Direct Partial Logic Derivatives in Analysis of Boundary States of Multi-State System

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Abstract. *Multi-State System* (MSS) is mathematical model that is used in reliability engineering for the representation of initial investigated object (system). In a MSS, both the system and its components may experience more than two states (performance levels). One of possible description of MSS is a structure function that is defined correlation between a system components states and system performance level. The investigation of a structure function allows obtaining different properties, measures and indices for MSS reliability. For example, boundary system's states, probabilities of a system performance levels and other measures are calculated based a structure function. In this paper mathematical approach of Direct Partial Logical Derivatives is proposed for calculation of boundary states of MSS.

Keywords. Multi-State System, Multiple-Valued Logic, Direct Partial Logic Derivatives, Boundary State

Key Terms. Reliability, Model, Approach, Methodology, ScientificField

1 Introduction

Reliability is a principal attribute for the operation of any modern technological system. A principal issue in reliability analysis is the uncertainty in the failure occurrences and consequences. With respect to the complexity of the system and the modeling of their behavior, a distinctive feature of the system reliability analysis is a comprehensive and integrated manner [1]. Focusing on safety, reliability engineering methods aim at the quantification of the probability of failure of the system. In paper [1] presents detail analysis of reliability engineering state and define principal problems in this scientific discipline. According to [1] one direction of reliability engineering is estimation of a complex system based on *Multi-State System* (MSS) reliability analysis methods.

MSS is mathematical model used in reliability analysis for system with some (more that two) levels of performance [1, 2]. This mathematical model has been exploited in reliability engineering since 1975 [3-5]. Principal advantage of this mathematical model is detail description of investigated object. MSS permits to define and investigate several performance levels: from perfect function to fault. The typical Binary-State System allows evaluating only two system states as functioning and failure. Other states, for example, as partly functioning or functioning with restrictions are not analyzed in case of Binary-State System use. But extra states and performance levels in the mathematical model dramatically increase size of this model and computational complexity of its analysis. Therefore the MSS has not been used intensively in reliability analysis. There is other aspect to restrict the application of MSS. It is absent of effective methods for MSS analysis.

According to analysis in paper [2] there are four groups of methods for MSS analysis that are based on different mathematical approach: an extension of Boolean models to the multi-valued case, the stochastic process as Markov process, the universal generation function methods and the Monte-Carlo simulation techniques. For example, Markov processes are used to analyze the system state transition process [6]. The universal generation function application is useful in optimization problem [7]. The Monte-Carlo simulation as a rule is used for reliability assessment of system with large number of components [8]. The methods based on extension of Boolean models to the multi-valued case were developed historically the first [9, 10]. According to these methods MSS is represented and defined by the structure function. This function is defined the conformance of the system performance level and components states. As a rule for the structure function definition and representation is used Boolean functions [10, 11]. Only in separated publications the structure function with some values has been considered [9, 12]. In papers [13, 14] the correlation of *Miltiple-Valued Logic* (MVL) function and structure function was analyzed. The interpretation of the structure function as the MVL function allowed using the mathematical approach of MVL in the analysis of the MSS structure function. In paper [14] the application of Logical Differential Calculus for MSS reliability analysis has been proposed. The Logic Differential Calculus is used for analysis of dynamic properties of MVL function and this approach can be applied for analysis of dynamic behaviour of MSS that is determined by the structure function.

In the paper [14] the basic conception of application of Direct Partial Logic Derivatives (it is part of Logical Differential Calculus) in MSS reliability analysis has been considered. The proposed method permits to investigate the influence of one system component state change to the system performance level. The new indices for quantitative analysis of such influence have been defined. The application of these derivatives for calculation of importance measures (Structural Importance, Birnbaum Importance and Criticality Importance) has been investigated in paper [15, 16]. The algorithm for calculation of Fussell-Vesely Importance based on Direct Partial Logic Derivatives has been proposed in [17].

