Contextual Reasoning in Concept Spaces

Stijn De Saeger and Atsushi Shimojima

Japan Advanced Institute of Science and Technology (JAIST)
{stijn,ashimoji}@jaist.ac.jp

Abstract. This paper presents ongoing work on a modular theory for contextual reasoning and the formalization of context. We introduce a two-tiered knowledge representation formalism that grounds an agent's pre-symbolic understanding of the world in a state space, and go on to develop a notion of concept spaces as its logical abstraction. After discussing some advantages of this model, we show how it accounts quite naturally for certain high-level forms of context-dependent inference.

1 Introduction

The program behind our research was originally put forward by Barwise and Seligman in "Information Flow" ([1]). Arguing about the relevance of state spaces as models for human reasoning, they state:

> Within the recent cognitive science literature, logic is often seen as irrevocably wed to what is percieved to be an outdated symbol-processing model of cognition. ... Perhaps the use of state spaces might allow a marriage of logic with continuous methods like those used in dynamical systems and so provide a toehold for those who envision a distinctly different model of human reasoning.

Our point of departure is the representation of some agent's empirically acquired knowledge in a real-valued state space, where states represent feasible combinations of values for the various components in a distributed system. Connecting this non-symbolic layer of knowledge representation up with ideas from Formal Concept Analysis [3] and more recently Boolean Concept Logic [2] allows us to speak about state spaces and their inhabitants at a higher level of abstraction, a propositional language of concepts. This two-tiered architecture reflects a cognitive agent's attunement to law-like regularities in some part of the world perceived – there and then – as a system of interacting components, for all practical purposes a *context*. We discuss some basic properties of this logic and go on to show it can be used to model forms of inference that are traditionally dealt with in the field of pragmatics. More concretely, we look at phenomena like context-dependent inference and restricted forms of presupposition accommodation. The theory and all examples in this paper were implemented in the functional programming language *Haskell*, and source code can be made available to interested readers upon request.

2 State Spaces

State spaces offer a means to view any number of arbitrary world entities as parameters in some complex, distributed system. Mapping those entities and their interrelation onto an n-dimensional mathematical space therefore gives a sub-symbolic representation of the state the agent perceived that part of the world to be in.

Dimensions and States Let a dimension δ be a pair $(i, \mathcal{R}_i)$ where i is a natural number taken from a set of indices $\mathcal{I} = \{1, \ldots, n\}$ and $\mathcal{R}_i \subseteq \mathbb{R}$, the reals, is a *range*. We will not assume that $\mathcal{R}_i$ is finite, only bounded. An n-dimensional state space Σ on a sequence of dimensions $\mathcal{D}im = \{(1, \mathcal{R}_1), \ldots, (n, \mathcal{R}_n)\}$ then is a subset of the product space $\prod_{i=1}^{n} \mathcal{R}_i$.

Alternatively, one can think of $\mathcal{D}im$ as a mapping from domain $\mathcal{I}$ to ranges $\mathcal{R} \subseteq \mathbb{R}$ such that $\mathcal{D}im = \{1 \mapsto \mathcal{R}_1, \ldots, n \mapsto \mathcal{R}_n, (n+1) \mapsto \emptyset, \ldots\}$. Since for named dimensions δ both the dimension name and the index i serve to uniquely determine what dimension we are referring to, we will often abbreviate $\delta = (i, \mathcal{R}_i)$ to $(\delta_i, \mathcal{R}_i)$ when no confusion is likely, simply replacing the dimension's index with the indexed dimension name.

For any dimension δ_i with $(1 \leq i \leq n)$, the i-th coordinate r_i of an n-tuple $\sigma = (r_1, \ldots, r_n)$ with $r_1, \ldots, r_n \in \mathbb{R}$ is called the dim_{δ_i} value of σ and written as $dim_{\delta_i}(\sigma)$. Given an n-dimensional state space Σ over a set of dimensions $\mathcal{D}im$, σ is called a *state in* Σ (written $\sigma \in \Sigma$) if $dim_{\delta_i}(\sigma) \in \mathcal{R}_i$ for every $\delta_i \in \mathcal{D}im$.

An Example In good tradition, our state space example will be the light circuit from [1] and [4]. Let the four-dimensional space Σ_{circ} represent an agent A's understanding of an integrated light circuit inside her house, consisting of two switches, a slider and a bulb. The bulb is on whenever just one of the switches is, but the slider can be used to adjust the bulb's brightness from 25 to 100 percent when it is on. Σ_{circ} is defined on the following dimensions:

$sw1 = (1, \mathcal{R}_1)$ where $\mathcal{R}_1 = \{0, 1\}$
$sw2 = (2, \mathcal{R}_2)$ where $\mathcal{R}_2 = \{0, 1\}$
$slid = (3, \mathcal{R}_3)$ where $\mathcal{R}_3 = \{0 \ldots 1\}$
$bulb = (4, \mathcal{R}_4)$ where $\mathcal{R}_4 = \{0 \ldots 1\}$

Dimension *bulb* is an output observable, its value is uniquely determined by the values of $sw1$, $sw2$ and $slid$: the equation

$$dim_{bulb}(\sigma) = |dim_{sw1}(\sigma) - dim_{sw2}(\sigma)| * (3 * dim_{slid}(\sigma) + 1)/4$$

models A's intuitive understanding of the light circuit where the intensity of the light is a linear function of the two switches and the slider. Note that this effectively limits the range of $\mathcal{R}_4$ to $\{0, \ 0.25 \ldots 1\}$ instead of $\{0 \ldots 1\}$. This is why we define state spaces in terms of *subsets* of the original product space.

