Subset-generated complete sublattices as concept lattices*

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Abstract. We present a solution to the problem of finding the complete sublattice of a given concept lattice generated by given set of elements. We construct the closed subrelation of the incidence relation of the corresponding formal context whose concept lattice is equal to the desired complete sublattice. The construction does not require the presence of the original concept lattice. We introduce an efficient algorithm for the construction and give an example and experiments.

1 Introduction and problem statement

One of the basic theoretical results of Formal Concept Analysis (FCA) is the correspondence between closed subrelations of a formal context and complete sublattices of the corresponding concept lattice [2]. In this paper, we study a related problem of constructing the closed subrelation for a complete sublattice generated by given set of elements.

Let $\langle X, Y, I \rangle$ be a formal context, $\mathcal{B}(X, Y, I)$ its concept lattice. Denote by V the complete sublattice of $\mathcal{B}(X, Y, I)$ generated by a set $P \subseteq \mathcal{B}(X, Y, I)$. As it is known [2], there exists a closed subrelation $J \subseteq I$ with the concept lattice $\mathcal{B}(X, Y, J)$ equal to V. We show a method of constructing J without the need of constructing $\mathcal{B}(X, Y, I)$ first. We also provide an efficient algorithm (with polynomial time complexity), implementing the method. The paper also contains an illustrative example and results of experiments, performed on the *Mushroom* dataset from the UCI Machine Learning Repository.

2 Complete lattices and Formal Concept Analysis

Recall that a partially ordered set U is called a *complete lattice* if each its subset $P \subseteq U$ has a supremum and infimum. We denote these by $\bigvee P$ and $\bigwedge P$, respectively. A subset $V \subseteq U$ is a $\bigvee$-*subsemilattice* (resp. $\bigwedge$-*subsemilattice*, resp. *complete sublattice*) *of* U, if for each $P \subseteq V$ it holds $\bigvee P \in V$ (resp. $\bigwedge P \in V$, resp. $\{\bigvee P, \bigwedge P\} \subseteq V$).

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For a subset $P \subseteq U$ we denote by $\mathrm{C}_{\bigvee} P$ the $\bigvee$-subsemilattice of U *generated by* P, i.e. the smallest (w.r.t. set inclusion) $\bigvee$-subsemilattice of U containing P. $\mathrm{C}_{\bigvee} P$ always exists and is equal to the intersection of all $\bigvee$-subsemilattices of U containing P. The $\bigwedge$-subsemilattice of U generated by P and the complete sublattice of U generated by P are defined similarly and are denoted by $\mathrm{C}_{\bigwedge} P$ and $\mathrm{C}_{\bigvee\bigwedge} P$, respectively.

The operators $\mathrm{C}_{\bigvee}$, $\mathrm{C}_{\bigwedge}$, and $\mathrm{C}_{\bigvee\bigwedge}$ are closure operators on the set U. Recall that a *closure operator on a set* X is a mapping $\mathrm{C}\colon 2^X \to 2^X$ (where 2^X is the set of all subsets of X) satisfying for all sets $A, A_1, A_2 \subseteq X$

1. $A \subseteq \mathrm{C}\,A$,
2. if $A_1 \subseteq A_2$ then $\mathrm{C}\,A_1 \subseteq \mathrm{C}\,A_2$,
3. $\mathrm{C}\mathrm{C}\,A = \mathrm{C}\,A$.

Concept lattices have been introduced in [4], our basic reference is [2]. A (*formal*) *context* is a triple $\langle X, Y, I\rangle$ where X is a set of objects, Y a set of attributes and $I \subseteq X \times Y$ a binary relation between X and Y specifying for each object which attributes it has.