In this paper new application of Direct Partial Logic Derivatives for MSS reliability analysis based on the structure function is considered. The new algorithm for calculation of boundary states of MSS is proposed. This algorithm is based on representation of MSS by the structure function (section 2). This section includes the conception of boundary states of MSS for every performance level. The special structures of MSS (parallel and series) and their structure function are considered in the section 2 too. These types of system are typically employed in reliability analysis [18]. The short description of conceptions of Direct Partial Logic Derivatives with respect to one variable and with respect to variable vector are presented in the section

3. The calculation of MSS boundary states for every system performance level is considered section 4.

2 MSS Structure Function

2.1 Structure Function of MSS

Consider the system of n components. Each component of this system has m states: from the complete failure (it is 0) to the perfect functioning (it is m-1). The i-th system component state is denoted as x_i (i = 1, …, n). Suppose, that this system has m performance level too: from the complete failure (it is 0) to the perfect functioning (it is m-1). The dependence of the system performance level on components states is defined by the structure function $\phi(\boldsymbol{x})$ identically:

$$\phi(x_1, x_2,\ldots, x_n) = \phi(\boldsymbol{x}): \{0, 1, \ldots, m\text{-}1\}^n \rightarrow \{0, 1, \ldots, m\text{-}1\}. \quad \textbf{(1)}$$

The function (1) agrees with the definition of a MVL function [19]. Therefore the mathematical approaches of MVL can be used in quantification analysis of MSS. But the structure function (1) allows representing the very small class of real system for which the number of system performance levels and number of every component states are equal. As a rule, the real-world system has different numbers of states for different components. And the number of performance levels can be different too. Therefore the structure function of real-word system must be defined as:

$$\phi(\boldsymbol{x}): \{0, 1, \ldots, m_1\text{-}1\}\times\ldots\times\{0, 1, \ldots, m_n\text{-}1\} \rightarrow \{0, 1, \ldots, M\text{-}1\}, \quad \textbf{(2)}$$

where m_i is number of states for i-th system component (i = 1, …, n) and M is number of a system performance levels.

The equation (2) can be interpreted as a MVL function. But some formal changes allow transforming this structure function definition into an incompletely specified MVL function. This transformation suppose the interpretation of the function (2) as incompletely specified MVL function for maximal value of number m_i and M: $m_{\max}$=MAX{m_1, …, m_n, M}. In this case, the structure function (2) is defined as:

$$\phi`(\boldsymbol{x}): \{0, 1, \ldots, m_{\max}\text{-}1\}^n \rightarrow \{0, 1, \ldots, m_{\max}\text{ -}1\}. \quad \textbf{(3)}$$

The interpretation of the structure function (2) as an incompletely specified MVL function (3) permits to use mathematical approaches of MVL without principal restriction for analysis of properties of the structure function (2).

For example, consider the simple service system (Fig. 1) in a region from paper [17]. This system consists of three components (n = 3) – service point 1 (x_1), service point 2 (x_2) and infrastructure (x_3). This system has four performance levels: 0 – non-operational (no customer is satisfied), 1 – partially operational (some customers are satisfied), 2 – partially non-operational (some customers are not satisfied), 3 – fully

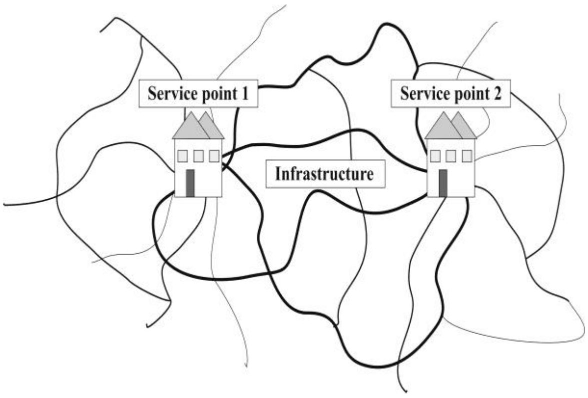

Fig. 1. A simple service system.

Table 1. The structure function of the simple service system

Components states		x_3			
x_1	x_2	0	1	2	3
0	0	**0**	**0**	**0**	**0**
0	1	**0**	**1**	**1**	**2**
1	0	**0**	**1**	**1**	**2**
1	1	**0**	**2**	**3**	**3**

Table 2. The structure function of the simple service system represented as an incompletely specified MVL function

Components states		x_3			
x_1	x_2	0	1	2	3
0	0	**0**	**0**	**0**	**0**
0	1	**0**	**1**	**1**	**2**
0	2	–	–	–	–
0	3	–	–	–	–
1	0	**0**	**1**	**1**	**2**
1	1	**0**	**2**	**3**	**3**
1	2	–	–	–	–
1	3	–	–	–	–
2	0	–	–	–	–
2	1	–	–	–	–

2	2	–	–	–	–
2	3	–	–	–	–
3	0	–	–	–	–
3	1	–	–	–	–
3	2	–	–	–	–
3	3	–	–	–	–

operational (all customers are satisfied). Next, we assume that the service points are only functional (state 1) or dysfunctional (state 0). The infrastructure can be modelled by 4 quality levels, i.e. from 0 (the quality of the infrastructure is poor) to 3 (the quality is perfect). The structure function of this system according to (2) is defined in Table 1 ($m_1 = m_2 = 2$, $m_3 = 4$ and $M = 4$). The structure function of this system as an incompletely specified MVL function is shown in Table 2 ($m_{\max} = 4$).