The range of the first two dimensions of Σ_{circ} consists of just two values, 0 or 1, representing the corresponding component being *off* or *on*. The latter

two dimensions are actually closed intervals of real values between 0 and 1, showing the position of the slider and the brightness of the bulb respectively. We assume that for our present objectives the notation "$r_k \ldots r_n$" represents the closed interval of reals from r_k to r_n.

A state $\sigma = (1, 0, 0.3, 0.475)$ for example, would be a state in Σ_{circ}.

3 Concept Spaces

A point σ in a given dimension space is a sub-symbolic representation of the state an agent perceives some part of the world to be in. The range of values of the various dimensions of the space and their interdependence reflects the agent's systematic understanding of her surroundings.

The way this allows for a subtle and fine-grained representation of the agent's reality is typical for non-symbolic knowledge representation formalisms like connectionist models and such. At the same time, state spaces can't escape the evident criticisms of non-symbolic approaches either: the lack of principled abstraction that symbolic approaches excel at. If logical abstraction is a cognitive reality then it has to be accounted for.[1]

Constraint Types We define a notion of *constraint types* as the way in which an agent distinguishes semantically meaningful patterns in collections of states. Formally, constraint types bear much resemblance to dimensions. Given a dimension $\delta = (i, \mathcal{R}_i)$ taken from a set $\mathcal{D}im$, a constraint type c is again a pair $(\delta_i, \mathcal{R}_c)$, where $1 \leq i \leq |\mathcal{D}im|$ is a unique dimension index and $\mathcal{R}_c$ is a subset of $\mathbb{R}$. Note that we do not stipulate the stronger requirement that $\mathcal{R}_c$ be a subset of $\mathcal{R}_i$: in essence c merely partitions a set of states into those that fall into $\mathcal{R}_c$, and those that do not. What it means for a state σ to satisfy a constraint type $c = (\delta_i, \mathcal{R}_c)$, written $\sigma \vDash c$, is :

$$\sigma \vDash (\delta_i, \mathcal{R}_c) \iff dim_{\delta_i}(\sigma) \in \mathcal{R}_c$$

where $dim_{\delta_i}(\sigma)$ is the respective value of σ for dimension δ_i.

Concept Spaces Several contemporary theories[2] handle knowledge representation by referring to classification objects as first-class citizens in the theory. Called *formal contexts* in formal concept analysis ([3]), *classifications* in channel theory ([1]) or *Chu spaces* in category theory, the central idea is that of a binary incidence relation defined on two sets, objects (or *tokens*), and attributes (or *types*). For good introductions and representative examples of such theories, we refer to [1] and [3].

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[1] Cognitive models that bring the best of both worlds together form an interesting research area. A very notable recent attempt is the S^3 inference engine in ([4]).

[2] Think for instance of channel theory, formal concept analysis and some other information theories based on category theory.

In the same spirit, we conceive of a formal context $\mathcal{C} = (\Sigma_{\mathcal{C}}, cons_{\mathcal{C}}, \models_{\mathcal{C}})$ on $\mathcal{D}im$ where $\Sigma_{\mathcal{C}}$ represents the associated state space, $cons_{\mathcal{C}}$ is a set of constraint types over $\mathcal{D}im$, and $\models_{\mathcal{C}} \subseteq \Sigma_{\mathcal{C}} \times cons_{\mathcal{C}}$, i.e. for any $\sigma \in \Sigma_{\mathcal{C}}$, $c \in cons_{\mathcal{C}}$: $(\sigma, c) \in \models_{\mathcal{C}}$ if $\sigma \vDash c$.

For the sake of our light circuit example, let $\mathcal{C}_{circ}$ be the formal context $(\Sigma_{circ}, cons_{circ}, \models_{circ})$. The constraint types in $cons_{circ}$ are straightforward:

Constraint Type	Intuitive Meaning
$(sw1_1, \{1\})$	Switch 1 is up.
$(sw1_1, \{0\})$	Switch 1 is down.
$(sw2_2, \{1\})$	Switch 2 is up.
$(sw2_2, \{0\})$	Switch 2 is down.
$(slid_3, \{0 \ldots 0.1\})$	Slider is down.
$(slid_3, \{0.9 \ldots 1\})$	Slider is up.
$(bulb_4, \{0 \ldots 0.33\})$	Light is dim.
$(bulb_4, \{0.33 \ldots 0.66\})$	Light is medium.
$(bulb_4, \{0.66 \ldots 1\})$	Light is bright.
$(bulb_4, \{0.1 \ldots 1\})$	Bulb is on.
$(bulb_4, \{0\})$	Bulb is off.