For subsets $A \subseteq X$ and $B \subseteq Y$ we set

$$A^{\uparrow_I} = \{y \in Y \mid \text{for each } x \in A \text{ it holds } \langle x, y\rangle \in I\},$$
$$B^{\downarrow_I} = \{x \in X \mid \text{for each } y \in B \text{ it holds } \langle x, y\rangle \in I\}.$$

The pair $\langle {}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ is a Galois connection between sets X and Y, i.e. it satisfies

1. If $A_1 \subseteq A_2$ then $A_2^{\uparrow_I} \subseteq A_1^{\uparrow_I}$, if $B_1 \subseteq B_2$ then $B_2^{\downarrow_I} \subseteq B_1^{\downarrow_I}$.
2. $A \subseteq A^{\uparrow_I\downarrow_I}$ and $B \subseteq B^{\downarrow_I\uparrow_I}$.

The operator ${}^{\uparrow_I\downarrow_I}$ is a closure operator on X and the operator ${}^{\downarrow_I\uparrow_I}$ is a closure operator on Y.

A pair $\langle A, B\rangle$ satisfying $A^{\uparrow_I} = B$ and $B^{\downarrow_I} = A$ is called a (*formal*) *concept* of $\langle X, Y, I\rangle$. The set A is then called *the extent of* $\langle A, B\rangle$, the set B *the intent of* $\langle A, B\rangle$. When there is no danger of confusion, we can use the term "an extent of I" instead of "the extent of a concept of $\langle X, Y, I\rangle$", and similarly for intents.

A partial order $\leq$ on the set $\mathcal{B}(X, Y, I)$ of all formal concepts of $\langle X, Y, I\rangle$ is defined by $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (iff $B_2 \subseteq B_1$). $\mathcal{B}(X, Y, I)$ along with $\leq$ is a complete lattice and is called *the concept lattice of* $\langle X, Y, I\rangle$. Infima and suprema in $\mathcal{B}(X, Y, I)$ are given by

$$\bigwedge_{j\in J} \langle A_j, B_j\rangle = \left\langle \bigcap_{j\in J} A_j, \left(\bigcup_{j\in J} B_j\right)^{\downarrow_I\uparrow_I} \right\rangle, \tag{1}$$

$$\bigvee_{j\in J} \langle A_j, B_j\rangle = \left\langle \left(\bigcup_{j\in J} A_j\right)^{\uparrow_I\downarrow_I}, \bigcap_{j\in J} B_j \right\rangle. \tag{2}$$

One of immediate consequences of (1) and (2) is that the intersection of any system of extents, resp. intents, is again an extent, resp. intent, and that it can be expressed as follows:

$$\bigcap_{j\in J} B_j = \left(\bigcup_{j\in J} A_j\right)^{\uparrow_I}, \quad \text{resp.} \quad \bigcap_{j\in J} A_j = \left(\bigcup_{j\in J} B_j\right)^{\downarrow_I},$$

for concepts $\langle A_j, B_j\rangle \in \mathcal{B}(X,Y,I)$, $j \in J$.

Concepts $\langle \{y\}^{\downarrow_I}, \{y\}^{\downarrow_I\uparrow_I}\rangle$ where $y \in Y$ are *attribute concepts*. Each concept $\langle A, B\rangle$ is infimum of some attribute concepts (we say the set of all attribute concepts is $\bigwedge$*-dense in* $\mathcal{B}(X,Y,I)$). More specifically, $\langle A, B\rangle$, is infimum of attribute concepts $\langle \{y\}^{\downarrow_I}, \{y\}^{\downarrow_I\uparrow_I}\rangle$ for $y \in B$ and $A = \bigcap_{y\in B}\{y\}^{\downarrow_I}$.

Dually, concepts $\langle \{x\}^{\uparrow_I\downarrow_I}, \{x\}^{\uparrow_I}\rangle$ for $x \in X$ are *object concepts*, they are $\bigvee$-dense in $\mathcal{B}(X,Y,I)$ and for each concept $\langle A, B\rangle$, $B = \bigcap_{x\in A}\{x\}^{\uparrow_I}$.

