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Stability of Extents in One-Sided Fuzzy Concept Lattices*

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Abstract: The efficient selection of relevant extents is an important issue for investigation in formal concept analysis. The notion of stability has been adopted for this reasoning. We present three different methods for evaluation of stability and we summarize the comparative remarks.

1 Introduction

The efficient selection of relevant formal concepts is an interesting and important issue for investigation and several studies have focused on this scalability question in formal concept analysis. The stability index [27] represents the proportion of subsets of attributes of a given concept whose closure is equal to the extent of this concept (in an extensional formulation). A high stability index signalizes that extent does not disappear if the extent of some of its attributes is modified. It helps to isolate concepts that appear because of noisy objects in [22] and the complete restoring of the original concept lattice is possible with combination of two other indices. The phenomenon of the basic level of concepts is advocated to select important formal concepts in [11]. Five quantitative approaches on the basic level of concepts and their metrics are comparatively analyzed in [12]. The approaches on selecting of the formal concepts and simplifying the concept lattices are examined in [15], as well.

In this paper, we present three methods which concern the selection of the relevant formal concepts from the set of all one-sided fuzzy formal concepts. We recall the modified Rice-Siff algorithm, extend the results on the quality subset measure and propose a new index for the stability of one-sided fuzzy formal concepts taking into account the probabilistic aspects in the fuzzy formal contexts. We would like to emphasize that the best results one can obtain by the combination of various methods.

2 Classical approach

A central role in this section will be played by the notions of a formal context (Fig. 1), a polar (Fig. 2), a formal concept and a concept lattice (Fig. 3). We recall the definitions and we refer to [18] for more details.

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Definition 1. *Let B and A be the nonempty sets and let $R \subseteq B \times A$ be a relation between B and A. A triple $\langle B, A, R\rangle$ is called a formal context, the elements of set B are called objects, the elements of set A are called attributes and the relation R is called incidence relation.*

	i	ii	iii
a	×		
b	×	×	
c		×	

Figure 1: A formal context $\langle \{a,b,c\}, \{i,ii,iii\}, R\rangle$

Definition 2. *Let $\langle B, A, R\rangle$ be a formal context and $X \in \mathscr{P}(B)$, $Y \in \mathscr{P}(A)$. Then the maps $\nearrow\colon \mathscr{P}(B) \to \mathscr{P}(A)$ and $\swarrow\colon \mathscr{P}(A) \to \mathscr{P}(B)$ defined by*

$$\nearrow(X) = X^{\nearrow} = \{y \in A : (\forall x \in X)\langle x, y\rangle \in R\}$$

and

$$\swarrow(Y) = Y^{\swarrow} = \{x \in B : (\forall y \in Y)\langle x, y\rangle \in R\}$$

are called concept-forming operators (also called derivation operators or polars) of a given formal context.

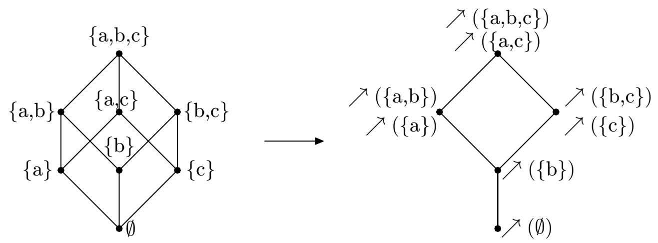

Figure 2: Polar $\nearrow$ of $(\mathscr{P}(B), \subseteq)$

Definition 3. *Let $\langle B, A, R\rangle$ be a formal context, $\nearrow$ and $\swarrow$ are concept-forming operators and $X \in \mathscr{P}(B)$, $Y \in \mathscr{P}(A)$. A pair $\langle X, Y\rangle$ such that $X^{\nearrow} = Y$ and $Y^{\swarrow} = X$ is called a formal concept of a given formal context. The set X is called extent of a formal concept and the set Y is called intent of a formal concept. The set of all formal concepts of a formal context $\langle B, A, R\rangle$ is a set*

$$\mathscr{C}(B, A, R) = \{\langle X, Y\rangle \in \mathscr{P}(B) \times \mathscr{P}(A) : X^{\nearrow} = Y,\ Y^{\swarrow} = X\}.$$