A state $\sigma = (0, 1, 0.9, 0.925)$ is thus classified as follows: $\sigma \vDash_{circ} \{(sw1_1, \{0\}),$ $(sw2_2, \{1\}), (slid_3, \{0.9 \ldots 1\}), (bulb_4, \{0.66 \ldots 1\}), (bulb_4, \{0.1 \ldots 1\})\}$.

We assume basic familiarity with the ideas of Formal Concept Analysis [3] and the like. To recall, for a formal context $(\mathcal{O}, \mathcal{A}, \models)$, a formal concept (X, Y) consists of an extent $X \subseteq \mathcal{O}$ and an intent $Y \subseteq \mathcal{A}$, and furthermore:

$X = Y' = \{x \in \mathcal{O} \mid \forall y \in Y \ : \ x \vDash y\}$

$Y = X' = \{y \in \mathcal{A} \mid \forall x \in X \ : \ x \vDash y\}$

Note that if (X, Y) is a concept in $\mathcal{C}$, $(X, Y) = (X'', X') = (Y', Y'')$. A semi-concept, on the other hand, is a pair (X, Y) with $X \subseteq \mathcal{O}$ and $Y \subseteq \mathcal{A}$ that satisfies either $X' = Y$ or $Y' = X$. The notion of semiconcept, as developed in [2], is a generalization of formal concepts, and so the set of semiconcepts of $\mathcal{C}$ is a superset of the set of formal concepts of $\mathcal{C}$, which are derived by computing the full concept closure (X'', X') from semiconcept (X, X') (or dually, (Y', Y'') from (Y', Y)).

We call the semiconcept lattice associated with a formal context a *concept space*, and write $\mathcal{S}(\mathcal{C})$ to denote the concept space obtained from a formal context $\mathcal{C} = (\Sigma_{\mathcal{C}}, cons_{\mathcal{C}}, \models_{\mathcal{C}})$. Being a lattice structure, there is a natural order relation associated with $\mathcal{S}(\mathcal{C})$: i.e. for each $c_1 = (e_1, i_1)$, $c_2 = (e_2, i_2) \in \mathcal{S}(\mathcal{C})$, $c_1 \sqsubseteq c_2 \iff e_1 \subseteq e_2$ and $i_1 \supseteq i_2$.

4   The Logic $\mathfrak{L}_{\mathcal{C}}$

Concept spaces give us that logical abstraction layer over the dynamical system modeled by a state space. Semantically then, we treat the concepts in the concept space $\mathcal{S}(\mathcal{C})$ as *propositions*. Taken as the atoms of some agent's propositional

language, concepts consist of a relevant set of states for which given proposition holds, and the set of meaningfull properties (the *constraint types*) based upon which this is so. Intuitively, a concept c is a proposition in the sense that it 'predicates' something of a given situation (state of the system) or some relevant part of it. Put differently, it asserts that the state σ currently under discussion is partly described by the constraint types of c, or conversely, that σ belongs to their extent. As a proposition, a concept $c = (X, Y)$ represents a statement about a state σ, so we write '$\sigma \triangleright c$' to denote that 'σ *belongs to* c', or 'c *holds of* σ'. Formally, $\sigma \triangleright c$ whenever $\sigma \in X$. Note that this implies that $\forall y \in Y : \sigma \vDash y$.[3] Also, we assume two auxiliary functions *Ext* and *Int* to access the respective extent and intent of a concept c.

We could define the logic $\mathfrak{L}(\mathcal{C})$ in terms of a concept space $\mathcal{S}(\mathcal{C})$ and a set of operations over $\mathcal{S}(\mathcal{C})$:

$$\mathfrak{L}(\mathcal{C}) = (\mathcal{S}(\mathcal{C}), \{\sqcap, \sqcup, \neg, \sim, \bot, \top\})$$

These operations are defined in [2] as :

Meet:	$(X_1, Y_1) \sqcap (X_2, Y_2)$	$=$	$(X_1 \cap X_2, (X_1 \cap X_2)')$
Join:	$(X_1, Y_1) \sqcup (X_2, Y_2)$	$=$	$((Y_1 \cap Y_2)', Y_1 \cap Y_2)$
Negation:	$\neg\, (X, Y)$	$=$	$(\mathcal{O} \setminus X, (\mathcal{O} \setminus X)')$
Opposition:	$\sim (X, Y)$	$=$	$((\mathcal{A} \setminus Y)', \mathcal{A} \setminus Y)$
Bottom:	$\bot$	$=$	$(\emptyset, \mathcal{A})$
Top:	$\top$	$=$	$(\mathcal{O}, \emptyset)$

A semiconcept c satisfies $c \sqcap c = c$ or $c \sqcup c = c$, depending on which $'$ operation was used to derive the concept. Accordingly, semiconcepts are called $\sqcap$-semiconcepts or $\sqcup$-semiconcepts, and their intersection gives exactly the formal concepts of $\mathcal{C}$. From the definitions above, it should be clear that the set of $\sqcap$-semiconcepts $\mathcal{S}_\sqcap(\mathcal{C})$ is closed under the operations $\sqcap$, $\neg$ and $\top$, and similarly $\mathcal{S}_\sqcup(\mathcal{C})$ is closed under operations $\sqcup$, $\sim$ and $\bot$.[4] Because all regularity (*knowledge*) in the system is contained in Σ_{circ}, we wish to define logical operations on concepts strictly in terms of their *extents*. The constraint types in $cons_\mathcal{C}$ provide a way to *discriminate between* chunks of knowledge, but not the knowledge itself. For example, there is nothing in the constraint types that says that one switch being up and one being down implies that the light is on – that is not their role.

