Fast Infinitesimal Fourier Transform for Signal and Image Processing via Multiparametric and Fractional Fourier Transforms

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Abstract. The fractional Fourier transforms (FrFTs) is one-parametric family of unitary transformations $\{\mathcal{F}^{\alpha}\}_{\alpha=0}^{2\pi}$. FrFTs found a lot of applications in signal and image processing. The identical and classical Fourier transformations are both the special cases of the FrFTs. They correspond to $\alpha = 0$ ($\mathcal{F}^0 = I$) and $\alpha = \pi/2$ ($\mathcal{F}^{\pi/2} = \mathcal{F}$), respectively. Up to now, the fractional Fourier spectra $F^{\alpha_i} = \mathcal{F}^{\alpha_i}\{f\}$, $i = 1, 2, ..., M$, has been digitally computed using classical approach based on the fast discrete Fourier transform. This method maps the N samples of the original function f to the N samples of the set of spectra $\{F^{\alpha_i}\}_{i=1}^{M}$, which requires $MN(2 + \log_2 N)$ multiplications and $MN \log_2 N$ additions. This paper develops a new numerical algorithm, which requires $2MN$ multiplications and $3MN$ additions and which is based on the infinitesimal Fourier transform.

Keywords: Fast fractional Fourier transform, infinitesimal Fourier transform, Schrödinger operator, signal and image analysis

1 Introduction

The idea of fractional powers of the Fourier operator $\{\mathcal{F}^{a}\}_{a=0}^{4}$ appeared in the mathematical literature [1,2,3,4]. The idea is to consider the eigen-value decomposition of the Fourier transform $\mathcal{F}$ in terms of the eigen-values $\lambda_n = e^{jn\pi/2}$ and eigen-functions in the form of the Hermite functions. The family of FrFT $\{\mathcal{F}^{a}\}_{a=0}^{4}$ is constructed by replacing the n-th eigen-value $\lambda_n = e^{jn\pi/2}$ by its a-th power $\lambda_n^a = e^{jn\pi a/2}$ for a between 0 and 4. This value is called the transform order. There is the angle parameterization $\{\mathcal{F}^{\alpha}\}_{\alpha=0}^{2\pi}$, where $\alpha = \pi a/2$ is a new angle parameter. Since this family depends on a single parameter, the fractional operators $\{\mathcal{F}^a\}_{a=0}^4$ (or $\{\mathcal{F}^\alpha\}_{\alpha=0}^{2\pi}$) form the Fourier-Hermite one-parameter strongly continuous unitary multiplicative group $\mathcal{F}^a\mathcal{F}^b = \mathcal{F}^{a \underset{4}{\oplus} b}$ (or $\mathcal{F}^\alpha\mathcal{F}^\beta = \mathcal{F}^{\alpha \underset{2\pi}{\oplus} \beta}$), where $a \underset{4}{\oplus} b = (a+b)\,\mathrm{mod}4$ (or $\alpha \underset{2\pi}{\oplus} \beta = (\alpha+\beta)\,\mathrm{mod}2\pi$) and $\mathcal{F}^0 = I$. The identical and classical Fourier transformations are both the special cases of the FrFTs. They correspond to $\alpha = 0$ ($\mathcal{F}^0 = I$) and $\alpha = \pi/2$ ($\mathcal{F}^{\pi/2} = \mathcal{F}$), respectively.

In 1980, Namias reinvented the fractional Fourier transform (FrFT) again in his paper [6]. He used the FrFT in the context of quantum mechanics as a way to solve certain problems involving quantum harmonic oscillators. He not only stated the standard definition for the FrFT, but, additionally, developed an operational calculus for this new transform. This approach was extended by McBride and Kerr [7]. Then Mendlovic and Ozaktas introduced the FrFT into the field of optics [8] in 1993. Afterwards, Lohmann [9] reinvented the FrFT based on the Wigner-distribution function and opened the FrFT to bulk-optics applications. It has been rediscovered in signal and image processing [10]. In these cases, the FrFT allows us to extract time-frequency information from the signal. A recent state of the art can be found in [11]. In the series of papers [12,13,14,15,16], we developed a wide class of classical and quantum fractional transforms.