Definition 4. *Let $\langle X_1,Y_1\rangle, \langle X_2,Y_2\rangle \in \mathrm{C}(B,A,R)$ be two formal concepts of a formal context $\langle B,A,R\rangle$. Let $\preceq$ be a partial order in which $\langle X_1,Y_1\rangle \preceq \langle X_2,Y_2\rangle$ if and only if $X_1 \subseteq X_2$. A partially ordered set $(\mathscr{C}(B,A,R),\preceq)$ is called a concept lattice of a given context and is denoted by* $\mathrm{CL}(B,A,R)$.

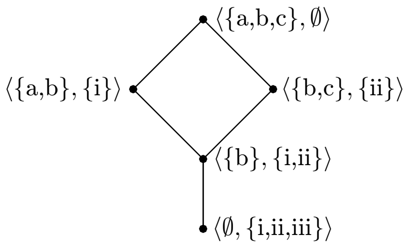

Figure 3: A concept lattice of $\langle\{\mathrm{a,b,c}\},\{\mathrm{i,ii,iii}\},R\rangle$

3 One-sided fuzzy approach

The statements that people use to communicate facts about the world are usually not bivalent. The truth of such statements is a matter of degree, rather than being only true or false. Fuzzy logic and fuzzy set theory are frameworks which extend formal concept analysis in various independent ways [5, 6, 9, 23]. Here, we recall the basic definitions of fuzzy formal context. The structures of partially ordered set, complete lattice or residuated lattice are applied here to represent data. The last one allows to speed up the computing.

Definition 5. *Consider two nonempty sets B a A, a set of truth degrees T and a mapping R such that $R: B\times A \longrightarrow T$. Then the triple $\langle B,A,R\rangle$ is called a (T)-fuzzy formal context, the elements of the sets B and A are called objects and attributes, respectively. The mapping R is a fuzzy incidence relation.*

In the definition of (T)-fuzzy formal context, we often take the interval $T = [0,1]$, because it is a scale of truth degrees commonly used in many applications. For such replacement, the terminology of $[0,1]$-fuzzy formal context has been adopted. Analogously, one can define the (more general) notion of L-fuzzy formal context, or P-fuzzy formal context, having replaced the interval $[0,1]$ by the algebraic structures of complete residuated lattice L, partially ordered set P or other plausible scale of truth degrees. Several extensions were advocated by the authors to provide the knowledge extraction from (T)-fuzzy formal contexts, whereby the set of truth degrees $T \in \{L; P; [0,1]; \{0,0.5,1\}; \{a_1,\dots,a_n\}; \dots\}$ is frequently selected.

The one-sided approach [23] one can represent by the concept-forming operators in a non-symmetric way. For [0,1]-fuzzy formal context and for every crisp subset of

R	a_1	a_2	a_3
b_1	1	0.9	0.8
b_2	0.8	0.7	0.7
b_3	0.3	0.3	0.3
b_4	0.8	0.6	0.9

Figure 4: Example of $[0,1]$-fuzzy formal context

objects X, the first function assigns the specific truth degree of the attribute (each object from X has this attribute at least in this specific truth degree):

Definition 6. *Let $X \subseteq B$ and $\uparrow: \mathscr{P}(B) \longrightarrow [0,1]^A$. Then $\uparrow$ is a mapping that assigns to every crisp set X of objects a fuzzy membership function $X^\uparrow$ of attributes, such that a value in a point $a \in A$ is:*

$$X^\uparrow(a) = \inf\{R(b,a) : b \in X\}. \tag{1}$$

Conversely, for each fuzzy membership function of attributes, the second concept-forming operator assigns the specific crisp set of objects (each included object has all attributes at least in a truth degree given by this fuzzy membership function):