Thus, operating strictly within $\mathcal{S}_\sqcap(\mathcal{C})$, we need to redefine disjunctive behaviour in our logic so as to have the result of a join operation return a $\sqcap$-semiconcept as well. Using an alternative join $c_1 \sqcup^* c_2 = \neg\, (\neg\, c_1 \sqcap \neg\, c_2)$, we will restrict our logic to $\mathfrak{L}(\mathcal{C}) = (\mathcal{S}(\mathcal{C}), \{\sqcap, \sqcup^*, \neg, \bot, \top\})$. This new definition

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[3] For convenience we overload this symbol so that $\Sigma \triangleright c$ iff $\sigma \triangleright c$ for every σ in Σ.

[4] Whenever we use the unspecified term 'concepts' in what follows, we take it to mean concepts in the most general guise, that is, semiconcepts drawn from $\mathcal{S}(\mathcal{C})$ that may or may not be formal concepts. We will call them *concepts* when discussing $\mathcal{S}(\mathcal{C})$, and *propositions* when talking about $\mathfrak{L}(\mathcal{C})$.

allows us to talk about the conjunction and disjunction of concepts nicely within the set of $\sqcap$-semiconcepts of $\mathcal{S}(\mathcal{C})$. [5]

Back To The Example We obtain the desired logic $\mathfrak{L}(\mathcal{C}_{circ})$ from the concept space $\mathcal{S}(\mathcal{C}_{circ})$ for $\mathcal{C}_{circ} = (\Sigma_{circ}, cons_{circ}, \models_{circ})$, together with the above operations. $\mathcal{S}(\mathcal{C}_{circ})$ is a dense, information-packed space of interrelated concepts, the number of which can be up to $2^{(|\Sigma_{circ}|+|cons_{circ}|)}$. Within this huge set, we name a few meaningful concepts as propositions.

Name	Concept	Intuitive Meaning
$up1$	$(\{(sw1_1, \{1\})\}', \{(sw1_1, \{1\})\}'')$	Switch 1 is up.
$dn1$	$(\{(sw1_1, \{0\})\}', \{(sw1_1, \{0\})\}'')$	Switch 1 is down.
$up2$	$(\{(sw2_2, \{1\})\}', \{(sw2_2, \{1\})\}'')$	Switch 2 is up.
$dn2$	$(\{(sw2_2, \{0\})\}', \{(sw2_2, \{0\})''$	Switch 2 is down.
$slidUp$	$(\{(slid_3, \{0.9\ldots1\})\}', \{(slid_3, \{0.9\ldots1\})\}'')$	The slider is up.
$slidDn$	$(\{(slid_3, \{0\ldots0.1\})\}', \{(slid_3, \{0\ldots0.1\})\}'')$	The slider is down.
$dark$	$(\{(bulb_4, \{0\ldots0.3\})\}', \{(bulb_4, \{0\ldots0.3\})\}'')$	The light is dark.
$medium$	$(\{(bulb_4, \{0.33\ldots0.6\})\}', \{(bulb_4, \{0.33\ldots0.6\})\}'')$	The light is medium
$bright$	$(\{(bulb_4, \{0.66\ldots1\})\}', \{(bulb_4, \{0.66\ldots1\})\}'')$	The light is bright.
off	$(\{(bulb_4, \{0\})\}', \{(bulb_4, \{0\})\}'')$	The light is off.
on	$\neg\ off$	The light is on.

Note that we could have chosen to express $medium$ as $\neg(bright \sqcup dark)$ to mean anything that is not dark or bright. The point is that propositions, however complex, are always expressible as single concepts in $\mathcal{S}(\mathcal{C}_{circ})$.

Sequent Calculus in $\mathfrak{L}(\mathcal{C})$ We already mentioned the natural subsumption relation between semiconcepts in $\mathcal{S}(\mathcal{C})$. For our purposes, this subsumption hierarchy gives us a semantic *implication relation*.

$$\forall\ x, y \in \mathcal{S}(\mathcal{C})\ :\ x \vdash_{\mathfrak{L}(\mathcal{C})} y \iff x \sqsubseteq_{\mathcal{S}(\mathcal{C})} y$$

Hence, if $x \sqsubseteq_{\mathcal{S}(\mathcal{C})} y$ then $\sigma \rhd x$ implies $\sigma \rhd y$. With this notion of entailment in place, we can define a sequent calculus for $\mathfrak{L}(\mathcal{C})$.