In this paper, the infinitesimal Fourier transforms are introduced, and the relationship of the fractional Fourier transform with the Schrödinger operator of the quantum harmonic oscillator is discussed. Up to now, the fractional Fourier spectra $F^{\alpha_i} = \mathcal{F}^{\alpha_i}\{f\}$, $i = 1, 2, ..., M$, have been digitally computed using classical approach based on the fast discrete Fourier transform. This method maps the N samples of the original function f to the NM samples of the set of spectra $\{F^{\alpha_i}\}_{i=1}^M$, which requires $MN(2+\log_2 N)$ multiplications and $MN\log_2 N$ additions. This paper develops a new numerical algorithm, which requires $2MN$ multiplications and $3MN$ additions and which is based on the infinitesimal Fourier transform.

2 Eigen-decomposition and Fractional Discrete Transforms

Let $\mathcal{F} = [F_k(i)]_{k,i=0}^{N-1}$ be an arbitrary discrete unitary $(N \times N)$-transform, λ_n and $\Psi_n(t)$ $n = 0, 1, \ldots, N-1$ be its eigen-values and eigen-vectors, respectively. Let $\mathbf{U} = \left[\Psi_0(i)|\Psi_1(i)|\vdots|\Psi_{N-1}(i)\right]$ be the matrix of the $\mathcal{F}$-transform eigen-vectors. Then $\mathbf{U}^{-1}\cdot\mathcal{F}\cdot\mathbf{U} = \mathbf{Diag}\{\lambda_n\}$. Hence, we have the following eigen-decomposition: $\mathcal{F} = [F_k(i)] = \mathbf{U}\cdot\mathbf{\Lambda}\cdot\mathbf{U}^{-1} = \mathbf{U}\cdot\mathbf{Diag}\{\lambda_n\}\cdot\mathbf{U}^{-1}$.

Definition 1. [12]. *For an arbitrary real numbers $a_0, \ldots, a_{N-1}$, we introduce the multi-parametric $\mathcal{F}$-transform*

$$\mathcal{F}^{(a_0,\ldots,a_{N-1})} := \mathbf{U}\left\{\mathbf{diag}\left(\lambda_0^{a_0}, \ldots, \lambda_{N-1}^{a_{N-1}}\right)\right\}\mathbf{U}^{-1}. \tag{1}$$

If $a_0 = \ldots = a_{N-1} \equiv a$ then this transform is called fractional $\mathcal{F}$-transform [12,13,14,15,16]. For this transform we have

$$\mathcal{F}^a := \mathbf{U}\left\{\mathbf{diag}\left(\lambda_0^a, \ldots, \lambda_{N-1}^a\right)\right\}\mathbf{U}^{-1} = \mathbf{U}\boldsymbol{\Lambda}^a\mathbf{U}^{-1}. \quad (2)$$

The zero-th-order fractional $\mathcal{F}$-transform is equal to the identity transform: $\mathcal{F}^0 = \mathbf{U}\boldsymbol{\Lambda}^0\mathbf{U}^{-1} = \mathbf{U}\mathbf{U}^{-1} = \mathbf{I}$, and the first-order fractional Fourier transform operator $\mathcal{F}^1 = \mathcal{F}$ is equal to the initial $\mathcal{F}$-transform $\mathcal{F}^1 = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1}$.

The families $\left\{\mathcal{F}^{(\alpha_0,\ldots,\alpha_{N-1})}\right\}_{(\alpha_0,\ldots,\alpha_{N-1})\in\mathbf{R}^N}$ and $\{\mathcal{F}^a\}_{a\in\mathbf{R}}$ form multi- and one-parameter continuous unitary groups, respectively, with multiplication rules

$$\mathcal{F}^{(a_0,\ldots,a_{N-1})}\mathcal{F}^{(b_0,\ldots,b_{N-1})} = \mathcal{F}^{(a_0+b_0,\ldots,a_{N-1}+b_{N-1})} \quad \text{and} \quad \mathcal{F}^a\mathcal{F}^b = \mathcal{F}^{a+b}.$$