Definition 7. *Let $f : A \to [0,1]$ and $\downarrow: [0,1]^A \longrightarrow \mathscr{P}(B)$. Then $\downarrow$ is a mapping that assigns to every fuzzy membership function f of attributes a crisp set $\downarrow(f)$ of objects, such that:*

$$f^\downarrow = \{b \in B : (\forall a \in A) R(b,a) \geq f(a)\}. \tag{2}$$

Lemma 1. *The pair $\langle\uparrow,\downarrow\rangle$ forms a Galois connection.*

Proof. Take $X, X_1, X_2 \subseteq B$ and $f, f_1, f_2 \in [0,1]^A$. The inequality $f_1 \leq f_2$ expresses that $f_1(a) \leq f_2(a)$ for all $a \in A$. From Eq. (1) and (2), it holds that

- $X_1 \subseteq X_2$ implies that $X_1^\uparrow \geq X_2^\uparrow$,
- $f_1 \leq f_2$ implies that $f_1^\downarrow \supseteq f_2^\downarrow$,
- $X \subseteq X^{\uparrow\downarrow}$,
- $f \leq f^{\downarrow\uparrow}$,

which are the assumptions on the pair of mappings to be a Galois connection. □

In addition, the composition of Eq. (1) and (2) allows us to define the notion of one-sided fuzzy concept.

Definition 8. *Let $X \subseteq B$ and $f \in [0,1]^A$. The pair $\langle X,f\rangle$ is called a one-sided fuzzy concept, if $X^\uparrow = f$ and $f^\downarrow = X$. The crisp set of objects X is called the extent and the fuzzy membership function $X^\uparrow$ is called the intent of one-sided fuzzy concept.*

The set of all one-sided fuzzy concepts ordered by inclusion of extents forms a complete lattice, called one--sided fuzzy concept lattice, as introduced in [23]. This construction is a generalization of classical approach from [18]. The one-sided fuzzy concept lattices for a fuzzy formal context from Fig. 4 is illustrated in Fig. 5.

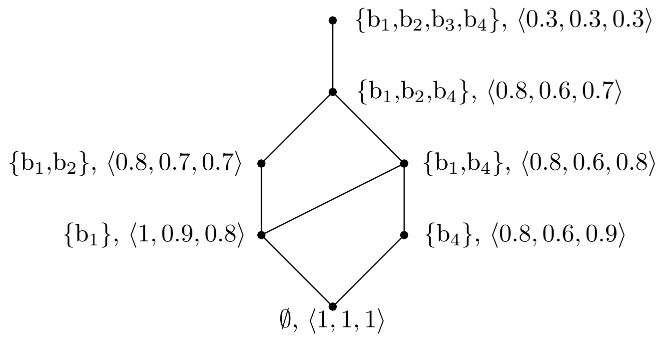

Figure 5: One-sided fuzzy concept lattice for $[0,1]$-fuzzy formal context from Fig. 4

3.1 Modified Rice-Siff algorithm

In effort to reduce the number of one-sided fuzzy formal concepts, the Rice-Siff algorithm was modified and applied in [23, 25]. The method focuses on the distance function and its metric properties. The distance function $\rho : \mathscr{P}(B) \times \mathscr{P}(B) \to \mathbb{R}$ is defined for $X_1, X_2 \subseteq B$ by:

$$\rho(X_1,X_2) = 1 - \frac{\sum_{a\in A} \min\{\uparrow(X_1)(a), \uparrow(X_2)(a)\}}{\sum_{a\in A} \max\{\uparrow(X_1)(a), \uparrow(X_2)(a)\}}.$$

This function is a metric on the set of all extents. The function is a cornerstone of Alg. 1.