A subrelation $J \subseteq I$ is called a *closed subrelation of* I if each concept of $\langle X, Y, J\rangle$ is also a concept of $\langle X, Y, I\rangle$. There is a correspondence between closed subrelations of I and complete sublattices of $\mathcal{B}(X,Y,I)$ [2, Theorem 13]: For each closed subrelation $J \subseteq I$, $\mathcal{B}(X,Y,J)$ is a complete sublattice of $\mathcal{B}(X,Y,I)$, and to each complete sublattice $V \subseteq \mathcal{B}(X,Y,I)$ there exists a closed subrelation $J \subseteq I$ such that $V = \mathcal{B}(X,Y,J)$.

3 Closed subrelations for generated sublattices

Let us have a context $\langle X, Y, I\rangle$ and a subset P of its concept lattice. Denote by V the complete sublattice of $\mathcal{B}(X,Y,I)$ generated by P (i.e. $V = \mathrm{C}_{\bigvee\bigwedge} P$). Our aim is to find, without computing the lattice $\mathcal{B}(X,Y,I)$, the closed subrelation $J \subseteq I$ whose concept lattice $\mathcal{B}(X,Y,J)$ is equal to V.

If $\mathcal{B}(X,Y,I)$ is finite, V can be obtained by alternating applications of the closure operators $\mathrm{C}_{\bigvee}$ and $\mathrm{C}_{\bigwedge}$ to P: we set $V_1 = \mathrm{C}_{\bigvee} P$, $V_2 = \mathrm{C}_{\bigwedge} V_1$, ..., and, generally, $V_i = \mathrm{C}_{\bigvee} V_{i-1}$ for odd $i > 1$ and $V_i = \mathrm{C}_{\bigwedge} V_{i-1}$ for even $i > 1$. The sets V_i are $\bigvee$-subsemilattices (for odd i) resp. $\bigwedge$-subsemilattices (for even i) of $\mathcal{B}(X,Y,I)$. Once $V_i = V_{i-1}$, we have the complete sublattice V.

Note that for infinite $\mathcal{B}(X,Y,I)$, V can be infinite even if P is finite. Indeed, denoting $FL(P)$ the free lattice generated by P [3] and setting $X = Y = FL(P)$, $I = \leq$ we have $FL(P) \subseteq V \subseteq \mathcal{B}(X,Y,I)$. ($\mathcal{B}(X,Y,I)$ is the Dedekind-MacNeille completion of $FL(P)$ [2], and we identify P and $FL(P)$ with subsets of $\mathcal{B}(X,Y,I)$ as usual.) Now, if $|P| > 2$ then $FL(P)$ is infinite [3], and so is V.

We always consider sets V_i together with the appropriate restriction of the ordering on $\mathcal{B}(X,Y,I)$. For each $i > 0$, V_i is a complete lattice (but not a complete sublattice of $\mathcal{B}(X,Y,I)$).

In what follows, we construct formal contexts with concept lattices isomorphic to the complete lattices V_i, $i > 0$. First, we find a formal context for the complete lattice V_1. Let $K_1 \subseteq P \times Y$ be given by

$$\langle\langle A, B\rangle, y\rangle \in K_1 \quad \text{iff} \quad y \in B. \tag{3}$$

As we can see, rows in the context $\langle P, Y, K_1\rangle$ are exactly intents of concepts from P.

Proposition 1. *The concept lattice $\mathcal{B}(P, Y, K_1)$ and the complete lattice V_1 are isomorphic. The isomorphism assigns to each concept $\langle B^{\downarrow_{K_1}}, B\rangle \in \mathcal{B}(P, Y, K_1)$ the concept $\langle B^{\downarrow_I}, B\rangle \in \mathcal{B}(X, Y, I)$.*

Proof. Concepts from V_1 are exactly those with intents equal to intersections of intents of concepts from P. The same holds for concepts from $\mathcal{B}(P, Y, K_1)$. ☐