Indeed, $\mathcal{F}^a\mathcal{F}^b = \mathbf{U}\boldsymbol{\Lambda}^a\mathbf{U}^{-1}\cdot\mathbf{U}\boldsymbol{\Lambda}^b\mathbf{U}^{-1} = \mathbf{U}\boldsymbol{\Lambda}^{a+b}\mathbf{U}^{-1} = \mathcal{F}^{a+b}$ and

$$\mathcal{F}^{(a_0,\ldots,a_{N-1})}\mathcal{F}^{(b_0,\ldots,b_{N-1})} =$$

$$= \mathbf{U}\left\{\mathbf{diag}\left(\lambda_0^{a_0}, \ldots, \lambda_{N-1}^{a_{N-1}}\right)\right\}\mathbf{U}^{-1}\cdot\mathbf{U}\left\{\mathbf{diag}\left(\lambda_0^{b_0}, \ldots, \lambda_{N-1}^{b_{N-1}}\right)\right\}\mathbf{U}^{-1} =$$

$$= \mathbf{U}\left\{\mathbf{diag}\left(\lambda_0^{a_0+b_0}, \ldots, \lambda_{N-1}^{a_{N-1}+b_{N-1}}\right)\right\}\mathbf{U}^{-1} = \mathcal{F}^{(a_0+b_0,\ldots,a_{N-1}+b_{N-1})}.$$

Let $\mathcal{F} = [F_k(i)]_{k,i=0}^{N-1}$ be a discrete Fourier $(N\times N)$-transform (DFT), then $\lambda_n = e^{j\pi n/2} \in \{\pm 1, \pm j\}$, where $j = \sqrt{-1}$ and $\{\Psi_n(t)\}_{n=0}^{N-1}$ are the Kravchuk polynomials.

Definition 2. *The multi-parametric and fractional DFT are*

$$\mathcal{F}^{(a_0,\ldots,a_{N-1})} := \mathbf{U}\left\{\mathbf{diag}\left(e^{j\pi 0 a_0/2}, e^{j\pi 1 a_1/2}, \ldots, e^{j\pi(N-1)a_{N-1}/2}\right)\right\}\mathbf{U}^{-1},$$

$$\mathcal{F}^a := \mathbf{U}\left\{\mathbf{diag}\left(e^{j\pi na/2}\right)\right\}\mathbf{U}^{-1}$$

and

$$\mathcal{F}^{(\alpha_0,\ldots,\alpha_{N-1})} := \mathbf{U}\left\{\mathbf{diag}\left(e^{j0\alpha_0}, e^{j1\alpha_1}, \ldots, e^{j(N-1)\alpha_{N-1}}\right)\right\}\mathbf{U}^{-1},$$

$$\mathcal{F}^\alpha := \mathbf{U}\left\{\mathbf{diag}\left(e^{jn\alpha}\right)\right\}\mathbf{U}^{-1}$$

in a- and α-parameterizations, respectively, where $\alpha = \pi a/2$.

The parameters $(a_0, \ldots, a_{N-1})$ and a can be any real values. However, the operators $\mathcal{F}^{(a_0,\ldots,a_{N-1})}$ and $\mathcal{F}^a$ are periodic in each parameter with period 4 since $\mathcal{F}^4 = I$. Hence, $\mathcal{F}^{(a_0,\ldots,a_{N-1})}\mathcal{F}^{(b_0,\ldots,b_{N-1}} = \mathcal{F}^{(a_0 \underset{4}{\oplus} b_0,\ldots,a_{N-1}\underset{4}{\oplus} b_{N-1})}$ and $\mathcal{F}^a\mathcal{F}^b = \mathcal{F}^{a\underset{4}{\oplus} b}$, where $a_i \underset{4}{\oplus} b_i = (a_i+b_i)\,\mathrm{mod}4$, $\forall i = 0,1,...,N-1$. Therefore, the ranges of $(a_0,\ldots,a_{N-1})$ and a are $(\mathbf{Z}/4\mathbf{Z})^N = [0,4]^N = [-2,2]^N$ and $\mathbf{Z}/4\mathbf{Z} = [0,4] = [-2,2]$, respectively.

