Superimposed Codes and Query Algorithms

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Abstract. New superimposed codes based on finite projective planes are proposed. These codes allow to construct efficient randomized query algorithms for some functions.

1 Introduction

Linear codes is the simplest class of codes. The alphabet used is a fixed choice of a finite field $GF(q) = F_q$ with q elements. In most of applications a special case of $GF(2) = F_2$ is considered. These codes are binary codes. A generating matrix G for a linear $[n, k]$ code over F_q is a $k-\text{by}-n$ matrix with entries in the finite field F_q, whose rows are linearly independent. The linear code corresponding to the matrix G consists of all the q^k possible linear combinations of rows of G. The requirement of linear independence is equivalent to saying that all the q^k linear combinations are distinct.

The superimposed codes also can be considered as linear codes, only the linear operation "multiplication modulo 2" is substituted by the operation "disjunction".

Let X be an n-element set . For an integer k, $0 \leq k \leq n$ we denote by $\binom{X}{k}$ the collection of all the k-subsets of X, while 2^X denotes the power set of X. A family of subsets of X is a subset of 2^X. It is called k-uniform if it is a subset of $\binom{X}{k}$.

We call the family of sets $\mathcal{F}$ r-cover-free if $F_0 \nsubseteq F_1 \cup \cdots \cup F_r$ holds for all pairwise distinct $F_0, F_1, F_2, \cdots, F_r$ in $\mathcal{F}$. Let us denote by $f(n, k)$ the maximum cardinality of an r-cover-free family $\mathcal{F}$ in $\binom{X}{k}$, $\mid X \mid= n$.

Definition 1. *An r-cover-free family $\mathcal{F}$ in $\binom{X}{k}$, $\mid X \mid= n$ is called a $[r, k, n]$-superimposed code.*

W. H. Kautz and R. C. Singleton [13] introduced the notion of superimposed codes, proved some properties, demonstrated possible applications and noted that relatively good superimposed codes can be obtained by taking random subsets of $\{1, 2, \cdots, n\}$ as members F_i of the family $\mathcal{F}$. In 1985 P. Erdös, P. Frankl

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and Z. Füredi [8] proved upper and lower bounds for $f(n,k)$ and noted relation of existence of superimposed codes with existence of Steiner systems with certain parameters (projective planes also are Steiner systems). The bounds were later improved by several authors and the notion of superimposed codes was generalized in [5–7, 12, 15, 17, 20] but very many important problems are still widely open.

2 Projective Planes

A projective plane consists of a set of lines, a set of points, and a relation between points and lines called incidence, having the following properties:

1) Given any two distinct points, there is exactly one line incident with both of them.
2) Given any two distinct lines, there is exactly one point incident with both of them.
3) There are four points such that no line is incident with more than two of them.

The second condition means that there are no parallel lines. The term "incidence" is used to emphasize the symmetric nature of the relationship between points and lines. Thus the expression "point p is incident with line l " is used instead of either "p is on l " or "l passes through p ".

A finite projective plane of order n is formally defined as a set of n^2+n+1 points with the properties that:

1) Any two points determine a line,
2) Any two lines determine a point,
3) Every point has $n+1$ lines on it, and
4) Every line contains $n+1$ points.

The number n here is called *the order of the projective plane.* It is proved that a finite projective plane can exist only when the order n is a power of a prime. Existence of projective planes with certain parameters in many cases is an open problem. (Properties of projective planes are described in [4].)

The projective plane of order 2, also known as the Fano plane, has 7 points, 7 lines and it is defined by the incidence matrix

	l_1	l_2	l_3	l_4	l_5	l_6	l_7
p_1	1	0	0	0	1	0	1
p_2	1	1	0	0	0	1	0
p_3	0	1	1	0	0	0	1
p_4	1	0	1	1	0	0	0
p_5	0	1	0	1	1	0	0
p_6	0	0	1	0	1	1	0
p_7	0	0	0	1	0	1	1

Definition 2. *For arbitrary prime number* q *we define a collection* $Proj(q)$ *of points* $(p_1, p_2, \cdots, p_{q^2+q+1})$ *and lines* $(l_1, l_2, \cdots, l_{q^2+q+1})$ *where a relation* point p_i is incident to line l_j *is defined by the following rule:*
1) If $i \in \{1, 2, \cdots, q+1\}$ *then the point* p_i *is incident to the lines* $l_{(i-1)q+2}$, $l_{(i-1)q+3}$, $\cdots, l_{iq+1}$ *and the line* l_i *is incident to the points* $p_{(i-1)q+2}, p_{(i-1)q+3}$, $\cdots, p_{iq+1}$.
2) If $i \in \{q+2, q+3, \cdots, q^2+q+1\}$ *and* $\lfloor \frac{i-(q+1)}{q} \rfloor = a$ *and* $\lfloor \frac{j-(q+1)}{q} \rfloor = b$ *then the point* p_i *is incident to the line* l_j *iff*

$$(\frac{i-(q+2)}{q} - \lfloor \frac{i-(q+2)}{q} \rfloor) =$$

$$= (\frac{j-(q+2)}{q} - \lfloor \frac{j-(q+2)}{q} \rfloor) + (\frac{j-(q+2)}{q} - \lfloor \frac{j-(q+2)}{q} \rfloor) \cdot (\lfloor \frac{i-(q+2)}{q} \rfloor).$$

