The Logical Difference for EL: from Terminologies towards TBoxes*

Shasha Feng[1], Michel Ludwig[2], and Dirk Walther[2]

[1] Jilin University, China
fengss@jlu.edu.cn
[2] Theoretical Computer Science
TU Dresden, Germany
{michel,dirk}@tcs.inf.tu-dresden.de

Abstract. In this paper we are concerned with the logical difference problem between ontologies. The logical difference is the set of subsumption queries that follow from a first ontology but not from a second one. We revisit our solution to logical difference problem for $\mathcal{EL}$-terminologies based on finding simulations between hypergraph representations of the terminologies, and we investigate a possible extension of the method to general $\mathcal{EL}$-TBoxes.

1 Introduction

Ontologies are widely used to represent domain knowledge. They contain specifications of objects, concepts and relationships that are often formalised using a logic-based language over a vocabulary that is particular to an application domain. Ontology languages based on description logics [2] have been widely adopted, e.g., description logics are underlying the Web Ontology Language (OWL) and its profiles.[3] Numerous ontologies have already been developed, in particular, in knowledge intensive areas such as the biomedical domain, and they are made available in dedicated repositories such as the NCBO bioportal.[4]

Ontologies constantly evolve, they are regularly extended, corrected and refined. As the size of ontologies increases, their continued development and maintenance becomes more challenging as well. In particular, the need to have automated tool support for detecting and representing differences between versions of an ontology is growing in importance for ontology engineering.

The logical difference is taken to be the set of queries that produce different answers when evaluated over distinct versions of an ontology. The language and vocabulary of the queries can be adapted in such a way that exactly the differences of interest become visible, which can be independent of the syntactic representation of the ontologies. We consider ontologies formulated in the

* The second and third authors were supported by the German Research Foundation (DFG) within the Cluster of Excellence 'Center for Advancing Electronics Dresden'.

[3] http://www.w3.org/TR/owl2-overview/

[4] http://bioportal.bioontology.org

lightweight description logic $\mathcal{EL}$ [1, 3] and queries that are $\mathcal{EL}$-concept inclusions. The relevance of $\mathcal{EL}$ for ontologies is emphasised by the fact that many ontologies are largely formulated in $\mathcal{EL}$. For instance, the dataset of the ORE 2014 reasoner evaluation comprises 8 805 OWL-EL ontologies.[5]

The logical difference problem was introduced in [7] and investigated for $\mathcal{EL}$-terminologies [6]. A hypergraph-based approach for $\mathcal{EL}$-terminologies was presented in [4], which was subsequently extended to $\mathcal{EL}$-terminologies with additional role inclusions, domain and range restrictions of roles in [8]. In this paper we investigate a possible extension of the method to general $\mathcal{EL}$-TBoxes. Clearly, such an extension needs to account for the additional expressivity of general TBoxes w.r.t. terminologies. After normalisation, a terminology may contain at most one axiom of the form $\exists r.A \sqsubseteq X$ or $A_1 \sqcap \ldots \sqcap A_n \sqsubseteq X$ for any concept name X, whereas a general TBox does not impose such a restriction.

We first show that for every concept inclusion $C \sqsubseteq D$ that follows from a TBox $\mathcal{T}$, there exists a concept name X in $\mathcal{T}$ that acts as an *interpolant* between the concepts C and D, i.e., we have that $\mathcal{T} \models C \sqsubseteq X$ and $\mathcal{T} \models X \sqsubseteq D$. Then we describe the set of all subsumees C of X in $\mathcal{T}$ using a concept of $\mathcal{EL}$ extended with disjunction and a least fixpoint operator, and the set of all subsumers D of X in $\mathcal{T}$ using a concept of $\mathcal{EL}$ extended with greatest fixpoint operators. Finally, we reduce the problem of deciding the logical difference between two $\mathcal{EL}$-TBoxes to fixpoint reasoning w.r.t. TBoxes in a hybrid μ-calculus [10].

The paper is organised as follows. We start by recalling some notions regarding the description logic $\mathcal{EL}$ and its extensions with disjunction and fixpoint operators. In Section 3, we discuss how the logical difference problem for $\mathcal{EL}$-terminologies could be extended to general $\mathcal{EL}$-TBoxes, and we establish a witness theorem for general $\mathcal{EL}$-TBoxes. In Section 4, we show how fixpoint reasoning can be used to decide whether two general $\mathcal{EL}$-TBoxes are logically different, and how witnesses to the logical difference can be computed. Finally we conclude the paper.

2 Preliminaries

We start by briefly reviewing the lightweight description logic $\mathcal{EL}$ and some notions related to the logical difference, together with some basic results.

Let $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_V}$ be mutually disjoint sets of concept names, role names, and variable names, respectively. We assume these sets to be countably infinite. We typically use A, B to denote concept names and r to denote role names.

The sets of *$\mathcal{EL}$-concepts* C, *$\mathcal{ELU}_\mu$-concepts* D, and *$\mathcal{EL}_\nu$-concepts* E are built according to the following grammar rules:

$$C ::= \top \mid A \mid C \sqcap C \mid \exists r.C$$
$$D ::= \top \mid A \mid D \sqcap D \mid D \sqcup D \mid \exists r.D \mid x \mid \mu x.D$$
$$E ::= \top \mid A \mid E \sqcap E \mid \exists r.E \mid x \mid \nu x.E$$

[5] `http://dl.kr.org/ore2014/`

where $A \in \mathsf{N_C}$, $r, s \in \mathsf{N_R}$, and $x \in \mathsf{N_V}$. For an $\mathcal{ELU}_\mu$-concept C, the set of *free variables* in C, denoted by $\mathrm{FV}(C)$ is defined inductively as follows: $\mathrm{FV}(\top) = \emptyset$, $\mathrm{FV}(A) = \emptyset$, $\mathrm{FV}(D_1 \sqcap D_2) = \mathrm{FV}(D_1) \cup \mathrm{FV}(D_2)$, $\mathrm{FV}(D_1 \sqcup D_2) = \mathrm{FV}(D_1) \cup \mathrm{FV}(D_2)$, $\mathrm{FV}(\exists r.D') = \mathrm{FV}(D')$, $\mathrm{FV}(x) = \{x\}$, $\mathrm{FV}(\mu x.D') = \mathrm{FV}(D') \setminus \{x\}$. The set $\mathrm{FV}(E)$ of free variables occurring in an $\mathcal{EL}_\nu$-concept E can be defined analogously. An $\mathcal{ELU}_\mu$-concept C (an $\mathcal{EL}_\nu$-concept D) is *closed* if C (D) does not contain free occurrences of variables, i.e. $\mathrm{FV}(C) = \emptyset$ ($\mathrm{FV}(D) = \emptyset$). In the following we assume that every $\mathcal{ELU}_\mu$-concept C and every $\mathcal{EL}_\nu$-concept D is closed.