In the case of α-parameterization, we have $\alpha_i \underset{2\pi}{\oplus} \beta_i = (\alpha_i+\beta_i)\,\mathrm{mod}2\pi$, $\forall i = 0,1,...,N-1$. So, the ranges of $(\alpha_0,\ldots,\alpha_{N-1})$ and α are $(\mathbf{Z}/2\pi\mathbf{Z})^N = [0,2\pi]^N = [-\pi,\pi]^N$ and $\mathbf{Z}/2\pi\mathbf{Z} = [0,2\pi] = [-\pi,\pi]$, respectively.

3 Canonical FrFT

The continuous Fourier transform is a unitary operator $\mathcal{F}$ that maps square-integrable functions on square-integrable ones and is represented on these functions $f(x)$ by the well-known integral

$$F(y) = (\mathcal{F}f)(y) = \frac{1}{\sqrt{2\pi}} \int_{x\in\mathbf{R}} f(x) e^{-jyx} dx. \tag{3}$$

Relevant properties are that the square $\left(\mathcal{F}^2 f\right)(x) = f(-x)$ is the inversion operator, and that its fourth power $\left(\mathcal{F}^4 f\right)(x) = f(x)$ is the identity. Hence, $\mathcal{F}^3 = \mathcal{F}^{-1}$. Thus, the operator $\mathcal{F}$ generates a cyclic group of the order 4. In 1961, Bargmann extended the Fourier transform in his paper [5] where he gave definition of the FrFT that was based on the Hermite polynomials as an integral transformation. If $H_n(x)$ is a Hermite polynomial of order n, where $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{x^2}$, then for $n \in \mathbf{N}_0$, functions $\Psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-x^2/2}$ are the eigen-functions of the Fourier transform

$$\mathcal{F}\left[\Psi_n(x)\right] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Psi_n(x) e^{2\pi \mathrm{j} yx} dx = \lambda_n \Psi_n(y) = e^{-j\frac{\pi}{2}n} \Psi_n(y)$$

with $\lambda_n = j^n = e^{-j\frac{\pi}{2}n}$ being the eigen-value corresponding to the n-th eigen-function. According to Bargmann, the fractional Fourier transform $\mathcal{F}^\alpha = \left[K^\alpha(x,y)\right]$ is defined through its eigen-functions as

$$K^\alpha(x,y) := \mathbf{U}\left\{\mathbf{diag}\left(\mathbf{e}^{-\mathbf{j}\alpha\mathbf{n}}\right)\right\}\mathbf{U}^{-1} = \sum_{n=0}^{\infty} e^{-j\alpha n} \Psi_n(x) \Psi_n(y). \tag{4}$$

Hence,

$$K^\alpha(x,y) := \sum_{n=0}^{\infty} e^{-j\alpha n} \Psi_n(x) \Psi_n(y) = e^{-\left(x^2+y^2\right)} \sum_{n=0}^{\infty} \frac{e^{-j\alpha n} H_n(x) H_n(y)}{2^n n! \sqrt{\pi}} =$$

$$= \frac{1}{\sqrt{\pi}\sqrt{1-e^{-2j\alpha}}} \cdot \exp\left\{ \frac{2xye^{-j\alpha} - e^{-2j\alpha}\left(x^2+y^2\right)}{1-e^{-2j\alpha}} \right\} \exp\left\{ -\frac{\left(x^2+y^2\right)}{2} \right\}, \tag{5}$$

where $K^\alpha(x,y)$ is the kernel of the FrFT. In the last step we used the Mehler formula [19]

$$\sum_{n=0}^{\infty} \frac{e^{-j\alpha n} H_n(x) H_n(y)}{2^n n! \sqrt{\pi}} = \frac{1}{\sqrt{\pi}\sqrt{1-e^{-2j\alpha}}} \exp\left\{ \frac{2xye^{-j\alpha} - e^{-2j\alpha}\left(x^2+y^2\right)}{1-e^{-2j\alpha}} \right\}.$$