(see an example for $q = 5$ *in Table 1).*

Table 1.

	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}	l_{14}	l_{15}	l_{16}	l_{17}	l_{18}	l_{19}	l_{20}	l_{21}	l_{22}	l_{23}	l_{24}	l_{25}	l_{26}	l_{27}	l_{28}	l_{29}	l_{30}	l_{31}
p_{01}	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{02}	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{03}	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
p_{04}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
p_{05}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0
p_{06}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
p_{07}	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
p_{08}	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0
p_{09}	0	1	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0
p_{10}	0	1	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0
p_{11}	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
p_{12}	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
p_{13}	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0
p_{14}	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0
p_{15}	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1
p_{16}	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0
p_{17}	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0
p_{18}	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0
p_{19}	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1
p_{20}	0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0
p_{21}	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0
p_{22}	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0
p_{23}	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1
p_{24}	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0
p_{25}	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0
p_{26}	0	0	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0
p_{27}	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1
p_{28}	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0
p_{29}	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0
p_{30}	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0
p_{31}	0	0	0	0	0	1	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0

Lemma 1. *The collection* $Proj(q)$ *is a projective plane.*

Proof. By Definition 2. □

In order to explore the properties of this projective plane we introduce auxiliary notions.

Definition 3. Elementary area $(i_1, 1_2) \times (j_1, j_2)$ *is a part of the incidence table describing incidence of points* $p_{i_1}, p_{i_1+1}, \cdots, p_{i_2}$ *to lines* $l_{j_1}, l_{j_1+1}, \cdots, l_{j_2}$.

We will consider only elementary areas where $(i_1, 1_2)$ is taken from the set $\{(1, q+1), (q+2, 2q+1), (2q+2, 3q+1), \cdots, (q^2+2, q^2+q+1)\}$ and (j_1, j_2) is taken from the set $\{(1, q+1), (q+2, 2q+1), (2q+2, 3q+1), \cdots, (q^2+2, q^2+q+1)\}$.

The following 4 lemmas are immediately implied by Definition 2.

Lemma 2. *The elementary area* $(1, q+1) \times (1, q+1)$ *is such that* p_a *is incident to* l_b *iff either* $a = 1$ *or* $b = 1$.

Lemma 3. *The elementary area* $(i_1, i_2) \times (1, q+1)$ *where*

$$(i_1, i_2) \in \{(q+2, 2q+1), (2q+2, 3q+1), \cdots, (q^2+2, q^2+q+1)\}$$

is such that p_a *is incident to* l_b *iff* $b \in \{(b+1)q+2, \cdots, (b+2)q+1\}$.

Lemma 4. *The elementary area* $(1, q+1) \times (j_1, j_2)$ *where*

$$(j_1, j_2) \in \{(q+2, 2q+1), (2q+2, 3q+1), \cdots, (q^2+2, q^2+q+1)\}$$

is such that p_a *is incident to* l_b *iff* $a \in \{(a+1)q+2, \cdots, (a+2)q+1\}$.

Lemma 5. *The elementary area* $(i_1, i_2) \times (j_1, j_2)$ *where*

$$(i_1, i_2) \in \{(q+2, 2q+1), (2q+2, 3q+1), \cdots, (q^2+2, q^2+q+1)\}$$

and

$$(j_1, j_2) \in \{(q+2, 2q+1), (2q+2, 3q+1), \cdots, (q^2+2, q^2+q+1)\}$$

is such that p_a *is incident to* l_b *iff* $b \equiv a + d(\mod q)$ *where* $d = \frac{i_1-2}{q} \cdot \frac{j_1-2}{q}$.

We demanded that the parameter q is a prime number. Hence the property formulated in Lemma 5 shows that in this elementary area each value of a corresponds to exactly one value of b. Moreover, Lemma 5 shows that the characteristics d of an elementary area describes this elementary area in much detail.