An $\mathcal{EL}$*-TBox* $\mathcal{T}$ is a finite set of axioms, where an axiom can be a *concept inclusion* $C \sqsubseteq C'$, or a *concept equation* $C \equiv C'$, where C, C' range over $\mathcal{EL}$-concepts. An $\mathcal{EL}$*-terminology* $\mathcal{T}$ is an $\mathcal{EL}$-TBox consisting of axioms α of the form $A \sqsubseteq C$ and $A \equiv C$, where A is a concept name, C an $\mathcal{EL}$-concept and no concept name A occurs more than once on the left-hand side of an axiom.

The semantics of $\mathcal{EL}$, $\mathcal{ELU}_\mu$, and $\mathcal{EL}_\nu$ is defined using interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where the domain $\Delta^\mathcal{I}$ is a non-empty set, and $\cdot^\mathcal{I}$ is a function mapping each concept name A to a subset $A^\mathcal{I}$ of $\Delta^\mathcal{I}$ and every role name r to a binary relation $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$. Interpretations are extended to concepts using a function $\cdot^{\mathcal{I},\xi}$ that is parameterised by an *assignment function* that maps variables $x \in \mathsf{N_V}$ to sets $\xi(x) \subseteq \Delta^\mathcal{I}$. Given an assignment ξ, the extension of an $\mathcal{EL}$, $\mathcal{ELU}_\mu$, or $\mathcal{EL}_\nu$-concept is defined inductively as follows: $\top^{\mathcal{I},\xi} := \Delta^\mathcal{I}$, $x^{\mathcal{I},\xi} := \xi(x)$ for $x \in \mathsf{N_V}$, $(C_1 \sqcap C_2)^{\mathcal{I},\xi} := C_1^\mathcal{I} \cap C_2^\mathcal{I}$, $(\exists r.C)^{\mathcal{I},\xi} := \{\, x \in \Delta^\mathcal{I} \mid \exists y \in C^{\mathcal{I},\xi} : (x, y) \in r^\mathcal{I} \,\}$, $(\mu x.C)^{\mathcal{I},\xi} = \bigcap\{\, W \subseteq \Delta^\mathcal{I} \mid C^{\mathcal{I},\xi[x \mapsto W]} \subseteq W \,\}$, and $(\nu x.C)^{\mathcal{I},\xi} = \bigcup\{\, W \subseteq \Delta^\mathcal{I} \mid W \subseteq C^{\mathcal{I},\xi[x \mapsto W]} \,\}$, where $\xi[x \mapsto W]$ denotes the assignment ξ modified by mapping x to W.

An interpretation $\mathcal{I}$ *satisfies* a concept C, an axiom $C \sqsubseteq D$ or $C \equiv D$ if, respectively, $C^{\mathcal{I},\emptyset} \neq \emptyset$, $C^{\mathcal{I},\emptyset} \subseteq D^\mathcal{I}$, or $C^{\mathcal{I},\emptyset} = D^{\mathcal{I},\emptyset}$. We write $\mathcal{I} \models \alpha$ iff $\mathcal{I}$ satisfies the axiom α. An interpretation $\mathcal{I}$ *satisfies* a TBox $\mathcal{T}$ iff $\mathcal{I}$ satisfies all axioms in $\mathcal{T}$; in this case, we say that $\mathcal{I}$ is a *model* of $\mathcal{T}$. An axiom α *follows* from a TBox $\mathcal{T}$, written $\mathcal{T} \models \alpha$, iff for all models $\mathcal{I}$ of $\mathcal{T}$, we have that $\mathcal{I} \models \alpha$. Deciding whether $\mathcal{T} \models C \sqsubseteq C'$, for two $\mathcal{EL}$-concepts C and C', can be done in polynomial time in the size of $\mathcal{T}$ and C, C' [1, 3]. For an $\mathcal{ELU}_\mu$-concept D and an $\mathcal{EL}_\nu$-concept E, it is known that $\mathcal{T} \models D \sqsubseteq E$ can be decided in exponential time in the size of $\mathcal{T}$, D and E [10].

A signature Σ is a finite set of symbols from $\mathsf{N_C}$ and $\mathsf{N_R}$. The signature $\mathsf{sig}(C)$, $\mathsf{sig}(\alpha)$ or $\mathsf{sig}(\mathcal{T})$ of the concept C, axiom α or TBox $\mathcal{T}$ is the set of concept and role names occurring in C, α or $\mathcal{T}$, respectively. An $\mathcal{EL}_\Sigma$-concept C is an $\mathcal{EL}$-concept such that $\mathsf{sig}(C) \subseteq \Sigma$.

An $\mathcal{EL}$-TBox $\mathcal{T}$ is *normalised* if it only contains $\mathcal{EL}$-concept inclusions of the forms $\top \sqsubseteq B$, $A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B$, $A \sqsubseteq \exists r.B$, or $\exists r.A \sqsubseteq B$, where $A, A_i, B \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and $n \geq 1$. Every $\mathcal{EL}$-TBox $\mathcal{T}$ can be normalised in polynomial time in the size of $\mathcal{T}$ with a linear increase in the size of the normalised TBox w.r.t. $\mathcal{T}$ such that the resulting TBox is a conservative extension of $\mathcal{T}$ [6]. Note that in a normalised terminology $\mathcal{T}$, we have that for every axiom of the form $\exists r.A \sqsubseteq B \in \mathcal{T}$, there exists an axiom of the form $B \sqsubseteq \exists r.A \in \mathcal{T}$; similarly for axioms of the form $A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B$ with $n \geq 2$. When convenient, we will abbreviate two axioms $A \sqsubseteq \exists r.B$ and $\exists r.B \sqsubseteq A$ by the single axiom $A \equiv \exists r.B$; similarly for $A \equiv B_1 \sqcap \ldots \sqcap B_n$.

3 Towards Logical Difference between General $\mathcal{EL}$-TBoxes

The logical difference between two TBoxes witnessed by concept inclusions over a signature Σ is defined as follows.