2.2 Series and Parallel MSS

There are some typical structures in the reliability engineering: series, parallel, *k*-out-of-*n* and bridge. Every system of these structures has single valued definition for Binary-State System (Fig.2).

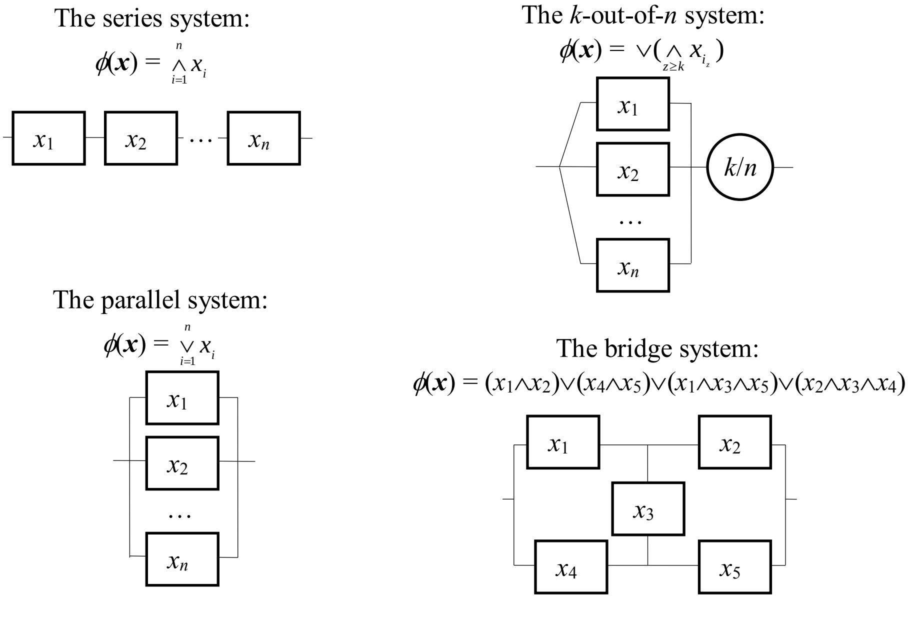

Fig. 2. Graphical and mathematical interpretation of typical structures of Binary-State System.

The extension of mathematical definition of the structure functions of these system for the MSS has some variants. For example, in paper [24] the structure functions of series, parallel and *k*-out-of-*n* MSS are defined based on the interpretation of mathematical equations of these system in terms of MVL. Structure functions of these MSS in MVL terms are declared by OR (∨) and AND (∧) MVL functions: OR(a, b) = MAX(a, b) and AND(a, b) = MIN(a, b). According to [24] structure functions parallel, series and k-out-of-n MSS are declared

- for parallel MSS as:

$$\phi(\boldsymbol{x}) = \bigvee_{i=1}^{n} x_i = \mathrm{MAX}(x_1, x_2, \ldots, x_n), \tag{4}$$

- for series MSS as:

$$\phi(\boldsymbol{x}) = \bigwedge_{i=1}^{n} x_i = \mathrm{MIN}(x_1, x_2, \ldots, x_n), \tag{5}$$

- for k-out-of-n MSS as:

$$\phi(\boldsymbol{x}) = \vee(\bigwedge_{z\geq k} x_{i_z}) = \mathrm{MAX}(\mathrm{MIN}_1(x_{i_1}, x_{i_2}, \ldots, x_{i_k}), \ldots, \mathrm{MIN}_w(x_{i_1}, x_{i_2}, \ldots, x_{i_k})), \tag{6}$$

$w = n!/(k!\cdot(n-k)!)$.

In paper [27] other declaration of these MSS are presented in which a system performance level depends on the number of functioning components. In Table 3 some structure functions of parallel MSS of two components (n = 2) with three states (m_1 = m_2 =3) and three performance level (M = 3) are shown. All these structure function in case of Binary-State System are parallel system. The series MSS can be defined similarly.