Lemma 6. *For arbitrary* (i_1, i_2) *from the set* $\{(q+2, 2q+1), (2q+2, 3q+1), \cdots, (q^2+2, q^2+q+1)\}$ *all the elementary areas* $(i_1, i_2) \times (q+2, 2q+1)$ *have characteristics* $d = 0$ *and the* $q-1$ *elementary areas* $(i_1, i_2) \times (j_1, j_2)$ *where* $(j_1, j_2) \in \{(q+2, 2q+1), (2q+2, 3q+1), \cdots, (q^2+2, q^2+q+1)\}$ *have* $q-1$ *distinct values of the characteristics* $d \in \{1, 2, \cdots, q\}$.

Proof. By definition 2, the elementary areas $(i_1, i_2) \times (2q+2, 3q+1)$ where

$$(i_1, i_2) \in \{(2q+2, 3q+1), (3q+2, 4q+1), \cdots, (q^2+2, q^2+q+1)\}$$

have characteristics d being $1, 2, \cdots, q$, respectively. It follows from Lemma 5 that the set of all characteristics for $(i_1, i_2) \times (kq+2, (k+1)q+1)$ can be obtained by multiplying all elements of $1, 2, \cdots, q$ to $k-1$. Since q is a prime number, the set $\{1, 2, \cdots, q\}$ does not change by such a multiplication. □

We need a much more strong property of $Proj(q)$.

Definition 4. *By S_j we define the set of all points incident to the line l_j.*

To simplify our notation we sometimes will not distinguish between a line l_j and the set S_j containing all the points incident to l_j.

Definition 5. *By S we denote the collection of all the sets S_1, S_2, S_{q^2+q+1}.*

Definition 6. *By Q we denote the set $\{1, 2, \cdots, q^2 + q + 1\}$.*

Definition 7. *By R_r we denote the set*

$$\{(u_1, u_2, \cdots, u_r) \mid (\forall i)(i \in Q) \text{ and } (\forall i, j)(i, j \in Q \text{ and } u_i \neq u_j)\}$$

Definition 8. *By $S_{(u_1,u_2,\cdots,u_r)}$ we denote the union of the sets $S_{u_1}, S_{u_2}, \cdots, S_{u_r}$.*

Definition 9. *By $U_{q,r}$ we denote the collection*

$$\{S_{(u_1,u_2,\cdots,u_r)} \mid (u_1, u_2, \cdots, u_r) \in R_r \text{ and } (\forall i)(S_{u_i} \in S)\}$$

Below we will consider only collections $U_{q,r}$ where $r = q - 1$.

Lemma 7. *Given $r \leq q - 1$, let $\{u_1, u_2, \cdots, u_r\}$ and $\{v_1, v_2, \cdots, v_r\}$ be two distinct subsets of the set Q. Then $S_{(v_1,v_2,\cdots,v_r)} \neq S_{(v_1,v_2,\cdots,v_r)}$.*

Proof. Assume from the contrary that there exist two distinct r-tuples $(u_1, u_2, \cdots, u_r)$ and $(v_1, v_2, \cdots, v_r)$ such that $S_{(u_1,u_2,\cdots,u_r)} = S_{(v_1,v_2,\cdots,v_r)}$.

Remember that every set S_j is a line, every $S_{(u_1,u_2,\cdots,u_r)}$ is a union of lines and the two sets $\{u_1, u_2, \cdots, u_r\}$ and $\{v_1, v_2, \cdots, v_r\}$ have the same cardinality. Since the sets $\{u_1, u_2, \cdots, u_r\}$ and $\{v_1, v_2, \cdots, v_r\}$ are distinct, there exists a number $j \in Q$ such that $j \in (u_1, u_2, \cdots, u_r) - (v_1, v_2, \cdots, v_r)$. Then $S_{(u_1,u_2,\cdots,u_r)}$ contains all the points from $\{p_{(j-i)q+2}, p_{(j-i)q+2}, \cdots, p_{jq+1}\}$. The line l_j is the only line which contains at least two of these points. If $j \notin \{v_1, v_2, \cdots, v_r\}$ then each of these q points enters $S_{(v_1,v_2,\cdots,v_r)}$ via a different line. Contradiction with the cardinality of the set $\{v_1, v_2, \cdots, v_r\}$. □

Lemma 8. *There are $2^{q \cdot \log q}$ distinct sets $S_{(u_1,u_2,\cdots,u_{q-1})}$ in $U_{q,q-1}$.*

Proof. By Lemma 7, the number of distinct sets $S_{(u_1,u_2,\cdots,u_{q-1})}$ is equal to the cardinality of R_{q-1}. i.e. to $\frac{(q^2+q+1)(q^2+q)(q^2+q-1)\cdots(q^2-1)}{(q-1)(q-2)\cdots 1}$. By Stirling formula, this is at least equal $\frac{q^{2q}e^q}{q^q} = 2^{q \cdot \log q}$. □

Comment. There are $q^2 + q + 1$ sets in the collection S. The number of distinct sets $S_{(u_1,u_2,\cdots,u_{q-1})}$ is quite impressive $2^{q \cdot \log q}$. However, Lemma becomes invalid if we substitute $r \leq q - 1$ by $r = q + 1$.