A *sequent* (Γ, Δ) is defined in the usual way as a pair of sets of propositions, and we say that (Γ, Δ) *is valid in* $\mathfrak{L}(\mathcal{C})$ iff there is no conceivable state σ such that both $\forall\ \gamma \in \Gamma\ :\ \sigma \rhd \gamma$ and $\nexists\ \delta \in \Delta\ :\ \sigma \rhd \delta$. In other words, the set Γ is interpreted conjunctively and Δ disjunctively, as usual. Moreover, conjunction and disjunction are easily expressed in $\mathcal{S}(\mathcal{C})$ using meet and join operations. If by doing so we are able to reduce Γ and Δ to single concepts in $\mathcal{S}(\mathcal{C})$, we can check the entailment relation $\vdash_{\mathfrak{L}(\mathcal{C})}$ between the unique resulting concepts. So, for any pair (Γ, Δ) of sets of concepts in $\mathcal{S}(\mathcal{C})$: [6]

$$\Gamma \vdash_{\mathfrak{L}(\mathcal{C})} \Delta \iff \sqcap\Gamma \sqsubseteq_{\mathcal{S}(\mathcal{C})} \sqcup^*\Delta$$

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[5] It is shown in [2] that $(\mathcal{S}(\mathcal{C}), \sqcap, \sqcup^*, \neg, \top)$ and its dual $(\mathcal{S}(\mathcal{C}), \sqcap^*, \sqcup, \sim, \bot)$ form a double Boolean algebra.

[6] See the appendix for a proof.

5 Context-based Reasoning

Context-based reasoning has become a bit of a household term in any discipline that deals with the non-trivial representation of meaning and the complex interactions it must support. The formalization of context is a point where research from such fields as artificial intelligence, cognitive science, logic and the philosophy of language converges. Nonetheless, an elusive concept such as context means different things to different people, and there seems to be no one definition of what constitutes contextual reasoning that researchers from all these disciplines can agree upon, let alone a unified theory.

In what follows we silently subscribe to the relevance-theoretic view on context and its role in supporting inferential communication (Sperber & Wilson, 1986 [5]), but will nevertheless use the term quite generally to refer to any form of reasoning in which some notion of *context* plays an dynamic role in the inference process. In any case, our work attempts to formalize the processes underlying such behaviour and how it follows straightforwardly from the cognitive architecture we described, rather than contributing to the literature on these subjects in terms of analysis. Specifically, we single out two related phenomena: context-dependent inference and accommodation.

5.1 Context As Background

Let's say that a set of background assumptions represents a body of (partial) knowledge about a given context that was established prior to the actual inference. In general we want to know whether some sequent holds for every conceivable state σ in the space. Often though we already have partial information about σ in the form of a set of propositions P, so determining the validity of (Γ, Δ) against P now comes down to checking (Γ, Δ) in the relevant subspace of $\mathcal{S}(\mathcal{C})$, where the meet of P becomes the new top concept $\top$. This new subspace we obtain through context transformations.

We call a *context transformation* any operation that turns a formal context $\mathcal{C}$ into a new context $\mathcal{C}^*$ such that there exists an infomorphism between $\mathcal{C}$ and $\mathcal{C}^*$. To clarify, *infomorphisms* are constructs from channel theory (see [1]) denoting a mapping between two formal contexts that preserves certain properties about their information structure. In short, for $\mathcal{C} = (\mathcal{O}, \mathcal{A}, \models_\mathcal{C})$ and $\mathcal{C}^* = (\mathcal{O}^*, \mathcal{A}^*, \models_{\mathcal{C}^*})$, an infomorphism $\mathbb{I}$ from $\mathcal{C}$ to $\mathcal{C}^*$ is a pair of contravariant functions $(f^\vee : \mathcal{O}^* \mapsto \mathcal{O}, \ f^\wedge : \mathcal{A} \mapsto \mathcal{A}^*)$ such that $\forall o \in \mathcal{O}^*, a \in \mathcal{A} \ : \ f^\vee(o) \models_\mathcal{C} a \iff o \models_{\mathcal{C}^*} f^\wedge(a)$.

Arguably, there is a fair number of such transformations for which this property could hold. However, keeping the definition sufficiently general at this point will allow for a variety of more specialized context transformations for doing the heavy lifting when dealing with other forms of context shifting behaviour later on. For the purpose of this paper however, we focus on one particularly simple transformation that restricts the state space Σ to the subspace of relevant states that satisfy some set of background conditions. Formally, given a formal context $\mathcal{C}$ and a proposition $p \in \mathcal{S}(\mathcal{C})$, a context transformation $\mathbb{T}$ on p and $\mathcal{C}$ is :

$$\mathbb{T}(\mathcal{C}, p) = (Ext(p), cons_\mathcal{C}, \models_\mathcal{C})$$

Let $\mathcal{C}^*$ be such $\mathbb{T}(\mathcal{C}, p)$ for some p. Then $(f^\wedge, f^\vee)$ taken as a pair of identity functions gives a straightforward infomorphism $\mathbb{I}$ from $\mathcal{C}$ to $\mathcal{C}^*$. Think of this $\mathbb{T}$ operation as restricting the focus of attention to a relevant subspace of possible states that all satisfy the background assumption p. Furthermore, since arbitrarily complex propositional formulas in $\mathfrak{L}(\mathcal{C})$ are reducible to a unique concept in $\mathcal{S}(\mathcal{C})$ through a series of meet operations, any set of background assumptions P can be mapped to a single concept in $\mathcal{S}(\mathcal{C})$ as well.