Algorithm 1. (Modified Rice-Siff algorithm)
input: $\langle B,A,R\rangle$
$\mathscr{C} \leftarrow \mathscr{D} \leftarrow \{\{b\}^{\uparrow\downarrow} : b \in B\}$;
while $(|\mathscr{D} > 1|)$ *do* {
$m \leftarrow \min\{\rho(X_1,X_2) : X_1, X_2 \in \mathscr{D}, X_1 \neq X_2\}$
$\Psi \leftarrow \{\langle X_1,X_2\rangle \in \mathscr{D}\times\mathscr{D} : \rho(X_1,X_2) = m\}$
$\mathscr{V} \leftarrow \{X \in \mathscr{D} : (\exists Y \in \mathscr{D})\langle X,Y\rangle \in \Psi\}$
$\mathscr{N} \leftarrow \{(X_1 \cup X_2)^{\uparrow\downarrow} : \langle X_1,X_2\rangle \in \Psi\}$
$\mathscr{D} \leftarrow (\mathscr{D} \setminus \mathscr{V}) \cup \mathscr{N}$
$\mathscr{C} \leftarrow \mathscr{C} \cup \mathscr{N}$
}
output: $\mathscr{C}$

Notice that set $\mathscr{D}$ is changed in each loop by excluding elements of $\mathscr{V}$ and joining a member of $\mathscr{N}$ in each loop. It assures that set $\mathscr{D}$ is still decreasing. More particular, two clusters with minimal distance are joined in each step of algorithm and the closure of their union is returned as the output. Such closures are gathered in a tree-based structure on the subset hierarchy with the cluster of all objects in the root. The zero iterations gather the closures of singletons, therefore the value of minimal distance function is not computed in the zero step. The more detailed properties of this clustering method with the special defined metric are described in [23, 24, 25].

3.2 Subset quality measure

Snášel et al. in [38] reflect the transformation of the original $[0,1]$-fuzzy formal context to the sequence of classical formal contexts (from Definition 1) using the binary relations called α-cuts for $\alpha \in [0,1]$. The core of our novel modification in this approach (i. e. lower cuts and interval cuts) follows and it can be fruitfully applied for real data.

Definition 9. *Let* $\langle B,A,R\rangle$ *be* $[0,1]$*-fuzzy formal context and let* $\alpha \in [0,1]$*. Then the binary relation* $R_\alpha \subseteq B\times A$ *is called*

- *the upper* α*-cut if* $\langle b,a\rangle \in R_\alpha$ *is equivalent to* $R(b,a) \geq \alpha$*,*
- *the lower* α*-cut if* $\langle b,a\rangle \in R_\alpha$ *is equivalent to* $R(b,a) \leq \alpha$*.*

The binary relation $R_{\alpha\beta} \subseteq B\times A$ *is called*

- *the interval* $\alpha\beta$*-cut if* $\langle b,a\rangle \in R_{\alpha\beta}$ *is equivalent to* $R(b,a) \in [\alpha,\beta]$*.*

It can be seen that the triple $\langle B,A,R_\alpha\rangle$ for every $\alpha \in [0,1]$ forms the formal context given by Definition 1. For each formal context, one can build the corresponding concept lattice $\mathrm{CL}(\langle B,A,R_\alpha\rangle)$ by Definition 4. With respect to the division of the interval $[0,1]$ into n parts, we can define the subset quality measure as follows.

Definition 10. *Let* $X \subseteq B$ *and* R_α *be the upper* α*-cuts for* $\alpha \in [0,1]$*. Then the upper quality measure of the subset* X *is the value*

$$q_{\mathrm{upp}}(X,n) = \frac{\left|\left\{p \in \{0,1,\dots,n\} : \left(\exists Y \subseteq A\right)\langle X,Y\rangle \in \mathrm{CL}\left(B,A,R_{\frac{p}{n}}\right)\right\}\right|}{n+1},$$

whereby $n+1$ *is the count of different values of* α *which divides interval* $[0,1]$ *into n partitions.*

The formula of the lower quality measure $q_{\mathrm{low}}(X,n)$ of the subset X one can build analogously. However, the slight modification is naturally needed if we consider the interval $\alpha\beta$-cuts.