Table 3. The structure functions of parallel MSS ($n = 2$, $m_1 = m_2 = M = 3$)

Components states		Structure function of parallel MSS			
x_1	x_2	$\phi_1(\boldsymbol{x})$	$\phi_2(\boldsymbol{x})$	$\phi_3(\boldsymbol{x})$	$\phi_4(\boldsymbol{x})$
0	0	**0**	**0**	**0**	**0**
0	1	**1**	**1**	**1**	**1**
0	2	**2**	**1**	**1**	**1**
1	0	**1**	**1**	**1**	**1**
1	1	**1**	**2**	**1**	**1**
1	2	**2**	**2**	**1**	**2**
2	0	**2**	**1**	**1**	**1**
2	1	**2**	**2**	**1**	**2**
2	2	**2**	**2**	**2**	**2**

Therefore the typical structures of MSS have no single valued mathematical definition, because there are the set of structure functions of MSS that can be agreed with one Binary-State System. The structure function allows defining MSS explicitly. Therefore the structure function is preferable form of MSS mathematical representation.

2.3 Boundary States of MSS

The conception of boundary states has been proposed for Binary-State System firstly. The boundary state is defined as state for which the failure of one system components or some components causes the fault of a system [20]. The boundary state of MSS must be defined for every system performance level [2]. In papers [21, 22] the boundary states of MSS are interpreted as minimal cut/path sets. Authors of [23] introduced conception of Lower (Upper) Boundary Points of MSS for system performance level j (j =0, …, M-1). The boundary states for system performance level j and component i (i = 1, …, n) (named as exact boundary states) has been proposed and considered in papers [24, 25]. In paper [26] and [17] the correlations of these boundary states with Lower (Upper) Boundary Points and minimal cut/path sets are shown accordingly.

The exact boundary states have been considered in paper [25]. These states are system states for which the change of the i-th component state from s to $\tilde{s}$ causes the system performance level change from j to $\tilde{j}$ (s, $\tilde{s} \in \{0,\dots, m_i -1\}$, $s \neq \tilde{s}$ and j, $\tilde{j} \in \{0,\dots, M -1\}$, $j \neq \tilde{j}$). The exact boundary state is defined by the exact boundary vector unambiguously. Illustrate the correlation of a system exact boundary state and an exact boundary vector by the example for the service system in Fig.1.

Determine the exact boundary states of this service system for which the failure of the first component causes the system failure as the change of the system performance level from state “1” to “0”. According to Table 1, there are two situations that agree to this condition. They are possible for the failure of the second component and the third component state “1”or “2”. These exact boundary states can be presented as vector states: $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \rightarrow 0,0,1)$ and $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \rightarrow 0,0,2)$. Note that the boundary state $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \rightarrow 0,0,3)$ does not satisfy the condition because the system performance level changes from “1” to “2” depending on the failure of the first component in this case.

One of possible mathematical approaches for the definition of the exact boundary states in MVL is Logical Differential Calculus, in particular the Direct Partial Logical Derivatives. Consider the application of this mathematical approach for analysis of structure function of MSS.

3 Direct Partial Logical Derivatives

3.1 Direct Partial Logical Derivative with respect to one variable

The mathematical tool of Direct Partial Logic Derivatives has been proposed in [25] for calculation of an exact boundary states of a MSS. The Direct Partial Logic Derivative with respect to variable x_i for the structure function (1) permits to analyze the system performance level change from j to $\tilde{j}$ when the i-th component state changes from s to $\tilde{s}$:

$$\partial\phi(j \to \tilde{j})/\partial x_i(s \to \tilde{s}) = \begin{cases} 1, \text{ if } \phi(s_i, \mathbf{x}) = j \text{ and } \phi(\tilde{s}_i, \mathbf{x}) = \tilde{j} \\ 0, \text{ other} \end{cases}, \tag{7}$$

where $\phi(s_i, \boldsymbol{x}) = \phi(x_1,\ldots, x_{i-1}, s, x_{i+1},\ldots, x_n)$; $\phi(\tilde{s}, \boldsymbol{x}) = \phi(x_1,\ldots, x_{i-1}, \tilde{s}, x_{i+1},\ldots, x_n)$; s, $\tilde{s} \in \{0,\ldots, m_i -1\}$, $s \neq \tilde{s}$ and j, $\tilde{j} \in \{0,\ldots, M -1\}$, $j \neq \tilde{j}$.