Expression (5) can be rewritten as

$$K^\alpha(x,y) = \sqrt{\frac{1-j\cot\alpha}{2\pi}} \exp\left\{ \frac{j}{2\sin\alpha} \left[\left(x^2+y^2\right)\cos\alpha - 2xy \right] \right\},$$

where $\alpha \neq \pi\mathbf{Z}$ (or $a \neq 2\mathbf{Z}$). Obviously, functions $\Psi_n(x)$ are eigen-functions of the fractional Fourier transform $\mathcal{F}^\alpha [\Psi_n(x)] = e^{jn\alpha}\Psi_n(x)$ corresponding to the n-th eigen-values $e^{jn\alpha}$, $n = 0, 1, 2, \ldots$. The FrFT $\mathcal{F}^\alpha$ is a unitary operator that maps square-integrable functions $f(x)$ on square-integrable ones

$$F^\alpha(y) = (\mathcal{F}^\alpha f)(y) = \int_{x\in\mathbf{R}} f(x)K^\alpha(x,y)dx =$$

$$= \frac{e^{-\frac{j}{2}\left(\frac{\pi}{2}\hat{\alpha}-\alpha\right)}}{\sqrt{2\pi\,|\sin\alpha|}} \int_{\mathbf{R}} f(x) \exp\left\{ \frac{j}{2\sin\alpha} \left[\left(x^2+y^2\right)\cos\alpha - 2xy\right] \right\} dx.$$

There exist several algorithms for fast calculation of spectrum of the fractional Fourier transform $F^\alpha(y)$. But all of them are based on the following transform of the FrFT:

$$F^\alpha(y) = (\mathcal{F}^\alpha f)(y) = \frac{e^{-\frac{j}{2}\left(\frac{\pi}{2}\hat{\alpha}-\alpha\right)} e^{jy^2\frac{\cos\alpha}{2\sin\alpha}}}{\sqrt{2\pi\,|\sin\alpha|}} \int_{\mathbf{R}} \left[f(x)e^{j\frac{x^2}{2}\cot\alpha}\right] e^{-jxy}dx =$$

$$= A_\alpha(y) \cdot \mathcal{F}\left\{f(x) \cdot B_\alpha(x)\right\}(y),$$

where $A_\alpha(y) = \frac{e^{-\frac{j}{2}\left(\frac{\pi}{2}\hat{\alpha}-\alpha\right)} e^{jy^2\frac{\cos\alpha}{2\sin\alpha}}}{\sqrt{2\pi|\sin\alpha|}}$, $B_\alpha(x) = e^{j\frac{x^2}{2}\cot\alpha}$.

Let us introduce the uniform discretization of the angle parameter α on M discrete values $\{\alpha_0, \alpha_1, ..., \alpha_i, \alpha_{i+1}, ..., \alpha_{M-1}\}$, where $\alpha_{i+1} = \alpha_i + \Delta\alpha$, $\alpha_i = i\Delta\alpha$ and $\Delta\alpha = 2\pi/M$.

The set of M spectra $\{F^{\alpha_0}(y), F^{\alpha_1}(y), ..., F^{\alpha_{M-1}}(y)\}$ can be computed by applying the following sequence of steps for all $\{\alpha_0, \alpha_1, ..., \alpha_{M-1}\}$:

1. Compute products $f(x)B_{\alpha_k}(x)$, which require N multiplications.
2. Compute the Fast Fourier Transform ($N\log_2 N$ multiplications and additions).
3. Multiply the result by $A_\alpha(y)$ (N multiplications).

This numerical algorithm requires $MN\log_2 N$ additions and $MN\left(2 + \log_2 N\right)$ multiplications.