Definition 11. *Let* $X \subseteq B$ *and* $R_{\alpha\beta}$ *be the interval* $\alpha\beta$*-cuts for* $\alpha,\beta \in [0,1]$*. Then the interval quality measure of the subset* X *is the value*

$$q_{\mathrm{int}}(X,n) = \frac{\left|\left\{\langle p,r\rangle \in J : \left(\exists Y \subseteq A\right)\langle X,Y\rangle \in \mathrm{CL}\left(B,A,R_{\frac{p}{n}\frac{r}{n}}\right)\right\}\right|}{|J|},$$

whereby $J = \left\{\langle p,r\rangle \in \{0,1,\dots,n\} \times \{0,1,\dots,n\} \wedge p < r\right\}$ *and* $n+1$ *is the count of different values of* α *and simultaneously* β *which divides interval* $[0,1]$ *into n partitions.*

In this paper, we omit the definitions of α-concepts, since more details about the properties of these structures can be found in [26, 2]. Moreover, the reduction of concepts from generalized one-sided concept lattices based on the method of upper α-cuts is introduced in the recently published book chapter of Butka et al. [14]. A method for α,β-cut of bipolar fuzzy formal contexts with illustrative examples is proposed in [36, 37].

3.3 Gaussian probabilistic index

In our recent work [3], the notions of $[0,1]$-fuzzy formal contexts and random variables are connected in effort to define the randomized formal contexts and to explore the stability of extents of one-sided fuzzy formal concepts.

We will consider the sample space Ω as a set of all possible finite or infinite outcomes of a random study. An event T is an arbitrary subset of Ω. The probability function p on a finite ($\{\omega_1,\dots,\omega_n\}$) or infinite (e. g. interval of real numbers) sample space Ω assigns to each event $T \subseteq \Omega$ a number $p(T) \in [0,1]$ such that $p(\Omega) = 1$ and $p(T_1 \cup T_2 \cup \dots) = p(T_1) + p(T_2) + \dots$ for $T_1, T_2, \dots$ which are disjoint. From $T \cup T^{\mathrm{c}} = \Omega$, we deduce that $p(T^{\mathrm{c}}) = 1 - p(T)$. Events $T_1, T_2, \dots T_m$ are called independent if $p(T_1 \cap T_2 \cap \dots \cap T_m) = \prod_{i=1}^{m} p(T_i)$.

Definition 12. *Let $\langle B,A,R\rangle$ be $[0,1]$-fuzzy formal context. For $i \in \{1,\dots,n\}$, consider the system of $[0,1]$-fuzzy formal contexts $\langle B,A,R_i\rangle$ such that*

$$R_i(b,a) = \min\Big\{1, \max\big\{0, R(b,a) + \varepsilon_{b,a,i}\big\}\Big\},$$

whereby $\varepsilon_{b,a,i}$ is a normally distributed value of a random variable $\mathscr{E}_{b,a}$ with the mean 0 and variance σ^2, i. e. $\mathscr{E}_{b,a} \sim \mathrm{N}(0,\sigma^2)$, for all $b \in B, a \in A$.

Let $X \subseteq B$. The Gaussian probability index gpi : $\mathscr{P}(B) \times \mathbb{R}^{+} \to [0,1]$ *is the function given by*

$$\mathrm{gpi}(X,\sigma) = p(X \textit{ is an extent of } \langle B,A,R_i\rangle)$$

for an arbitrary subset of objects X, an arbitrary standard deviation σ and mean 0. The [0,1]-fuzzy formal context $\langle B,A,R_i\rangle$ will be called the randomized (fuzzy) formal context for each $i \in \{1,\dots,n\}$.