For example, investigate the influence of the first component failure to the fault of the simple service system in Fig.1. The Direct Partial Logic Derivative $\partial\phi(1\to0)/\partial x_1(1\to0)$ allows to calculate the system state for which this failure causes the system break down. The calculation of this derivative is shown in Fig.3 in form of flow graph. The derivative $\partial\phi(1\to0)/\partial x_1(1\to0)$ has two non-zero values that agrees with state vectors: $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0,0,1)$ and $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0,0,2)$. According to definition of the Direct Partial Logic Derivative (7) for these system state the failure of the first system component causes the system failure too. Therefore the service system fails after the failure of the first service point if the second service point isn't functioning and the functioning of the infrastructure conforms to state one or two. The system states $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0,0,1)$ and $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0,0,2)$ are exact boundary states for the first system component failure and the system performance level 1.

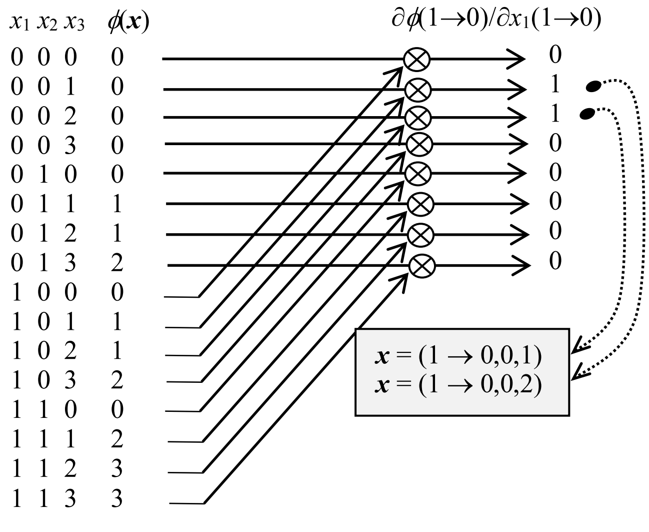

Fig. 3. Calculation of the Direct Partial Logic Derivative $\partial\phi(1\to0)/\partial x_1(1\to0)$.

The Direct Partial Logic Derivative (7) allows investigating boundary states of a MSS for which component state x_i change from s to $\tilde{s}$ causes the system performance level change from j to $\tilde{j}$. Therefore, this derivative allows calculating exact boundary states of the i-th system component for MSS performance level j that agree to state vectors $\boldsymbol{x} = (x_1, x_2,\ldots, x_n)$. All possible changes of the i-th system component and their influence to MSS performance level can be investigated based on the Direct Partial Logic Derivative (7). But this derivative permits to investigate the influence of one component only. The Direct Partial Logic Derivative with respect of variable vector investigate the system state changes depending on changes of states of some system components.

3.2 Direct Partial Logical Derivative with respect to variable vector

A Direct Partial Logic Derivatives of a structure function $\phi(\boldsymbol{x})$ of n variables with respect to variables vector $\boldsymbol{x}^{(p)} = (x_{i_1}, x_{i_2}, \ldots, x_{i_p})$ reflects the fact of changing of function from j to $\tilde{j}$ when the value of every variable of vector $\boldsymbol{x}^{(p)}$ is changing from s to $\tilde{s}$ [15]:

$$\partial\phi(j \to \tilde{j})/\partial x_i(\boldsymbol{s}^{(p)} \to \tilde{\boldsymbol{s}}^{(p)}) = \begin{cases} 1, \text{ if } \phi(s_{i_1},\ldots,s_{i_p},\boldsymbol{x}) = j \text{ and } \phi(\tilde{s}_{i_1},\ldots,\tilde{s}_{i_p},\boldsymbol{x}) = \tilde{j} \\ 0, \text{ other} \end{cases} \quad (8)$$

In (8) a change of value of i_q-th variable x_{i_q} form s_{i_q} to $\tilde{s}_{i_q}$ agrees with a change of i_q-th MSS component state form s_{i_q} to $\tilde{s}_{i_q}$ ($q = 1, \ldots, p$ and $p < n$). So, changes of some components states correspond with change of a variables vector $\boldsymbol{x}^{(p)} = (x_{i_1}, x_{i_2}, \ldots, x_{i_p})$. Every variable values of this vector changes form s_{i_q} to $\tilde{s}_{i_q}$. So, vector $\boldsymbol{x}^{(p)}$ can be interpreted as components states vector or components efficiencies vector.

