actions b and a and time moments 0 and 5, respectively, such that an action b causally precedes an action a, but it is not the case in $\mathcal{TN}_4$.

Fourth, $\mathcal{TN}_4$ and $\mathcal{TN}_5$ are $\equiv_{les\text{–}pr}$–equivalent but not $\equiv_{c\text{–}pr}$–equivalent because, for example, the time C-processes of $\mathcal{TN}_4$ and $\mathcal{TN}_5$ w.r.t. their initial states, containing events (with its input and output conditions) labeled by actions a and time moments 0, are not isomorphic.

Finally, $\mathcal{TN}_5$ and $\mathcal{TN}_6$ are $\leftrightarrows_{c\text{–}pr}$–equivalent but not $\equiv_{les\text{–}pr}$–equivalent because it is easy to see that the time event structure, corresponding to any maximal time O-processes of $\mathcal{TN}_6$ w.r.t. its initial states, contains two conflicting events labeled by actions b and time moments 0, but it is not the case in $\mathcal{TN}_5$.

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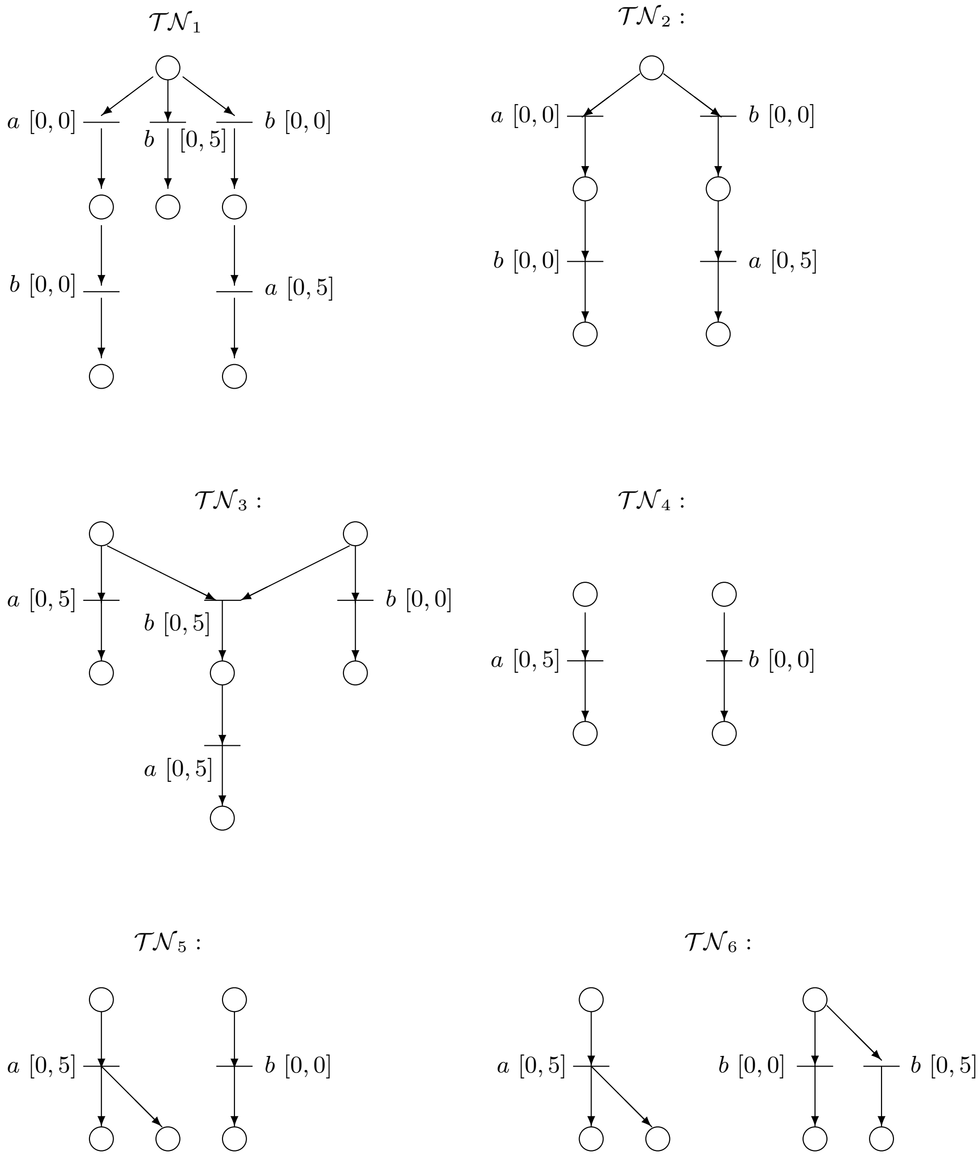
$\mathcal{TN}_1$
a $[0,0]$
b $[0,5]$
b $[0,0]$
b $[0,0]$
a $[0,5]$
$\mathcal{TN}_2$:
a $[0,0]$
b $[0,0]$
b $[0,0]$
a $[0,5]$
$\mathcal{TN}_3$:
a $[0,5]$
b $[0,5]$
b $[0,0]$
a $[0,5]$
$\mathcal{TN}_4$:
a $[0,5]$
b $[0,0]$
$\mathcal{TN}_5$:
a $[0,5]$
b $[0,0]$
$\mathcal{TN}_6$:
a $[0,5]$
b $[0,0]$
b $[0,5]$

Figure 4.