Definition 1. *The* Σ-concept difference *between two* $\mathcal{EL}$*-TBoxes* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *for a signature* Σ *is the set* $\mathsf{cDiff}_\Sigma(\mathcal{T}_1, \mathcal{T}_2)$ *of all* $\mathcal{EL}$*-concept inclusions* α *such that* $\mathsf{sig}(\alpha) \subseteq \Sigma$*,* $\mathcal{T}_1 \models \alpha$*, and* $\mathcal{T}_2 \not\models \alpha$*.*

$\mathcal{EL}$-TBoxes can be translated into directed hypergraphs by taking the signature symbols as nodes and treating the axioms as hyperedges connecting the nodes. For normalised $\mathcal{EL}$-TBoxes, the axiom $\top \sqsubseteq B$ is translated into the hyperedge $(\{x_\top\}, \{x_B\})$, the axiom $A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B$ into the hyperedge $(\{x_{B_1}, \ldots, x_{B_n}\}, \{x_A\})$, the axiom $A \sqsubseteq \exists r.B$ into the hyperedge $(\{x_A\}, \{x_r, x_B\})$, and the axiom $\exists r.A \sqsubseteq B$ into the hyperedge $(\{x_r, x_B\}, \{x_A\})$, where each node x_Y corresponds to the signature symbol Y, respectively. A feature of the translation of axioms into hyperedges is that all information about the axiom and the logical operators in it is preserved. In fact we can treat the ontology and its hypergraph representation interchangeably. The existence of certain simulations between hypergraphs for $\mathcal{EL}$-terminologies characterises the fact that the corresponding terminologies are logically equivalent and, thus, no logical difference exists [4,8].

As the set $\mathsf{cDiff}_\Sigma(\mathcal{T}_1, \mathcal{T}_2)$ is infinite in general, we make use of the following "primitive witnesses" theorem from [6] that states that we only have to consider two specific types of concept differences. If there is an inclusion $C \sqsubseteq D \in \mathsf{cDiff}_\Sigma(\mathcal{T}_1, \mathcal{T}_2)$ for two terminologies $\mathcal{T}_1$ and $\mathcal{T}_2$, then we know that there is a concept name $A \in \Sigma$ such that A occurs either on the left-hand or the right-hand side of an inclusion in the set $C \sqsubseteq D \in \mathsf{cDiff}_\Sigma(\mathcal{T}_1, \mathcal{T}_2)$. For checking whether $\mathsf{cDiff}_\Sigma(\mathcal{T}_1, \mathcal{T}_2) = \emptyset$, we only have to consider such simple inclusions. However, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are general $\mathcal{EL}$-TBoxes, the situation is different.

Example 1. Let $\mathcal{T}_1 = \{X \equiv A_1 \sqcap A_2, X \sqsubseteq \exists r.\top\}$, $\mathcal{T}_2 = \emptyset$, and let $\Sigma = \{A_1, A_2, r\}$. Note that $\mathcal{T}_1$ is not a terminology as the concept name X occurs twice on the left-hand side of an axiom. Then every difference $\alpha \in \mathsf{cDiff}_\Sigma(\mathcal{T}_1, \mathcal{T}_2)$ is equivalent to the inclusion $A_1 \sqcap A_2 \sqsubseteq \exists r.\top$. In particular, there does not exist a difference of the form $\psi \sqsubseteq \theta$, where ψ or θ is a concept name from Σ.

As illustrated by the example, we need to account for a new kind of differences $C \sqsubseteq C' \in \mathsf{cDiff}_\Sigma(\mathcal{T}_1, \mathcal{T}_2)$ which are induced by a concept name $X \in \mathsf{sig}(\mathcal{T}_1)$ such that $X \notin \Sigma$, $\mathcal{T}_1 \models C \sqsubseteq X$, and $\mathcal{T}_1 \models X \sqsubseteq C'$. We obtain the following witness theorem for $\mathcal{EL}$-TBoxes as an extension of the witness theorem for $\mathcal{EL}$-terminologies.

Theorem 1 (Witness Theorem). *Let $\mathcal{T}_1$, $\mathcal{T}_2$ be two normalised $\mathcal{EL}$-TBoxes and let Σ be a signature. Then,* $\mathsf{cDiff}_\Sigma(\mathcal{T}_1,\mathcal{T}_2) \neq \emptyset$ *iff there exists an $\mathcal{EL}_\Sigma$-inclusion $\alpha = \varphi \sqsubseteq \psi$ such that $\mathcal{T}_1 \models \alpha$ and $\mathcal{T}_2 \not\models \alpha$, where*

(i) φ is an $\mathcal{EL}_\Sigma$-concept and $\psi = A \in \Sigma$,
(ii) $\varphi = A \in \Sigma$ and ψ is an $\mathcal{EL}_\Sigma$-concept, or
(iii) there exists $X \in \mathsf{sig}(\mathcal{T}_1) \setminus \Sigma$ such that $\mathcal{T}_1 \models \varphi \sqsubseteq X$ and $\mathcal{T}_1 \models X \sqsubseteq \psi$.

The proof of the witness theorem for terminologies [6] is based on analysing the subsumption $\mathcal{T}_1 \models \varphi \sqsubseteq \psi$ syntactically, using a sequent calculus [5]. A similar technique can be used for the proof of Theorem 2.

For deciding whether $\mathsf{cDiff}_\Sigma(\mathcal{T}_1,\mathcal{T}_2) = \emptyset$ in the case of general TBoxes, we now have to additionally consider differences of Type (*iii*). Differences of types (*i*) or (*ii*) can be checked by using forward or backward simulations adapted to normalised $\mathcal{EL}$-TBoxes, respectively, whereas Type (*iii*) differences require a combination of both techniques.

Before we illustrate how Type (*iii*) differences can be dealt with, we first introduce some auxiliary notions. We define $\mathsf{cWtn}^{\mathsf{lhs}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2)$ as the set of all concept names A from Σ such that there exists an $\mathcal{EL}_\Sigma$-concept C with $A \sqsubseteq C \in \mathsf{cDiff}_\Sigma(\mathcal{T}_1,\mathcal{T}_2)$. Similarly, $\mathsf{cWtn}^{\mathsf{rhs}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2)$ is the set of all concept names $A \in \Sigma$ such that there exists an $\mathcal{EL}_\Sigma$-concept C with $C \sqsubseteq A \in \mathsf{cDiff}_\Sigma(\mathcal{T}_1,\mathcal{T}_2)$. The concept names in $\mathsf{cWtn}^{\mathsf{lhs}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2)$ are called *left-hand side witnesses* and the concept names in $\mathsf{cWtn}^{\mathsf{rhs}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2)$ *right-hand side witnesses.* Additionally, we define $\mathsf{cWtn}^{\mathsf{mid}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2)$ as the set of all concept names X from $\mathsf{sig}(\mathcal{T}_1)$ but not from Σ such that there exists $C \sqsubseteq C' \in \mathsf{cDiff}_\Sigma(\mathcal{T}_1,\mathcal{T}_2)$ and $\mathcal{T}_1 \models C \sqsubseteq X$ and $\mathcal{T}_1 \models X \sqsubseteq C'$. The concept names in $\mathsf{cWtn}^{\mathsf{mid}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2)$ are called *interpolating witnesses.* To summarise, we have the following sets:

$$\begin{aligned}
\mathsf{cWtn}^{\mathsf{lhs}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2) &= \{\, A \in \Sigma \mid \exists C \in \mathcal{EL}_\Sigma \colon A \sqsubseteq C \in \mathsf{cDiff}_\Sigma(\mathcal{T}_1,\mathcal{T}_2) \,\}\\
\mathsf{cWtn}^{\mathsf{rhs}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2) &= \{\, A \in \Sigma \mid \exists C \in \mathcal{EL}_\Sigma \colon C \sqsubseteq A \in \mathsf{cDiff}_\Sigma(\mathcal{T}_1,\mathcal{T}_2) \,\}\\
\mathsf{cWtn}^{\mathsf{mid}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2) &= \{\, X \in \mathsf{sig}(\mathcal{T}_1) \setminus \Sigma \mid \exists C, C' \in \mathcal{EL}_\Sigma \colon \mathcal{T}_1 \models C \sqsubseteq X,\\
&\qquad\qquad \mathcal{T}_1 \models X \sqsubseteq C', \mathcal{T}_2 \not\models C \sqsubseteq C' \,\}
\end{aligned}$$

We illustrate the witness sets with the following example.

Example 2. Let $\mathcal{T}_1 = \{X \equiv A_1 \sqcap A_2,\ X \sqsubseteq \exists r.\top, A_3 \sqsubseteq A_2, A_3 \sqsubseteq \exists r.A_2\}$, $\mathcal{T}_2 = \{A_2 \sqsubseteq \exists r.\top\}$, and let $\Sigma = \{A_1, A_2, r\}$. Then it holds that $\mathsf{cWtn}^{\mathsf{lhs}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2) = \{A_3\}$ (e.g. $\{A_3 \sqsubseteq A_2, A_3 \sqsubseteq \exists r.A_2\} \subseteq \mathsf{cDiff}_\Sigma(\mathcal{T}_1,\mathcal{T}_2)$), $\mathsf{cWtn}^{\mathsf{rhs}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2) = \{A_2\}$ (e.g. $A_3 \sqsubseteq A_2 \in \mathsf{cDiff}_\Sigma(\mathcal{T}_1,\mathcal{T}_2)$), and $\mathsf{cWtn}^{\mathsf{mid}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2) = \{X\}$ (e.g. $\mathcal{T}_1 \models A_1 \sqcap A_3 \sqsubseteq X$, $\mathcal{T}_1 \models X \sqsubseteq \exists r.\top$ and $\mathcal{T}_2 \not\models A_1 \sqcap A_3 \sqsubseteq \exists r.\top$).

We obtain as a corollary of Theorem 2 that, to decide the logical difference between two $\mathcal{EL}$-TBoxes, it is sufficient to check the emptiness of the witness sets.

Corollary 1. *Let $\mathcal{T}_1, \mathcal{T}_2$ be two normalised $\mathcal{EL}$-TBoxes and let Σ be a signature. Then it holds that* $\mathsf{cDiff}_\Sigma(\mathcal{T}_1,\mathcal{T}_2) = \emptyset$ *iff* $\mathsf{cWtn}^{\mathsf{lhs}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2) = \mathsf{cWtn}^{\mathsf{rhs}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2) = \mathsf{cWtn}^{\mathsf{mid}}_\Sigma(\mathcal{T}_1,\mathcal{T}_2) = \emptyset$.

To characterise an interpolating witness of a Type-(*iii*) difference, we use the sets of its of subsumees and subsumers formulated using certain signature symbols only. A similar approach was used for the construction of uniform interpolants of $\mathcal{EL}$-TBoxes in [9].

Definition 2. *Let* $\mathcal{T}$ *be an* $\mathcal{EL}$*-TBox, let* Σ *be a signature and let* C *be an* $\mathcal{EL}$*-concept. We define* $\mathrm{Premises}^{\Sigma}_{\mathcal{T}}(C) := \{ E \in \mathcal{EL}_{\Sigma} \mid \mathcal{T} \models E \sqsubseteq C \}$ *and* $\mathrm{Conclusions}^{\Sigma}_{\mathcal{T}}(C) := \{ E \in \mathcal{EL}_{\Sigma} \mid \mathcal{T} \models C \sqsubseteq E \}$.

The set $\mathrm{Premises}^{\Sigma}_{\mathcal{T}}(C)$ contains all $\mathcal{EL}$-concepts formulated using Σ-symbols only that entail C w.r.t. $\mathcal{T}$; or are entailed by C in the case of $\mathrm{Conclusions}^{\Sigma}_{\mathcal{T}}(C)$. The elements of $\mathrm{Premises}^{\Sigma}_{\mathcal{T}}(C)$ are also called Σ-implicants or Σ-subsumees of C w.r.t. $\mathcal{T}$, and the elements of $\mathrm{Conclusions}^{\Sigma}_{\mathcal{T}}(C)$ are also named Σ-implicates or Σ-subsumers of C w.r.t. $\mathcal{T}$.

In [4], it was established that a concept name X is forward simulated by a concept name Y in an $\mathcal{EL}$-terminology $\mathcal{T}$ iff it holds that $\mathrm{Conclusions}^{\Sigma}_{\mathcal{T}}(X) \subseteq \mathrm{Conclusions}^{\Sigma}_{\mathcal{T}}(Y)$; and similary, X is backward simulated by Y iff $\mathrm{Premises}^{\Sigma}_{\mathcal{T}}(X) \subseteq \mathrm{Premises}^{\Sigma}_{\mathcal{T}}(Y)$. We aim now at lifting this result to general $\mathcal{EL}$-TBoxes.