For example, consider the simple service system (Fig.1) failure depending on fault of the first service point and reduction of functioning of infrastructure from state 2 to 1. This system behavior can be presented by the Direct Partial Logic Derivative $\partial\phi(1\to0)/\partial x_1(1\to0)\ \partial x_3(2\to1)$. The calculation of this derivative is shown in Fig.4 by the flow graph.

The derivative $\partial\phi(1\to0)/\partial x_1(1\to0)\ \partial x_3(2\to1)$ has two values and one of them is non-zero value that agrees with state vector: $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0, 0, 2 \to 1)$. This state vector define of the service system failure depending on the failure of the first service point and deterioration of the infrastructure functioning from state 2 to state 1. Therefore the system state $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0, 0, 2 \to 1)$ can be interpreted as exact boundary state for the first and the third system components of the system performance level 1.

The Direct Partial Logic Derivative with respect to variable vector (8) allows investigating boundary states of a MSS for which simultaneous changes of p components states from s_{i_q} to $\tilde{s}_{i_q}$ ($q = 1, \ldots, p$ and $p < n$) causes the system performance level change from j to $\tilde{j}$. Therefore, the Direct Partial Logic Derivative with respect to variable vector allows calculating exact boundary states for MSS performance level j of the i-th system component.

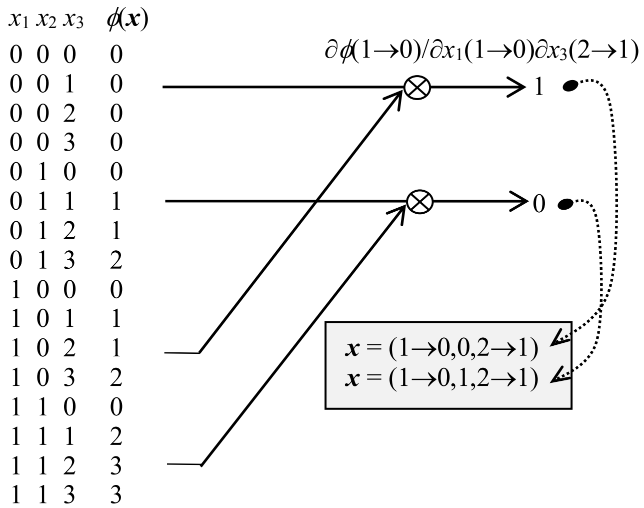

Fig. 4. Calculation of the Direct Partial Logic Derivative $\partial\phi(1\to 0)/\,\partial x_1(1\to 0)\,\partial x_3(2\to 1)$.

4 The Calculation and Estimation of Exact Boundary States of MSS based on Direct Partial Logic Derivatives

The exact boundary state of MSS are defined based on the condition that fixed system performance level change depending on the appointed change of one system component state or specified changes of some components states. The Direct Partial Logic Derivative with respect to one variable (7) and the Direct Partial Logic Derivative with respect to variable vector (8) can be used to investigate change of the system performance level from j to $\tilde{j}$ that are caused by specified changes of one or some system components states. These derivatives have non-zero values of the structure function for system states that satisfy for specified condition: the system performance level change from j to $\tilde{j}$ depending on specified changes of one or some system components states. *Therefor the exact boundary states can be defined as system states that conform to non-zero values of derivatives (7) and (8).*

The exact boundary state for MSS performance level j depending on the i-th system component $\begin{pmatrix} {}^{j\to\tilde{j}} \\ x_i \\ {}_{s\to\tilde{s}} \end{pmatrix}$ is indicated by vector state $\boldsymbol{x} = (x_1,\ldots, x_i,\ldots, x_n) = (a_1,\ldots, s_i,\ldots, a_n)$ for which $\phi(a_1,\ldots, s_i,\ldots, a_n) = j$ and $\phi(a_1,\ldots, \tilde{s}_i ,\ldots, a_n) = \tilde{j}$. This state is calculate as non-zero value of Direct Partial Logic Derivative (7). The exact boundary state for MSS performance level j depending on p components x_{i_1}, x_{i_2}, …,

x_{i_p} $\begin{pmatrix} j \to \tilde{j} \\ x_{i_1} .. x_{i_p} \\ s_{i_1} \to \tilde{s}_{i_1} s_{i_p} \to \tilde{s}_{i_p} \end{pmatrix}$ is indicated by vector state $\boldsymbol{x} = (x_1,\ldots, x_{i_1},\ldots, x_{i_p},\ldots, x_n) =$ $(a_1,\ldots, s_{i_1},\ldots, s_{i_p},\ldots, a_n)$ for which $\phi(a_1,\ldots, s_{i_1},\ldots, s_{i_p},\ldots, a_n) = j$ and $\phi(a_1,\ldots,\tilde{s}_{i_1},\ldots,\tilde{s}_{i_p},\ldots,a_n) = \tilde{j}$. This state is calculate as non-zero value of Direct Partial Logic Derivative (8).