Theorem 1. *For arbitrary prime q, the family $S = \{S_1, S_2, \ldots, S_{q^2+q+1}\}$ is a $[q - 1, q, q^2 + q + 1]$-superimposed code.*

Proof. By Lemmas 6 and 7. □

3 Decision Trees

We wish to use superimposed codes to prove advantages of probabilistic decision trees over deterministic decision trees.

A deterministic decision tree is a rooted ordered binary tree T. Each internal node of T is labeled with a variable x_i and each leaf is labeled with a value 0 or 1. Given an input $x \in \{0,1\}^n$, the tree is evaluated as follows. Start at the root. If this is a leaf then stop. Otherwise, query the variable x_i that labels the root. If $x_i = 0$ then recursively evaluate the left subtree, if $x_i = 1$, then recursively evaluate the right subtree. The output of the tree is the value (0 or 1) of the leaf that is reached eventually. Note that an input x deterministically determines the leaf, and thus the output, that the procedure ends up in.

We say that a decision tree computes f if its output equals $f(x)$, for all $x \in \{0,1\}^n$. Clearly there are many different decision trees that compute the same f. The complexity of such a tree is its depth, i.e., the number of queries made on the worst-case input. We define $D(f)$, the decision tree complexity of f as the depth of an optimal (= minimal-depth) decision tree that computes f.

As in many other models of computation, we can add the power of randomization to decision trees. We add coin flips as internal nodes to the tree. That is, the tree may contain internal nodes labeled by a bias $p \in \{0,1\}$, and when the evaluation procedure reaches such a node, it will flip a coin with bias p and will go to the left child on outcome "heads" and to the right child on "tails". Now an input x no longer determines with certainty which leaf of the tree will be reached, but instead induces a probability distribution over the set of all leaves. Thus, the tree outputs 0 or 1 with a certain probability. The complexity of the tree is the number of queries on the worst-case input and worst-case outcome of the coin flips.

Definition 10. *We say that a randomized decision tree computes f with bounded-error if its output equals $f(x)$ with probability exceeding $\frac{1}{2}$, for all $x \in \{0,1\}^n$. $R(f)$ denotes the complexity of the optimal randomized decision tree that computes f with bounded error.*

We introduce 2 functions for which we consider deterministic and randomized decision trees. Let $q \geq 2$ be a prime number. We denote q^2+q+1 by Q. As in Definition 2, the projective plane $Proj(q)$ consists of points $(p_1, p_2, \cdots, p_Q)$ and lines $(l_1, l_2, \cdots, l_Q)$.

The Boolean function $F_1^Q(x_1, x_2, \ldots, x_Q)$ equals 1 if and only if there exists a line $l_i \in Proj(q)$ such that for every point $p_j \in l_i$ the value $x_j = 1$.

The function $F_2^Q(x_1, x_2, \ldots, x_Q)$ equals i if and only if there exists a line $l_i \in Proj(q)$ such that for all $x_j = 1$ if and only if $j \in l_i$. Otherwise $F_2^Q(x_1, x_2, \ldots, x_Q) = Q + 1$.

Theorem 2. *(trivial) $D(F_1^Q) = Q$ and $D(F_2^Q) = Q$.*

Theorem 3. *$R(F_1^7) \leq 3$.*

Theorem 4. $R(F_1^Q) \leq q+1$.

Theorem 5. $R(F_2^Q) \leq q-1$.

R. Freivalds [11] introduced a new type of automata and algorithms, called ultrametric automata and ultrametric algorithms where p-adic numbers are used to replace real numbers, called probabilities, as measures of indeterminism. More detailed description of this notion can be found in [1]. The properties of ultrametric algorithms corresponding to distinct primes p may be surprisingly different.

Definition 11. *We say that an p–ultrametric decision tree computes f with bounded-error if for all $x \in \{0,1\}^n$ its output equals $f(x)$ with probability exceeding $\frac{1}{2}$.*
$U_p(f)$ denotes the complexity of the optimal p–ultrametric decision tree that computes f with bounded error.

Theorem 6. *For arbitrary odd prime number p the complexity $U_p(F_1^Q) \leq q+1$.*

Theorem 7. *For arbitrary odd prime number p the complexity $U_p(F_2^Q) \leq q-1$.*

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