4 Infinitesimal Fourier Transform

In order to construct fast multi-parametric $\mathcal{F}$-transform and fractional Fourier transform algorithms we turn our attention to notion of a semigroup and its generator (infinitesimal operator). Let $\mathrm{L}_2(\mathbf{R},\mathbf{C})$ be a space of complex-valued functions (signals), and let $\mathbf{Op}(\mathrm{L}_2)$ be the Banach algebra of all bounded linear operators on $\mathrm{L}_2(\mathbf{R},\mathbf{C})$ endowed with the operator norm. A family $\{\mathbf{U}(\alpha)\}_{\alpha\in\mathbf{R}} \subset \mathbf{Op}(\mathrm{L}_2)$ is called the Hermite group on $\mathrm{L}_2(\mathbf{R},\mathbf{C})$ if it satisfies the Abel functional equations $\mathbf{U}(\alpha+\beta) = \mathbf{U}(\alpha)\mathbf{U}(\beta)$, $\alpha, \beta \in \mathbf{R}$ and $\mathbf{U}(0) = \mathbf{I}$, and the orbit maps $\alpha \to F^\alpha = \mathbf{U}(\alpha)\{f\}$ are continuous from $\mathbf{R}$ into $\mathrm{L}_2(\mathbf{R},\mathbf{C})$ for every $f \in \mathrm{L}_2(\mathbf{R},\mathbf{C})$.

Definition 3. *The infinitesimal generator* $\mathbf{A}(0)$ *of the group* $\{\mathbf{U}(\alpha)\}_{\alpha\in\mathbf{R}}$ *and the infinitesimal transform* $\mathbf{U}(d\alpha)$ *are defined as follows* [18,19]*:*

$$\mathbf{A}(0) = \left.\frac{\partial \mathbf{U}(\alpha)}{\partial \alpha}\right|_{\alpha=0}, \quad \mathbf{U}(d\alpha) = \mathbf{I} + d\mathbf{U}(0) = \mathbf{I} + \mathbf{A}(0)d\alpha.$$

Obviously,

$$\mathbf{U}(\alpha_0 + d\alpha) = \mathbf{U}(\alpha_0) + d\mathbf{U}(\alpha_0) = \mathbf{U}(\alpha_0) + \left.\frac{\partial \mathbf{U}(\alpha)}{\partial \alpha}\right|_{\alpha_0} d\alpha =$$

$$= \mathbf{U}(\alpha_0) + \mathbf{A}(\alpha_0)d\alpha.$$

But

$$\mathbf{U}(\alpha_0 + d\alpha) = \mathbf{U}(d\alpha_0)\mathbf{U}(\alpha_0) = \left[\mathbf{I} + d\mathbf{U}(0)\right]\mathbf{U}(\alpha_0) =$$

$$= \mathbf{U}(\alpha_0) + \left.\frac{\partial \mathbf{U}(\alpha)}{\partial \alpha}\right|_{\alpha=0} \mathbf{U}(\alpha_0)d\alpha =$$

$$= \mathbf{U}(\alpha_0) + \mathbf{A}(0)\mathbf{U}(\alpha_0)d\alpha = \left[\mathbf{I} + \mathbf{A}(0)\right]\mathbf{U}(\alpha_0)d\alpha.$$

Hence, $\mathbf{A}(\alpha_0) = \mathbf{A}(0)\mathbf{U}(\alpha_0)$ and $F^{\alpha_0+d\alpha}(y) = \Big[\mathbf{I} + \mathbf{A}(0)\Big]F^{\alpha_0}(y)d\alpha$.

Define now the linear operator $\mathcal{H} = \frac{1}{2}\left(\frac{d^2}{dx^2} - x^2 + 1\right)$. It is known that

$$\mathcal{H}\Psi_n(x) = \frac{1}{2}\left(\frac{d^2}{dx^2} - x^2 + 1\right)\Psi_n(x) = n\Psi_n(x). \tag{6}$$

From (4) and (6) we have

$$j\frac{\partial}{\partial\alpha}F^\alpha(y)\Big|_{\alpha=0} = j\frac{\partial}{\partial\alpha}\left\{\mathcal{F}^\alpha F\right\}(y)\Big|_{\alpha=0} = \sum_{n=0}^{\infty} n\Psi_n(y)\int_{\mathbf{R}}\Psi_n(x)f(x)dx,$$

$$\mathcal{H}F^\alpha(x) = \sum_{n=0}^{\infty} n\Psi_n(y)\int_{\mathbf{R}}\Psi_n(x)f(x)dx.$$