Consider estimation of the system boundary states for coherent MSS. There are next assumptions for structure function of coherent MSS [2]: (*a*) the structure function is monotone and $\phi(\boldsymbol{s})=s$ ($s\in\{0, \ldots, m\text{-}1\}$) and (*b*) all components are *s*-independent and are relevant to the system.

Every system component is characterized by the probabilities of its state:

$$p_{i,s} = \Pr\{x_i = s\},\ s = 0, \ldots, m_i - 1. \tag{9}$$

The probability of every boundary state (a_1, …, s_i, …, a_n) for MSS performance level j depending on the i-th system component change from s to $\tilde{s}$ is calculated based on the probabilities of components states:

$$p_{(a_1 \ldots s_i \ldots a_n)}\begin{pmatrix} j \to \tilde{j} \\ x_i \\ s \to \tilde{s} \end{pmatrix} = p_{1,a_1} \cdot \ldots \cdot p_{i-1,a_{i-1}} \cdot p_{i,s_i} \cdot p_{i+1,a_{i+1}} \cdot \ldots p_{n,a_n} . \tag{10}$$

The probability of boundary state for MSS performance level j depending on the i-th system component states changes is calculated as:

$$p\begin{pmatrix} j \to \tilde{j} \\ x_i \\ s \to \tilde{s} \end{pmatrix} = \sum_{s,\tilde{s}} p_{(a_1 \ldots s_i \ldots a_n)}\begin{pmatrix} j \to \tilde{j} \\ x_i \\ s \to \tilde{s} \end{pmatrix} . \tag{11}$$

Next measure is defined the probability of boundary state of MSS performance level j depending of all component state change from s to $\tilde{s}$:

$$p\begin{pmatrix} j \to \tilde{j} \\ x \\ s \to \tilde{s} \end{pmatrix} = \sum_{i=1}^{n} p\begin{pmatrix} j \to \tilde{j} \\ x_i \\ s \to \tilde{s} \end{pmatrix} . \tag{12}$$

The probability of MSS performance level change depending on the change of the i-th system component from state s to $\tilde{s}$ is calculated according to:

$$p\begin{pmatrix} x_i \\ s\to\tilde{s} \end{pmatrix} = \sum_{j,\tilde{j}} p\begin{pmatrix} j\to\tilde{j} \\ x_i \\ s\to\tilde{s} \end{pmatrix}. \quad (13)$$

The probability of MSS performance level change depending on all change of the i-th system component state is generalization of previous equation:

$$p\begin{pmatrix} x_i \end{pmatrix} = \sum_{s,\tilde{s}} p\begin{pmatrix} x_i \\ s\to\tilde{s} \end{pmatrix}. \quad (14)$$

The similar measures to (10) - (14) can be defined for estimation of exact boundary state for MSS performance level j of p components $x_{i_1}, x_{i_2}, \ldots, x_{i_p}$ $\begin{pmatrix} j\to\tilde{j} \\ x_{i_1}..x_{i_p} \\ s_{i_1}\to\tilde{s}_{i_1} s_{i_p}\to\tilde{s}_{i_p} \end{pmatrix}$.

Consider some examples for calculation of measures (10) - (14) for the simple service system in Fig.1. The components states probabilities for this system are defined in Table 4. Consider this system failure depending to the first components. The Direct Partial Logic Derivative $\partial\phi(1\to0)/\partial x_1(1\to0)$ represents this system behavior (Fig.3). This derivative has two non-zero values that conform to two boundary states $\begin{pmatrix} 1\to0 \\ x_1 \\ 1\to0 \end{pmatrix}$: $\boldsymbol{x} = (1,0,1)$ and $\boldsymbol{x} = (1,0,2)$.