Example 3. Let $\mathcal{T}_1 = \{A \sqsubseteq X, \exists r.X \sqsubseteq X, X \sqsubseteq B_1, X \sqsubseteq B_2\}$ and $\mathcal{T}_2 = \{A \sqsubseteq Y, \exists r.Y \sqsubseteq Y', \exists r.Y' \sqsubseteq Y, \exists r.Y \sqsubseteq Z_1, \exists r.Y \sqsubseteq Z_2, Y \sqsubseteq B_1, Y \sqsubseteq B_2, Z_1 \sqsubseteq B_1, Z_2 \sqsubseteq B_2\}$ be two $\mathcal{EL}$-TBoxes. Let $\Sigma = \{A, B_1, B_2, r\}$ be a signature. Note that X in $\mathcal{T}_1$ is cyclic and intuitively, the interpretation of X in a model $\mathcal{I}$ of $\mathcal{T}_1$ contains all finite r-chains "ending in A". In $\mathcal{T}_2$ the concept name Y is cyclic and its interpretation contains all r-chains ending in A that are of *even* length, whereas the interpretations of the concept names Z_1 and Z_2 contain all r-chains ending in A that are of *odd* length. Formally, we have that:

$$\{A, \exists r.A, \exists r.\exists r.A, \dots\} \subseteq \mathrm{Premises}^{\Sigma}_{\mathcal{T}_1}(X)$$
$$\{A,\ \exists r.\exists r.A,\ \exists r.\exists r.\exists r.\exists r.A, \dots\} \subseteq \mathrm{Premises}^{\Sigma}_{\mathcal{T}_2}(Y)$$
$$\{\exists r.A,\ \exists r.\exists r.\exists r.A, \dots\} \subseteq \mathrm{Premises}^{\Sigma}_{\mathcal{T}_2}(Z_i) \text{ for } i \in \{1,2\}$$

In particular, for $i \in \{1,2\}$, we have

$$\mathrm{Premises}^{\Sigma}_{\mathcal{T}_1}(X) = \mathrm{Premises}^{\Sigma}_{\mathcal{T}_2}(Y) \cup \mathrm{Premises}^{\Sigma}_{\mathcal{T}_2}(Z_i).$$

Intuitively, the set of Σ-implicants of X in $\mathcal{T}_1$ are distributed over the concept names Y and Z_i in $\mathcal{T}_2$. Moreover it holds that

$$\mathrm{Conclusions}^{\Sigma}_{\mathcal{T}_1}(X) = \mathrm{Conclusions}^{\Sigma}_{\mathcal{T}_2}(Y) = \mathrm{Conclusions}^{\Sigma}_{\mathcal{T}_2}(Z_1 \sqcap Z_2).$$

The concept name X in $\mathcal{T}_1$ could be forward simulated either by Y or $Z_1 \sqcap Z_2$ in $\mathcal{T}_2$. Note that Z_1 or Z_2 individually are not sufficient. Analogously, X could be backward simulated by $Y \sqcup Z_1$ or $Y \sqcup Z_2$. None of the concept names X, Z_1, or Z_2 are sufficient individually for the backward simulation. Combining backward and forward simulation, X could be simulated by $Y \sqcup (Z_1 \sqcap Z_2)$.

In general, we hypothesise that non-Σ-concept names X in $\mathcal{T}_1$ need to be "simulated" by concepts of the form $\bigsqcup_{i=1}^{n} C_i$, where C_i are $\mathcal{EL}$-concepts.

4 Finding Logical Differences via Fixpoint Reasoning

We now show how fixpoint reasoning can be used to find difference witnesses between general $\mathcal{EL}$-TBoxes.

Given Theorem 2, we know that any difference $C \sqsubseteq C' \in \mathsf{cDiff}_\Sigma(\mathcal{T}_1, \mathcal{T}_2)$, for two $\mathcal{EL}_\Sigma$-concepts C and C', is connected to some concept name X occurring in $\mathcal{T}_1$ for which either $\mathcal{T}_1 \models C \sqsubseteq X$, or $\mathcal{T}_1 \models X \sqsubseteq C'$ (or both) holds. To check whether X is indeed a difference witness, we construct concepts $\mathrm{B}^\Sigma_{\mathcal{T}_1}(X)$ and $\mathrm{F}^\Sigma_{\mathcal{T}_1}(X)$ formulated in $\mathcal{ELU}^\Sigma_\mu$ and in $\mathcal{EL}^\Sigma_\nu$, respectively, to describe the (potentially infinite) disjunction of Σ-concepts that are subsumed by X w.r.t. $\mathcal{T}_1$, and the conjunction of all the Σ-concepts that subsume X w.r.t. $\mathcal{T}_1$, respectively. Note that the use of fixpoint allows for a finite description of infinite disjunctions or conjunctions. The $\mathcal{ELU}^\Sigma_\mu$-concept $\mathrm{B}^\Sigma_{\mathcal{T}_1}(X)$ hence is a finite representation of the set $\mathrm{Premises}^\Sigma_{\mathcal{T}_1}(X)$, whereas the $\mathcal{EL}^\Sigma_\nu$-concept $\mathrm{F}^\Sigma_{\mathcal{T}_1}(X)$ represents the set $\mathrm{Premises}^\Sigma_{\mathcal{T}_1}(X)$ in a finite way. Using the fixpoint descriptions of the premises and conclusions of X w.r.t. $\mathcal{T}_1$, we can verify whether X is a difference witness by checking $\mathcal{T}_2 \models \mathrm{B}^\Sigma_{\mathcal{T}_1}(X) \sqsubseteq \mathrm{F}^\Sigma_{\mathcal{T}_1}(X)$.

We first turn our attention to the set $\mathrm{Premises}^\Sigma_{\mathcal{T}_1}(X)$. Before we can give a formal definition for the concept $\mathrm{B}^\Sigma_{\mathcal{T}_1}(X)$, we have to introduce the following auxiliary notion to handle concept names X in the definition of $\mathrm{B}^\Sigma_{\mathcal{T}_1}(X)$ for which there exist axioms of the form $Z_1 \sqcap \ldots \sqcap Z_n \sqsubseteq Z$ in a normalised TBox $\mathcal{T}$ such that $\mathcal{T} \models Z \sqsubseteq X$. Intuitively, given a concept name X, we construct a set $\mathrm{Conj}_\mathcal{T}(X)$ containing sets of concept names which has the property that for every $\mathcal{EL}$-concept D with $\mathcal{T} \models D \sqsubseteq X$, there exists a set $S = \{Y_1, \ldots, Y_m\} \in \mathrm{Conj}_\mathcal{T}(X)$ such that $\mathcal{T} \models D \sqsubseteq Y_i$ follows without involving any axioms of the form $Z_1 \sqcap \ldots \sqcap Z_n \sqsubseteq Z$. Nested implications between such axioms also have to be taken into account.

Definition 3. *Let $\mathcal{T}$ be a normalised $\mathcal{EL}$-TBox and let $X \in \mathsf{N_C}$. We define the set $Conj_\mathcal{T}(X) \subseteq 2^{\mathsf{sig}(\mathcal{T}) \cap \mathsf{N_C}}$ to be smallest set inductively defined as follows:*

- *$\{X\} \in Conj_\mathcal{T}(X)$;*
- *if $S \in Conj_\mathcal{T}(X)$, $Y \in S$, and $Z_1 \sqcap \ldots \sqcap Z_n \sqsubseteq Z \in \mathcal{T}$ such that $n \geq 2$ and $\mathcal{T} \models Z \sqsubseteq Y$, then $S \setminus \{Y\} \cup \{Z_1, \ldots, Z_n\} \in Conj_\mathcal{T}(X)$.*

Note that for every concept name X the set $\mathrm{Conj}_\mathcal{T}(X)$ is finite as $\mathsf{sig}(\mathcal{T}) \cap \mathsf{N_C}$ is finite.