R

	a_1	a_2
b_1	0.8	0.1
b_2	0.5	0.7
b_3	0.1	0.9

$\longrightarrow$

R_1

	a_1	a_2
b_1	0.87	0.07
b_2	0.46	0.72
b_3	0	0.93

R_2

	a_1	a_2
b_1	1	0.11
b_2	0.53	0.82
b_3	0.08	0.78

$\dots$

R_n

	a_1	a_2
b_1	0.78	0
b_2	0.51	0.66
b_3	0.15	1

Figure 6: Example of randomized formal contexts

The values of Gaussian probability index express the probability of X being the extent of the arbitrary randomized formal context by supposing the standard deviation σ in the values of the incidence relation R_i in comparison with the original incidence relation R. Alternatively, the values of the Gaussian probability index one can compute by the following construction. Consider the randomized formal contexts $\langle B,A,R_1\rangle, \langle B,A,R_2\rangle \dots, \langle B,A,R_n\rangle$ for a large positive integer n (see Fig. 6). Then by the classical definition of probabilistic function p one can write

$$\mathrm{gpi}(X,\sigma) = \frac{|i, i \in \{1,2,\dots,n\} : X \text{ is an extent of } \langle B,A,R_i\rangle|}{n}. \tag{3}$$

The computation of Eq. (3) is described by Alg. 2.

Algorithm 2. (Algorithm of Gaussian probabilistic index)
input: $\langle B,A,R\rangle$, X, σ, n
  $k \leftarrow 0$;
  for $i := 1$ *to* n *do*
    {
    for all $b \in B$ *do*
      for all $a \in A$ *do*
        {
        $\varepsilon_{b,a,i} \leftarrow$ Random.nextGaussian() $* \sigma$;
        $R_i(b,a) \leftarrow \min\{1, \max\{0, R(b,a) + \varepsilon_{b,a,i}\}\}$;
        }
    if (X *is an extent of* $\langle B,A,R_i\rangle$) *then*
      $k \leftarrow k+1$;
    }
  $\mathrm{gpi}(X,\sigma) \leftarrow \dfrac{k}{n}$;
output: $\mathrm{gpi}(X,\sigma)$

In effort to express the values of Gaussian probabilistic index directly from the input $[0,1]$-fuzzy formal context, we explore the probabilistic aspects of randomized formal contexts including the boundary test conditions in [3].

Theorem 1. *Let $X \subseteq B$ and let $\langle B,A,R_i\rangle$ be a randomized formal context for some $i \in \{1,\dots,n\}$, i. e.*

$$R_i(b,a) = \min\Big\{1, \max\big\{0, R(b,a) + \varepsilon_{b,a,i}\big\}\Big\}$$

for the $[0,1]$-fuzzy formal context $\langle B,A,R\rangle$ and normally distributed value $\varepsilon_{b,a,i}$ of random variable $\mathscr{E}_{b,a} \sim \mathrm{N}(0,\sigma^2)$ for all $b \in B, a \in A$. Then the value of Gaussian probabilistic index for the subset $X \subseteq B$ and standard deviation σ is given by

$$\mathrm{gpi}(X,\sigma) = p\Big(\bigcap_{o \in B \setminus X} \Big(\bigcap_{a \in A} \Big(\Big(\bigcap_{x \in X} T_x\Big)^{\mathrm{c}}\Big)^{\mathrm{c}}\Big)\Big),$$

where T_x represents the event

$$\begin{aligned} \mathscr{E}_{o,a} - \mathscr{E}_{x,a} < R(x,a) - R(o,a) \quad &\wedge \\ \mathscr{E}_{o,a} < 1 - R(o,a) \quad &\wedge \\ \mathscr{E}_{x,a} > -R(o,a). & \end{aligned}$$

For more details, see the results from [3]. Here, we emphasize that the set of pairs $\{\langle X, \mathrm{gpi}(X,\sigma)\rangle : X \subseteq B\}$ for some σ can be ordered by the second coordinate, which gives the opportunity to use the Gaussian probabilistic index to select the relevant one-sided formal concepts in the applications.