Table 4. The components states probabilities of the simple service system

0	Components states							
	x_1		x_2		x_3			
	0	1	0	1	0	1	2	3
$p_{i,s}$	0.3	0.7	0.2	0.8	0.2	0.6	0.1	0.1

The probabilities of boundary states for the system failure depending the first component break down $\begin{pmatrix} 1\to0 \\ x_1 \\ 1\to0 \end{pmatrix}$ are calculate according to (10) and are:

$$p_{(101)}\begin{pmatrix} 1\to0 \\ x_1 \\ 1\to0 \end{pmatrix} = p_{1,1}\cdot p_{2,0}\cdot p_{3,1} = 0.084 \text{ and } p_{(102)}\begin{pmatrix} 1\to0 \\ x_1 \\ 1\to0 \end{pmatrix} = p_{1,1}\cdot p_{2,0}\cdot p_{3,2} = 0.014 \quad (15)$$

The probability of boundary state for this system failure depending on the first component is calculated based on (11) as:

$$p\begin{pmatrix}1\to 0\\ x_1\\ 1\to 0\end{pmatrix} = p_{(101)}\begin{pmatrix}1\to 0\\ x_1\\ 1\to 0\end{pmatrix} + p_{(102)}\begin{pmatrix}1\to 0\\ x_1\\ 1\to 0\end{pmatrix} = 0.098\,. \qquad \textbf{(16)}$$

Therefore according to (16) the service system failure depending on the breakdown of the first service point is 0.098. By the similar way the probability of this system failure depending on the breakdown of the second service point is calculated and this probability is $p\begin{pmatrix}1\to 0\\ x_2\\ 1\to 0\end{pmatrix} = 0.098$.

There are three boundary states for the system failure depending the fault of infrastructure (the third component): $p_{(011)}\begin{pmatrix}1\to 0\\ x_3\\ 1\to 0\end{pmatrix} = 0.144$, $p_{(101)}\begin{pmatrix}1\to 0\\ x_3\\ 1\to 0\end{pmatrix} = 0.084$ and $p_{(111)}\begin{pmatrix}2\to 0\\ x_1\\ 1\to 0\end{pmatrix} = 0.336$. Therefore the probability the service system fault caused by the failure of the infrastructure is 0.564.

Other probabilities of this system failure or deterioration of the performance level are calculated according to (10) – (14) similar.

5 Conclusion

The mathematical background for application of mathematical methods of MVL for reliability analysis of MSS is considered in this paper. The correlation of the structure function and MVL function are shown and proved by means of the conception of incompletely specified MVL function. This background allows using Direct Partial Logic Derivatives for analysis of MSS structure function.

Mathematical approach of Direct Partial Logic Derivatives in MVL is used for investigation of dynamic properties of MVL function. The analysis of boundary values of MVL function is possible based on these derivatives too. In this paper the investigation of boundary values of the structure function of MSS and definition of MSS exact boundary states based on these valued are considered. Conception of exact boundary states is defined as boundary states for fixed MSS performance level change depending on change of the appointed change components state. This conception is extended for exact boundary states depending on changes of some components states.

New measures for estimation of MSS boundary states estimation are introduced and considered in the paper. The analysis of MSS based on the exact boundary states has not limits for the numbers of components (n) and states for every component (m_i), and system performance levels (M) according to the theoretical background. But in real-world applications these numbers have important influence to the structure function dimension (number of structure function elements) that is calculated as:

$$N_{\text{structure function dimension}} = m_1 \times m_1 \times \ldots \times m_n$$

As a rule the number of system performance levels (M) and number of component states (m_i) are defined between two and seven. The increase of the structure function dimension depending on the number of components (*n*) is illustrate in Fig.5. According to the investigation in papers [15 – 17] the Direct Partial Logical Derivatives is applicable for systems which have dimension less than ten millions elements. So the proposed method can be used for the MSS at least ten components.

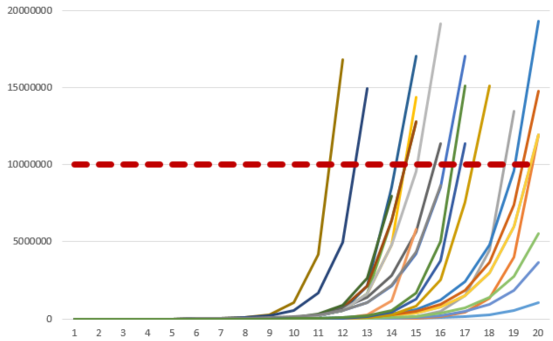

Fig. 5. Calculation of the Direct Partial Logic Derivative $\partial\phi(1\to 0)/\partial x_1(1\to 0)\ \partial x_3(2\to 1)$.

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