Example 4. Let $\mathcal{T}_1 = \{A \sqsubseteq X, \exists r.X \sqsubseteq X\}$. Then $\mathrm{Conj}_{\mathcal{T}_1}(X) = \{\{X\}\}$. For $\mathcal{T}_2 = \{X_1 \sqcap X_2 \sqsubseteq X,\ X_3 \sqcap X_4 \sqsubseteq X_1,\ Y_1 \sqcap Y_2 \sqsubseteq X\}$, we have that

$$\mathrm{Conj}_{\mathcal{T}_2}(X) = \{\{X\}, \{X_1, X_2\}, \{X_3, X_4, X_2\}, \{Y_1, Y_2\}\}.$$

We can now give a formal definition of the concept $\mathrm{B}^\Sigma_\mathcal{T}(X)$.

Definition 4. *Let $\mathcal{T}$ be a normalised $\mathcal{EL}$-TBox, let Σ be a signature, and let $X \in \mathsf{sig}(\mathcal{T})$. For a mapping $\eta\colon \mathsf{N_C} \to \mathsf{N_V}$, we define a closed $\mathcal{ELU}^\mu_\Sigma$-concept $B^\Sigma_\mathcal{T}(X, \eta)$ as follows. We set $B^\Sigma_\mathcal{T}(X, \eta) = \top$ if $\mathcal{T} \models \top \sqsubseteq X$; otherwise $B^\Sigma_\mathcal{T}(X, \eta)$ is defined recursively in the following way:*

– *If* $X \in \mathsf{dom}(\eta)$*, then*

$$B^{\Sigma}_{\mathcal{T}}(X, \eta) = \eta(X)$$

– *If* $X \notin \mathsf{dom}(\eta)$*, we set*

$$B^{\Sigma}_{\mathcal{T}}(X, \eta) = \mu x. \bigsqcup_{\substack{S \in Conj_{\mathcal{T}}(X) \\ S = \{Y_1, \ldots, Y_m\}}} (Y'_1 \sqcap \ldots \sqcap Y'_m)$$

where x *is a fresh variable, and* Y'_i $(1 \leq i \leq n)$ *is defined as follows for* $\eta' := \eta \cup \{X \mapsto x\}$*:*

$$Y'_i = \bigsqcup_{\substack{\mathcal{T} \models B \sqsubseteq Y_i \\ B \in \Sigma}} B \sqcup \bigsqcup_{\substack{\exists r.Z \sqsubseteq Y \in \mathcal{T} \\ r \in \Sigma \\ \mathcal{T} \models Y \sqsubseteq Y_i}} \exists r.B^{\Sigma}_{\mathcal{T}}(Z, \eta')$$

Finally, we set $B^{\Sigma}_{\mathcal{T}}(X) = B^{\Sigma}_{\mathcal{T}}(X, \emptyset)$*.*

Intuitively, the construction of $\mathrm{B}^{\Sigma}_{\mathcal{T}}(X)$ starts from X and recursively collects all the concept names contained in Σ and all the left-hand sides of axioms in $\mathcal{T}$ that could be relevant for X to be entailed by a concept w.r.t. $\mathcal{T}$. By taking into account all possible axioms that could lead to a logical entailment, it is guaranteed that we capture every Σ-concept from which X follows w.r.t. $\mathcal{T}$. Reasoning involving axioms of the form $Z_1 \sqcap \ldots \sqcap Z_n \sqsubseteq Z$ is handled by the set $\mathrm{Conj}_{\mathcal{T}}(X)$. Infinite recursion over concepts of the form $\exists r.C$ is avoided by keeping track of the concept names that been visited already using the mapping η.

We note that for a normalised $\mathcal{EL}$-terminology $\mathcal{T}$, the concept $\mathrm{B}^{\Sigma}_{\mathcal{T}}(X)$ is of a simpler form than for normalised $\mathcal{EL}$-TBoxes. This is because the concept name X can occur on the right-hand side of at most one axiom of the form $\exists r.A \sqsubseteq X$ or $A_1 \sqcap \ldots \sqcap A_n \sqsubseteq X$ with $n \geq 2$ in $\mathcal{T}$, whereas in a TBox several such axioms may occur.

We illustrate the concept $\mathrm{B}^{\Sigma}_{\mathcal{T}}(X)$ with the following examples.

Example 5. Let $\mathcal{T}_1 = \{A_1 \sqcap A_2 \sqsubseteq X,\ A_3 \sqsubseteq A_2, \exists r.A_2 \sqsubseteq A_1, \exists r.A_2 \sqsubseteq X\}$, $\mathcal{T}_2 = \mathcal{T}_1 \cup \{\exists r.X \sqsubseteq A_2\}$, and let $\Sigma = \{A_1, A_2, A_3, r\}$. We obtain the following $\mathcal{ELU}^{\mu}_{\Sigma}$-concepts. We write φ instead of $\mu x.\varphi$ if x does not occur freely in φ.

$$\mathrm{B}^{\Sigma}_{\mathcal{T}_1}(A_1) = A_1 \sqcup \exists r.(A_2 \sqcup A_3) \qquad \mathrm{B}^{\Sigma}_{\mathcal{T}_1}(A_2) = A_2 \sqcup A_3$$
$$\mathrm{B}^{\Sigma}_{\mathcal{T}_1}(X) = ((A_1 \sqcup \exists r.(A_2 \sqcup A_3)) \sqcap (A_2 \sqcup A_3)) \sqcup \exists r.(A_2 \sqcup A_3)$$
$$\mathrm{B}^{\Sigma}_{\mathcal{T}_2}(X) = \mu x.(((A_1 \sqcup \exists r.(A_2 \sqcup A_3 \sqcup \exists r.x)) \sqcap (A_2 \sqcup A_3 \sqcup \exists r.x)) \sqcup \exists r.(A_2 \sqcup A_3 \sqcup \exists r.x))$$