Let's look at some example sequents. We will freely simplify expressions like "$\{a, b\} \vdash_{\mathfrak{L}(\mathcal{C}_{circ})} \{c, d\}$" to "$a, b \vdash c, d$" whenever we mean the inference to take place in $\mathfrak{L}(\mathcal{C}_{circ})$, and explicitly mention the logic otherwise.

1. $up1, slidUp \vdash bright, up2$?
   This sequent asks whether it is true that in case both switch 1 and the slider are up, either the light bulb is bright or switch 2 is up as well (which would turn off the light). As understanding of the system suggests, this sequent is judged valid, as $(up1 \sqcap slidUp) \sqsubseteq_{\mathcal{S}(\mathcal{C}_{circ})} (bright \sqcup^* up2)$.

2. $dn1, dark \vdash_{\mathfrak{L}(\mathbb{T}(\mathcal{C}_{circ},\ up2))} on$?
   In the normal concept space $\mathcal{S}(\mathcal{C}_{circ})$ switch 1 being down and the light being dim does not entail that the bulb is on. That changes though when confronted with some new knowledge about the state of the light circuit. Suppose it is brought to our agent's attention that switch 2 is up. In that case it makes no sense to check any further sequent in the whole of $\mathcal{S}(\mathcal{C}_{circ})$, because the present context has shifted to a subspace of relevant states in which switch 2 is up. In this new space the top concept $\top$ becomes $up2$. As it turns out, this sequent is valid in $\mathfrak{L}(\mathbb{T}(\mathcal{C}_{circ}, up2))$, meaning that against the background condition that switch 2 is up, switch 1 being down *does* entail that the bulb is on, though the bulb may be dimmed by the slider switch.

3. $slidUp \vdash_{\mathfrak{L}(\mathbb{T}(\mathcal{C}_{circ},\ dark))} (dn1 \sqcap dn2), (up1 \sqcap up2)$?
   This sequent represents a rather complex inference. It says: "*Given the background knowledge that the light is dark, does the slider being all the way up imply that both switches are either down or up?*" This sequent again is valid in $\mathfrak{L}(\mathbb{T}(\mathcal{C}_{circ},\ dark))$, as can be confirmed by doing the step by step calculation.

5.2 Accommodation

Accommodation builds upon this notion of reasoning against background conditions. We need to qualify what we mean by accommodation though. In the literature on natural language semantics, accommodation was introduced by Lewis (1979) as a repair strategy used to salvage some necessary truth conditions associated with a certain class of lexical items (so-called *presupposition triggers*). As our theory is not designed to handle semantic presupposition, we will have little to say about those particular forms of accommodation.

Instead, we follow Thomason (1990) and the neo-Griceans by treating accommodation more broadly as a catering strategy for *pragmatic* presuppositions

– those presuppositions a cognitive agent is invited to supply so some new piece of information (typically an utterance) can be interpreted as making a meaningful contribution to the exchange. Hence, we think of accommodation quite generally as the process of supplying a minimal set of background conditions Ω in $\mathcal{S}(\mathcal{C})$ that make a sequent (Γ, Δ) true while remaining consistent with existing assumptions Ω^*. Since $\mathcal{S}(\mathcal{C})$ is a space of semantically interconnected propositions, this becomes a feasible operation.

This kind of inferential behaviour is arguably most prominent in communication. Imagine two agents A and B exchanging information about some situation under discussion, say, the light circuit in A's house. Whereas *full knowledge* about the current state of the system would be represented by a single unique state σ, *partial knowledge p* about the circuit state takes the form of a set of possible states σ^*, represented by some subspace $\mathbb{T}(\mathcal{C}_{circ}, p)$. [7]

Imagine at some point in the exchange that A is inquiring about the value $dim_{\delta_i}(\sigma)$ of the current state σ for a given dimension δ_i. Furthermore, assume that Ω represents the (possibly emtpy) set of background knowledge that was established throughout the conversation, the *common ground* so to speak. In this setting, let B's answer to A be some utterance u, taken to be just another proposition. At this point, several scenarios are thinkable. If u provides satisfactory information about δ_i then all is well. The more interesting case is when u does not seem to answer A's question in any obvious way. In accordance with Gricean pragmatics (notably Sperber and Wilson's relevance theory [5]) however, A is justified in assuming that B was trying to make a relevant contribution regardless, and that B communicated u in the belief that A's inferentially processing it will provide the required information and potentially other meaningful information along the way. In other words, A will need to accommodate B's utterance, which means determine the minimal background conditions Ω^* such that $u \vdash_{\mathfrak{L}(\mathbb{T}(\mathcal{C}_{circ}, \ \Omega \ \sqcap \ \Omega^*))} v$, for some v that provides conclusive information regarding δ_i.