Therefore, $j\frac{\partial F^\alpha(x)}{\partial\alpha} = \mathcal{H}F^\alpha(y)$, $\frac{\partial F^\alpha(x)}{F^\alpha(x)} = -j\mathcal{H}\partial\alpha$. The solution of this equation is given by $F^\alpha(x) = \left\{e^{-j\alpha\mathcal{H}}F\right\}$ and $\mathcal{F}^\alpha = e^{-j\alpha\mathcal{H}} = e^{-j\alpha\left[\frac{1}{2}\left(\frac{d^2}{dx^2} - x^2 + 1\right)\right]}$. Obviously,

$$\mathcal{F}^{\alpha+d\alpha} = \mathcal{F}^{d\alpha}\mathcal{F}^\alpha \simeq \left(\mathbf{I} + d\mathcal{F}^\alpha\right)\exp\left[-j\alpha\mathcal{H}\right] =$$

$$= \left(\mathbf{I} + \frac{\partial\mathcal{F}^\alpha}{\partial\alpha}d\alpha\right)\exp\left(-j\alpha\mathcal{H}\right) = \left(\mathbf{I} - j\mathcal{H}d\alpha\right)\exp\left(-j\alpha\mathcal{H}\right),$$

where the operator

$$\mathcal{F}^{d\alpha} = \left(\mathbf{I} - j\mathcal{H}d\alpha\right) = \mathbf{I} - j\frac{1}{2}\left(\frac{d^2}{dx^2} - x^2 + 1\right)d\alpha \tag{7}$$

is called the *infinitesimal Fourier transform* or the *generator* of the fractional Fourier transforms [17,18].

Let us introduce operators $(M_x f)(x) := xf(x)$ and $(M_y F)(y) := yF(y)$. Using the Fourier transform (3), the first of ones may be written as $M_x = \mathcal{F}^{-1}\left(j\frac{d}{dy}\right)\mathcal{F}$. Obviously, $x^2 = M_x^2 = -\mathcal{F}^1\left(\frac{d^2}{dy^2}\right)\mathcal{F}$. Then

$$\mathcal{F}^{d\alpha} = \mathbf{I} - j\frac{1}{2}\left(\frac{d^2}{dx^2} + \mathcal{F}^{-1}\left(\frac{d^2}{dy^2}\right)\mathcal{F} + 1\right)d\alpha.$$

Discretization of x-domain with the interval discretization Δx is equal to the periodization of y-domain

$$\frac{d^2}{dx^2} + \mathcal{F}^{-1}\left(\frac{d^2}{dy^2}\right)\mathcal{F} + 1 \longrightarrow D_{\Delta x}\left[\frac{d^2}{dx^2}\right] + \mathcal{F}^{-1}\left(P_{2\pi/\Delta x}\left[\frac{d^2}{dy^2}\right]\right)\mathcal{F} + 1.$$

Discretization of y-domain with the interval discretization Δy is equal to the periodization of x-domain

$$D_{\Delta x}\left[\frac{d^2}{dx^2}\right] + \mathcal{F}^{-1}\left(P_{2\pi/\Delta x}\left[\frac{d^2}{dy^2}\right]\right)\mathcal{F} + 1 \longrightarrow$$

$$\longrightarrow P_{2\pi/\Delta y}D_{\Delta x}\left[\frac{d^2}{dx^2}\right] + \mathcal{F}^{-1}\left(P_{2\pi/\Delta x}D_{\Delta y}\left[\frac{d^2}{dy^2}\right]\right)\mathcal{F} + 1.$$

An approximation for the second derivative can be given by the second order central difference operator

$$\frac{d^2}{dx^2}f(x) \approx f(n \underset{N}{\ominus} 1) - 2f(n) + F(n \underset{N}{\oplus} 1), \quad \frac{d^2}{dy^2}F(y) \approx F(k \underset{N}{\ominus} 1) - 2F(k) + F(k \underset{N}{\oplus} 1),$$

where $N = 2\pi/\Delta x\Delta y$. On the other hand,

$$\mathcal{F}^{-1}\left(\frac{d^2}{dy^2}F(y)\right)\mathcal{F} \approx \mathcal{F}^{-1}\left[F(k \underset{N}{\ominus} 1) - 2F(k) + F(k \underset{N}{\oplus} 1)\right]\mathcal{F} =$$

$$= \left(f(n)e^{-j\frac{2\pi}{N}n} - 2f(n) + f(n)e^{j\frac{2\pi}{N}n}\right) = 2f(n)\left(\cos\frac{2\pi}{N}n - 1\right).$$