Example 6. Let $\mathcal{T}_1$, $\mathcal{T}_2$ be defined as in Example 3, and let $\Sigma = \{A, B_1, B_2, r\}$. We have that for $i \in \{1, 2\}$:

$$\mathrm{B}^{\Sigma}_{\mathcal{T}_1}(X) = \mu x.(A \sqcup \exists r.x) \qquad \mathrm{B}^{\Sigma}_{\mathcal{T}_1}(B_i) = B_i \sqcup A \sqcup \exists r.\mu x.(A \sqcup \exists r.x)$$
$$\mathrm{B}^{\Sigma}_{\mathcal{T}_2}(Y) = \mu y.(A \sqcup \exists r.\exists r.y) \qquad \mathrm{B}^{\Sigma}_{\mathcal{T}_2}(Z_i) = \exists r.\mu y.(A \sqcup \exists r.\exists r.y)$$
$$\mathrm{B}^{\Sigma}_{\mathcal{T}_2}(B_i) = B_i \sqcup A \sqcup \exists r.\mu y_1.(\exists r.(A \sqcup \exists r.y_1)) \sqcup \exists r.\mu y_2.(A \sqcup \exists r.\exists r.y_2)$$

By inspecting Definition 4 it is easy to see that $\models \mathrm{B}_{\mathcal{T}}^{\Sigma}(X) \equiv \bot$ if there does not an $\mathcal{EL}_{\Sigma}$-concept C with $\mathcal{T} \models C \sqsubseteq X$. Overall, one can establish the following correctness and completeness properties.

Lemma 1. *Let $\mathcal{T}$ be a normalised $\mathcal{EL}$-TBox, let Σ be a signature, and let $X \in \mathsf{sig}(\mathcal{T})$. Then the $\mathcal{ELU}_{\mu}^{\Sigma}$-concept $B_{\mathcal{T}}^{\Sigma}(X)$ satisfies the following properties:*

(i) $\mathcal{T} \models B_{\mathcal{T}}^{\Sigma}(X) \sqsubseteq X$, *and*
(ii) *for every* $D \in \mathrm{Premises}_{\mathcal{T}}^{\Sigma}(X)$,

$$\mathcal{T} \models D \sqsubseteq X \text{ iff } \models D \sqsubseteq B_{\mathcal{T}}^{\Sigma}(X).$$

The following lemma states that the $\mathcal{ELU}_{\mu}$-concept $\mathrm{B}_{\mathcal{T}}^{\Sigma}(X)$ exactly captures the infinite set $\mathrm{Premises}_{\mathcal{T}}^{\Sigma}(X)$. More formally, the concept $\mathrm{B}_{\mathcal{T}}^{\Sigma}(X)$ is equivalent to the infinite disjunction over all the concepts contained in the set $\mathrm{Premises}_{\mathcal{T}}^{\Sigma}(X)$.

Lemma 2. *Let $\mathcal{T}$ be a normalised $\mathcal{EL}$-TBox, let Σ be a signature, and let $X \in \mathsf{sig}(\mathcal{T})$. Then for every interpretation $\mathcal{I}$ it holds that*

$$(B_{\mathcal{T}}^{\Sigma}(X))^{\mathcal{I},\emptyset} = \bigcup \{\, C^{\mathcal{I},\emptyset} \mid C \in \mathrm{Premises}_{\mathcal{T}}^{\Sigma}(X) \,\}.$$

Analogously to the concept $\mathrm{B}_{\mathcal{T}}^{\Sigma}(X)$, it is possible to construct an $\mathcal{EL}_{\nu}^{\Sigma}$-concept $\mathrm{F}_{\mathcal{T}}^{\Sigma}(X)$ which exactly captures the set $\mathrm{Conclusions}_{\mathcal{T}}^{\Sigma}(X)$ for a concept name X and an $\mathcal{EL}$-TBox $\mathcal{T}$. Due to lack of space, we cannot give a full definition of the concept $\mathrm{F}_{\mathcal{T}}^{\Sigma}(X)$. Instead, we state its existence and its essential property in the following lemma.

Lemma 3. *Let $\mathcal{T}$ be a normalised $\mathcal{EL}$-TBox, let Σ be a signature, and let $X \in \mathsf{sig}(\mathcal{T})$. Then there exists an $\mathcal{EL}_{\nu}$-concept $F_{\mathcal{T}}^{\Sigma}(X)$ such that for every interpretation $\mathcal{I}$ it holds that*

$$(F_{\mathcal{T}}^{\Sigma}(X))^{\mathcal{I},\emptyset} = \bigcup \{\, C^{\mathcal{I},\emptyset} \mid C \in \mathrm{Conclusions}_{\mathcal{T}}^{\Sigma}(X) \,\}.$$

We can now state the following theorem, which establishes how the concepts $\mathrm{B}_{\mathcal{T}}^{\Sigma}(X)$ and $\mathrm{F}_{\mathcal{T}}^{\Sigma}(X)$ can be used to search for difference witnesses.

Theorem 2. *Let $\mathcal{T}_1$, $\mathcal{T}_2$ be two normalised $\mathcal{EL}$-TBoxes. Then it holds that:*

(i) $A \notin \mathsf{cWtn}_{\Sigma}^{\mathsf{lhs}}(\mathcal{T}_1, \mathcal{T}_2)$ *iff* $\mathcal{T}_2 \models A \sqsubseteq F_{\mathcal{T}_1}^{\Sigma}(A)$, *for every* $A \in \Sigma$;
(ii) $A \notin \mathsf{cWtn}_{\Sigma}^{\mathsf{rhs}}(\mathcal{T}_1, \mathcal{T}_2)$ *iff* $\mathcal{T}_2 \models B_{\mathcal{T}_1}^{\Sigma}(A) \sqsubseteq A$, *for every* $A \in \Sigma$; *and*
(iii) $X \notin \mathsf{cWtn}_{\Sigma}^{\mathsf{mid}}(\mathcal{T}_1, \mathcal{T}_2)$ *iff* $\mathcal{T}_2 \models B_{\mathcal{T}_1}^{\Sigma}(X) \sqsubseteq F_{\mathcal{T}_1}^{\Sigma}(X)$, *for every* $X \in \mathsf{sig}(\mathcal{T}_1) \setminus \Sigma$.

Theorem 2 together with Corollary 1 give rise to an algorithm for deciding the logical difference between $\mathcal{EL}$-TBoxes. Procedure 1 is such an algorithm based on reasoning in the hybrid μ-calculus, which allows for fixpoint reasoning w.r.t. TBoxes [10].