The phrase *"minimal background conditions"* needs explaining. Is it possible to compare two propositions p and q in terms of informativity? Recall that concept space $\mathcal{S}(\mathcal{C})$ forms a lattice ordered by the $\sqsubseteq$ relation, so it seems so. When $p \sqsubseteq q$ (or, $Ext(p) \subseteq Ext(q)$ and $Int(p) \supseteq Int(q)$), we say that p is a more specific concept than q, and $(\mathcal{S}(\mathcal{C}), \sqsubseteq)$ forms an informational scale. The most general concept $\top$ has the entire state space Σ for its extent and $\emptyset$ as intent; it contains no information. Put differently, for any formal context $\mathcal{C}$, $\mathfrak{L}(\mathcal{C}) = \mathfrak{L}(\mathbb{T}(\mathcal{C}, \top))$.

Hence, if there is no relevant v such that $u \vdash_{\mathfrak{L}(\mathbb{T}(\mathcal{C}, \ \top))} v$, accommodating u means supplying the background assumption t such that we find a relevant concept v so that $u \vdash_{\mathfrak{L}(\mathbb{T}(\mathcal{C}, \ t))} v$ and there is no t^* with $u \vdash_{\mathfrak{L}(\mathbb{T}(\mathcal{C}, \ t^*))} v$ and $t \sqsubseteq t^*$. This gives us a very straightforward algorithm to accommodate u: look for a relevant proposition v entailed by u in concept space $\mathcal{S}(\mathcal{C})$. If found then

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[7] This is where having transformations on formal contexts pays off. Realistically, an agent is rarely if ever interested in full knowledge about some complex system, moreover unnecessarily representing it in full may be costly and inefficient.

return v, else calculate the next minimal subspace $\mathbb{T}(\mathcal{C}, t)$ where t is a concept one level down in the concept space ordered by $\sqsubseteq_{\mathcal{S}(\mathcal{C})}$, and repeat.

Example A small example of said behaviour might be illustrative at this point. Admittedly, a dialogue example about light circuits is a bit artificial, but if nothing else it may generate some basic intuitions about the phenomenon we are trying to capture. Imagine the following conversation taking place between A and B.

1. A : *A bit dim here, no? Could you check whether the light is on?*
   . . .
2. B : *It seems the slider was down.*

   In the framework of our model, A's utterance seems to be doing two things: by asserting that the place is dim it fixes a common background $\{dark\}$, and furthermore it asks for information regarding dimension $bulb_4$. Let's say B's answer translates to proposition $slidDn$. However, neither "$slidDn \vdash_{\mathfrak{L}(\mathbb{T}(\mathcal{C}_{circ},\ dark))} on$" nor "$slidDn \vdash_{\mathfrak{L}(\mathbb{T}(\mathcal{C}_{circ},\ dark))} off$" represents a valid inference. In other words, B's reply does not provide A's requested information. As a competent communicator though, B's answer holds the promise of a relevant interpretation, only the hearer must supply some additional background in order to retrieve it. Searching the concept space $\mathcal{S}(\mathbb{T}(\mathcal{C}_{circ},\ dark))$ from the most general concept $\top$ downwards, two concepts verify the sequent when added to the background: "$dn1 \sqcap up2$" and "$up1 \sqcap dn2$", not surprisingly the two switch configurations that turn the bulb on. Since either one of these will do, their disjunction represents the smallest commitment on A's part that accommodates the utterance. As our inference engine confirms, "$slidDn \vdash_{\mathfrak{L}(\mathbb{T}(\mathcal{C}_{circ},\ p))} on$", with $p = \{dark,\ ((up1 \sqcap dn2) \sqcup^{*} (dn1 \sqcap up2))\}$, is a valid inference.

5.3   Discussion

This line of thinking raises a number of interesting questions as well. For one, if B really intended A to arrive at this conclusion, why didn't she just give a straight answer by pointing out that the light is on − after all, that would have saved A some apparently unnecessary inferential processing. Clearly, B must have thought the extra inferential effort to be offset by some additional information import as a side effect. In this case, presumably B's utterance gives an additional explanation of what made the room dark, besides honoring A's question.

   This seems to make a lot of sense, as it also explains why for instance B's utterance is unlikely to get accommodated with the highly similar proposition $q = \{((up1 \sqcap up2) \sqcup^{*} (dn1 \sqcap dn2))\}$. This time, "$slidDn \vdash_{\mathfrak{L}(\mathbb{T}(\mathcal{C}_{circ},\ q))} off$" would make for an interpretation to the effect that the light is off. Realistically, q is neither more nor less informative than the above-mentioned p. What is different in this case is that it is unclear what the role of B's utterance ($slidDn$) would be as premise of this sequent. Indeed, "$\emptyset \vdash_{\mathfrak{L}(\mathbb{T}(\mathcal{C}_{circ},\ q))} off$" would be equally valid, and as such the added value of an utterance like "$slidDn$" is anything but

clear. Many researchers in the field of pragmatics have proposed sensible answers to these and similar questions, and future work will have to show whether the present model, suitably adapted, can stand up to these challenging issues.

6   Conclusion

We have presented work in progress on a theory of contextual inference based on a two-tiered model of knowledge representation. As a framework for contextual reasoning, it makes some important first steps towards implementing the dynamic program mentioned by Barwise and Seligman in [1]. With abstract, logical reasoning radically *grounded in* and *relative to* an agent's sub-symbolic understanding of her surroundings we are able to generate the sort of subtle inferential behaviour that tends to be hard to axiomatize. For the sake of discussion, we further gave some examples of what such context-dependent reasoning looks like in our model.