These allow one to give the approximation for $\mathcal{H} = \frac{1}{2}\left(\frac{d^2}{dx^2} - x^2 + 1\right)$ as follows:

$$\mathcal{H}f(x) = \left[\frac{1}{2}\left(\frac{d^2}{dx^2} - x^2 + 1\right)\right]f(x) \approx$$

$$\approx \frac{1}{2}\left\{\left[f(n \underset{N}{\ominus} 1) - 2f(n) + f(n \underset{N}{\oplus} 1)\right] + 2f(n)\left(\cos\frac{2\pi}{N}n - 1\right) + f(n)\right\} =$$

$$= -\left[\cos\frac{2\pi}{N}n - 3/2\right]f(n) + \frac{1}{2}\left[f(n \underset{N}{\ominus} 1) + f(n \underset{N}{\oplus} 1)\right].$$

In the N-diagonal basis we have

$$\mathcal{F}^{d\alpha}f(x) \approx \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix} + j\varDelta\alpha\times$$

$$\times \left[\begin{array}{c|c|c|c|c|c} -1/2 & 1/2 & . & . & . & 1/2 \\ \hline 1/2 & \cos(1\varOmega)-3/2 & 1/2 & . & . & . \\ \hline . & 1/2 & \cos(2\varOmega)-3/2 & 1/2 & . & . \\ \hline . & . & 1/2 & \cos(3\varOmega)-3/2 & 1/2 & . \\ \hline . & . & . & 1/2 & \ddots & 1/2 \\ \hline 1/2 & . & . & . & 1/2 & -1/2 \end{array}\right] \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix}, \tag{8}$$

where $\varOmega = 2\pi/N$.

Let us introduce the uniform discretization of the angle parameter α on M discrete values $\{\alpha_0, \alpha_1, ..., \alpha_i, \alpha_{i+1}, ..., \alpha_{M-1}\}$, where $\alpha_{i+1} = \alpha_i + \varDelta\alpha$, $\alpha_i = i\varDelta\alpha$ and $\varDelta\alpha = 2\pi/M$. Then

$$F^{\alpha_{i+1}}(y) = F^{\alpha_i+\varDelta\alpha}(y) \approx F^{\alpha_i}(k) + j\varDelta\alpha\times$$

$$\times\left\{\left[\cos\frac{2\pi}{N}k - 3/2\right]F^{\alpha_i}(k) + \frac{1}{2}\left[F^{\alpha_i}(k\underset{N}{\ominus}1) + F^{\alpha_i}(k\underset{N}{\oplus}+1)\right]\right\}. \tag{9}$$

It is easy to see that this algorithm requires only $2MN$ multiplications and $3MN$ additions vs. $MN\,(2+\log_2 N)$ multiplications and $MN\log_2 N$ additions in the classical algorithm. In (8), we used $\mathcal{O}(h^2)$ approximation $\left(\frac{d^2}{dx^2}f\right)(k) \approx (f(k-1) - 2f(k) + f(k+1))$. More fine approximations $\mathcal{O}(h^{2k})$ also can be used [19].

5 Conclusions

In this work, we introduce a new algorithm of computing for Fractional Fourier transforms based on the infinitesimal Fourier transform. It requires $2MN$ multiplications and $3MN$ additions vs. $MN\,(2+\log_2 N)$ multiplications and $MN\log_2 N$ additions in the classical algorithm. Presented algorithm can be utilized for fast computation in most applications of signal and image processing. We have presented a definition of the infinitesimal Fourier transform that exactly satisfies the properties of the Schrodinger Equation for quantum harmonic oscillator.

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