3.4 Comparative remarks

The relationship between the Gaussian probabilistic index and the methods from Subsection 3.1 and 3.2 is now briefly outlined:

- every cluster $\mathscr{N}$ obtained by modified Rice-Siff algorithm is the extent of one-sided fuzzy formal concept of the input formal context (because we have that $\mathscr{N} = \{(X_1 \cup X_2)^{\uparrow\downarrow} : \langle X_1, X_2\rangle \in \Psi\}$),

- modified Rice-Siff algorithm represents the crisp index for selection of one-sided concepts, the Gaussian probabilistic index is a fuzzy index,
- the subset quality measure and the Gaussian probabilistic index can be applied also for the subsets which are not the extents of the one-sided formal concepts of the input [0,1]-fuzzy formal context,
- the clusters obtained by modified Rice-Siff algorithm have mostly the higher $\mathrm{gpi}(X,\sigma)$ as the other extents of one-sided formal concepts, some exceptions exist,
- the Gaussian probabilistic index gpi works with data tables (relations) which need not to be ordinally equivalent. The relationship between the ordinally equivalent relations were explored by Bělohlávek [7],
- we conclude that it is important to understand the advantages of the available methods and to apply them separately or in their mutual combination.

The comparative example on the modified Rice-Siff algorithm and the Gaussian probabilistic index can be found in [3] including the interpretation and explanations.

4 Applications and future work

An extensive overview of papers which apply formal concept analysis in various domains including software mining, web analytics, medicine, biology and chemistry data is provided in [32]. Particularly, we mention the conceptual difficulties in the education of mathematics [34], the techniques for analyzing and improving integrated care pathways [33] or evaluation of questionnaires [10, 8]. In [20], formal concept analysis is applied as a tool for image processing and detection of inaccuracies. Recently, the morphological image and signal processing from the viewpoint of fuzzy formal concept analysis was presented in [1]. The main results offer the possibility to interpret the binary images as the classical formal concepts and open digital signals as fuzzy formal concepts.

Regarding one-sided fuzzy approach from Section 3, a set of representative symptoms for the disease are investigated in [21]. Furthermore, the application of fuzzy concepts clustering in the domain of text documents [13, 35] or attribute characterizations of cars in generalized one--sided concept lattices [19] are the subjects of study.

In our future work, our aim is to extend the results presented in [23, 24, 25] and to verify the methods in the applications from the educational area or in the area of social networks. More particular, for a given set of students from a longitudinal survey about the relationships between the students in the secondary school classes, we can compute

a) the clusters of students sensed similar by their schoolmates (by modified Rice-Siff algorithm from Subsection 3.1),

b) the clusters of more popular students or less popular students (by upper or lower cuts of subset quality measure from Subsection 3.2),

c) the stable clusters of students sensed similar due to random fluctuation of data (by Gaussian probabilistic index from Subsection 3.3).

The another possibility is to consider a set of students and their scores of the tests from different subjects (see Fig. 7). Take for example student b_2 and find the students with better results as b_2 in all subjects. From Section 3 we have that $\{\mathrm{b}_2\}^{\uparrow\downarrow} = \{\mathrm{b}_1, \mathrm{b}_2\}$. Will it be valid after the repeated exams? We suppose that student b_3 will not be better than b_1 or b_2. However, how about student b_4? What is the probability of that some other student will join the group $\{\mathrm{b}_1, \mathrm{b}_2\}$ in other testing?

R	a_1	a_2	a_3
b_1	1	0.9	0.8
b_2	0.8	0.7	0.7
b_3	0.3	0.3	0.3
b_4	0.8	0.6	0.9

Figure 7: Students and their scores

We can answer these question by Gaussian probabilistic index presented in Subsection 3.3. The Gaussian normal distribution one can replace by real observations of teachers who can estimate the standard deviations for each individual student. We can suppose that one of the students will obtain roughly 90% in the most of exams, but once a time it can happened that he/she will pass 70% for different reasons, otherwise will reach 98%.

In another way, the paper [31] compares several collaborative-filtering techniques on a dataset from courses with only a few of students. The random Galois lattices [16], the randomized formal contexts of a discrete random variable, a generalized probability framework [17] and stability in a multi-adjoint framework [28, 29, 30] or heterogeneous framework [4] will be the point of interest in our future work, as well.

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