Obviously a great many interesting questions still lie ahead. Nevertheless, the inference patterns we talked about clearly lie at the heart of contextual reasoning, and it is our contention that a theory that 'gets them right' can have many promising applications wherever the fields of logic, language and information meet.

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A   Appendix: Proof

This appendix contains a proof of our sequent calculus theorem:

$$\Gamma \vdash_{\mathfrak{L}(\mathcal{C}_{cl})} \Delta \iff \sqcap \Gamma \ \sqsubseteq_{\mathcal{S}(\mathcal{C}_{cl})} \ \sqcup^{*}\Delta$$

We need to prove both conditionals. Throughout the proof we omit the subscripts in $\vdash_{\mathfrak{L}(\mathcal{C}_{cl})}$ and $\sqsubseteq_{\mathcal{S}(\mathcal{C}_{cl})}$ for readability.

1. $\Gamma \vdash \Delta \Rightarrow \sqcap \Gamma \sqsubseteq \sqcup^{*}\Delta$.
   Assume $\Gamma \vdash \Delta$, that is, assume that $\forall \sigma \in \Sigma$: if $\sigma \rhd \gamma$ for every $\gamma \in \Gamma$, then $\sigma \rhd \delta$ for some $\delta \in \Delta$. We must show that the *meet* of Γ is a subconcept of the *join** of Δ. Let σ be an arbitrary state in Σ such that $\forall \gamma \in \Gamma : \sigma \rhd \gamma$. By the definition of $\rhd$ then, this means that $\forall \gamma \in \Gamma : \sigma \in Ext(\gamma)$. This implies that $\sigma \in \bigcap Ext(\Gamma)$ then, so $\sigma \rhd \sqcap \Gamma$. Furthermore, we know that $\exists\, \delta \in \Delta : \sigma \rhd \delta$. Then :
$$
\begin{aligned}
\exists\, \delta \in \Delta : \sigma \rhd \delta &= \exists\, \delta \in \Delta : \sigma \in Ext(\delta) \\
&= \sigma \in \bigcup_{\delta \in \Delta} Ext(\delta) \\
&= \sigma \notin \Sigma \setminus \left(\bigcup_{\delta \in \Delta} Ext(\delta) \right) \\
&= \sigma \notin \bigcap_{\delta \in \Delta} \Sigma \setminus Ext(\delta) \quad \text{(De Morgan)} \\
&= \sigma \in \Sigma \setminus \left(\bigcap_{\delta \in \Delta} \Sigma \setminus Ext(\delta) \right) \\
&= \sigma \rhd \sqcup^{*}\Delta
\end{aligned}
$$

   Discharging our initial assumption that $\Gamma \vdash \Delta$, we conclude that $\sqcap \Gamma \sqsubseteq \sqcup^{*}\Delta$ if $\Gamma \vdash \Delta$ as the first half of the biconditional. The second half is totally parallel.

2. $\sqcap \Gamma \sqsubseteq \sqcup^{*}\Delta \Rightarrow \Gamma \vdash \Delta$
   Similarly, we first assume that $\sqcap \Gamma \sqsubseteq \sqcup \Delta$, or that the *meet* of Γ is a subconcept of the *join** of Δ. Based on this, we show it follows that every state $\sigma \in \Sigma$ that belongs to all $\gamma \in \Gamma$ must necessarily belong to some $\delta \in \Delta$. Let σ be an arbitrary state such that $\sigma \rhd \sqcap \Gamma$. This again means that $\forall \gamma \in \Gamma : \sigma \in \bigcap Ext(\gamma)$. Also, by the assumption and the definition of the $\sqsubseteq$ relation, $\sigma \rhd \sqcup^{*}\Delta$. Then :
$$
\begin{aligned}
\sigma \rhd \sqcup^{*}\Delta &= \sigma \in \Sigma \setminus \left(\bigcap_{\delta \in \Delta} \Sigma \setminus Ext(\delta) \right) \\
&= \sigma \notin \bigcap_{\delta \in \Delta} \Sigma \setminus Ext(\delta) \\
&= \sigma \notin \Sigma \setminus \left(\bigcup_{\delta \in \Delta} Ext(\delta) \right) \quad \text{(De Morgan)} \\
&= \sigma \in \bigcup_{\delta \in \Delta} Ext(\delta) \\
&= \exists\, \delta \in \Delta : \sigma \in Ext(\delta) \\
&= \exists\, \delta \in \Delta : \sigma \rhd \delta
\end{aligned}
$$

We can again drop the original assumption and conclude that $\Gamma \vdash \Delta$ whenever $\sqcap \Gamma \sqsubseteq \sqcup^{*}\Delta$. Bringing both halfs together we can therefore conclude that $\Gamma \vdash_{\mathfrak{L}(\mathcal{C}_{cl})} \Delta \iff \sqcap \Gamma \ \sqsubseteq_{\mathcal{S}(\mathcal{C}_{cl})} \ \sqcup^{*}\Delta$.