Theorem 3. *Procedure 1 runs in ExpTime.*

Procedure 1 Deciding existence of logical difference

Input: Normalised $\mathcal{EL}$-TBoxes $\mathcal{T}_1$, $\mathcal{T}_2$ and signature Σ

Output: **true** or **false**

for every concept name $X \in \mathsf{sig}(\mathcal{T}_1) \cup \Sigma$ **do**
  $\mathcal{B} := \mathrm{B}^{\Sigma}_{\mathcal{T}_1}(X)$
  $\mathcal{F} := \mathrm{F}^{\Sigma}_{\mathcal{T}_1}(X)$
  if $X \in \Sigma$ **then**
    if $\mathcal{T}_2 \not\models X \sqsubseteq \mathcal{F}$ **or** $\mathcal{T}_2 \not\models \mathcal{B} \sqsubseteq X$ **then**
      return true
    end if
  else if $\mathcal{T}_2 \not\models \mathcal{B} \sqsubseteq \mathcal{F}$ **then**
    return true
  end if
end for
return false

We note that the lower bound for the running time of Procedure 1 may also be exponential as the underlying problem of deciding the logical difference of two $\mathcal{EL}$-TBoxes is ExpTime-complete [6, 7].

Example 7. Continue Example 3, where $\mathsf{sig}(\mathcal{T}_1) \cup \Sigma = \{A, X, B_1, B_2, r\}$. For A, $\mathcal{B} = \mathrm{B}^{\Sigma}_{\mathcal{T}_1}(A) = A$, and $\mathcal{F} = \mathrm{F}^{\Sigma}_{\mathcal{T}_1}(A) = B_1 \sqcap B_2$. As $\mathcal{T}_2 \models A \sqsubseteq \mathcal{F}$ and $\mathcal{T}_2 \models \mathcal{B} \sqsubseteq A$, the loop continues. Then for X, $\mathcal{B} = \mathrm{B}^{\Sigma}_{\mathcal{T}_1}(X) = \mu x.(A \sqcup \exists r.x)$ and $\mathcal{F} = \mathrm{F}^{\Sigma}_{\mathcal{T}_1}(X) = B_1 \sqcap B_2$ (cf. Example 6). Since it holds that $\mathcal{T}_2 \models \mathcal{B} \sqsubseteq \mathcal{F}$, the loop continues. For B_1, $\mathcal{B} = B_1 \sqcup A \sqcup \exists r.\mu x.(A \sqcup \exists r.x)$ and $\mathcal{F} = B_1$. As $\mathcal{T}_2 \models B_1 \sqsubseteq \mathcal{F}$ and $\mathcal{T}_2 \models \mathcal{B} \sqsubseteq B_1$, the loop continues. The case of B_2 is similar to that of B_1. Finally, the algorithm returns false.

Procedure 1 can be modified to obtain witnesses to difference subsumption.

Example 8. We run Procedure 1 on $\mathcal{T}_1$, $\mathcal{T}_3 = \mathcal{T}_2 \setminus \{Z_1 \sqsubseteq B_1\}$ and Σ, where $\mathcal{T}_1$, $\mathcal{T}_2$ and Σ are the same as in Example 3. Then, for X, we have that $\mathcal{B} = \mathrm{B}^{\Sigma}_{\mathcal{T}_1}(X) = \mu x.(A \sqcup \exists r.x)$ and $\mathcal{F} = \mathrm{F}^{\Sigma}_{\mathcal{T}_1}(X) = B_1 \sqcap B_2$. However, $\mathcal{T}_3 \not\models \mathcal{B} \sqsubseteq \mathcal{F}$, which means $X \in \mathsf{cWtn}^{\mathsf{mid}}_{\Sigma}(\mathcal{T}_1, \mathcal{T}_3)$. Similarly, it can readily be seen that $B_1 \in \mathsf{cWtn}^{\mathsf{rhs}}_{\Sigma}(\mathcal{T}_1, \mathcal{T}_3)$.

5 Conclusion

We have revisited our solution to logical difference problem for $\mathcal{EL}$-terminologies which was based on finding simulations between hypergraph representations of the terminologies [4]. We have shown that there is a new type of witness in the logical difference between two $\mathcal{EL}$-TBoxes. We have shown that deciding the logical difference between two $\mathcal{EL}$-TBoxes can be reduced to fixpoint reasoning w.r.t. TBoxes.

References

1. Baader, F., Brandt, S., Lutz, C.: Pushing the $\mathcal{EL}$ envelope. In: Proceedings of the Nineteenth International Joint Conference on Artificial Intelligence IJCAI-05. Morgan-Kaufmann Publishers, Edinburgh, UK (2005)
2. Baader, F., Calvanese, D., McGuinness, D.L., Nardi, D., Patel-Schneider, P.F. (eds.): The description logic handbook: theory, implementation, and applications. Cambridge University Press, 2 edn. (June 2010)
3. Brandt, S.: Polynomial time reasoning in a description logic with existential restrictions, GCI axioms, and—what else? In: de Mantáras, R.L., L. Saitta (eds.) Proceedings of the 16th European Conference on Artificial Intelligence (ECAI-2004). pp. 298–302. IOS Press (2004)
4. Ecke, A., Ludwig, M., Walther, D.: The concept difference for $\mathcal{EL}$-terminologies using hypergraphs. In: Proceedings of DChanges'13. CEUR-WS.org (2013)
5. Hofmann, M.: Proof-theoretic approach to description-logic. In: Proceedings of the 20th Annual IEEE Symposium on Logic in Computer Science. pp. 229–237. LICS '05, IEEE Computer Society, Washington, DC, USA (2005)
6. Konev, B., Ludwig, M., Walther, D., Wolter, F.: The logical difference for the lightweight description logic $\mathcal{EL}$. Journal of Artificial Intelligence Research (JAIR) 44, 633–708 (2012)
7. Konev, B., Walther, D., Wolter, F.: The logical difference problem for description logic terminologies. In: Proceedings of IJCAR'08: the 4th international joint conference on Automated Reasoning. pp. 259–274. Springer-Verlag, Berlin, Heidelberg (2008), http://dx.doi.org/10.1007/978-3-540-71070-7_21
8. Ludwig, M., Walther, D.: The logical difference for elhr-terminologies using hypergraphs. In: Schaub, T., Friedrich, G., O'Sullivan, B. (eds.) Proceedings of the 21st European Conference on Artifical Intelligence (ECAI 2014). Frontiers in Artificial Intelligence and Applications, vol. 263, pp. 555–560. IOS Press (2014)
9. Nikitina, N., Rudolph, S.: Expexpexplosion: Uniform interpolation in general EL terminologies. In: ECAI. pp. 618–623 (2012)
10. Sattler, U., Vardi, M.Y.: The hybrid μ-calculus. In: Proceedings of IJCAR'01: the First International Joint Conference on Automated Reasoning. Lecture Notes in Computer Science, vol. 2083, pp. 76–91. Springer (2001), http://dx.doi.org/10.1007/3-540-45744-5_7