Next we describe formal contexts for complete lattices V_i, $i > 1$. All of the contexts are of the form $\langle X, Y, K_i\rangle$, i.e. they have the set X as the set of objects and the set Y as the set of attributes (the relation K_1 is different in this regard). The relations K_i for $i > 1$ are defined in a recursive manner:

$$\text{for } i > 1,\ \langle x, y\rangle \in K_i \quad \text{iff} \quad \begin{cases} x \in \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I} \text{ for even } i, \\ y \in \{x\}^{\uparrow_{K_{i-1}}\downarrow_{K_{i-1}}\uparrow_I} \text{ for odd } i. \end{cases} \tag{4}$$

Proposition 2. *For each $i > 1$,*

1. *$K_i \subseteq I$,*
2. *$K_i \subseteq K_{i+1}$.*

Proof. We will prove both parts for odd i; the assertions for even i are proved similarly.

1. Let $\langle x, y\rangle \in K_i$. From $\{y\} \subseteq \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}$ we get $\{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I} \subseteq \{y\}^{\downarrow_I}$. Thus, $x \in \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I}$ implies $x \in \{y\}^{\downarrow_I}$, which is equivalent to $\langle x, y\rangle \in I$.

2. As $K_i \subseteq I$, we have $\{y\}^{\downarrow_{K_i}\uparrow_{K_i}\downarrow_I} \supseteq \{y\}^{\downarrow_{K_i}\uparrow_{K_i}\downarrow_{K_i}} = \{y\}^{\downarrow_{K_i}}$. Thus, $x \in \{y\}^{\downarrow_{K_i}}$ yields $x \in \{y\}^{\downarrow_{K_i}\uparrow_{K_i}\downarrow_I}$. ☐

We can see that the definitions of K_i for even and odd $i > 1$ are dual. In what follows, we prove properties of K_i for even i and give the versions for odd i without proofs.

First we give two basic properties of K_i that are equivalent to the definition. The first one says that K_i can be constructed as a union of some specific rectangles, the second one will be used frequently in what follows.

Proposition 3. *Let $i > 1$.*

1. *If i is even then $K_i = \bigcup_{y \in Y} \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I} \times \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}$. If i is odd then $K_i = \bigcup_{x \in X} \{x\}^{\uparrow_{K_{i-1}}\downarrow_{K_{i-1}}\uparrow_I} \times \{x\}^{\uparrow_{K_{i-1}}\downarrow_{K_{i-1}}}$.*
2. *If i is even then for each $y \in Y$, $\{y\}^{\downarrow_{K_i}} = \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I}$. If i is odd then for each $x \in X$, $\{x\}^{\uparrow_{K_i}} = \{x\}^{\uparrow_{K_{i-1}}\downarrow_{K_{i-1}}\uparrow_I}$.*

Proof. We will prove only the assertions for even i.

1. The "$\subseteq$" inclusion is evident. We will prove the converse inclusion. If $\langle x, y\rangle \in \bigcup_{y' \in Y} \{y'\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I} \times \{y'\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}$ then there is $y' \in Y$ such that $x \in \{y'\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I}$ and $y \in \{y'\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}$. The latter implies $\{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}} \subseteq$

$\{y'\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}$, whence $\{y'\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I} \subseteq \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I}$. Thus, x belongs to $\{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I}$ and by definition, $\langle x, y\rangle \in K_i$.

2. Follows directly from the obvious fact that $x \in \{y\}^{\downarrow_{K_i}}$ if and only if $\langle x, y\rangle \in K_i$. □

A direct consequence of 2. of Prop. 3 is the following.

Proposition 4. *If i is even then each extent of K_i is also an extent of I. If i is odd then each intent of K_i is also an intent of I.*

Proof. Let i be even. 2. of Prop. 3 implies that each attribute extent of K_i is an extent of I. Thus, the proposition follows from the fact that each extent of K_i is an intersection of attribute extents of K_i.

The statement for odd i is proved similarly except for $i = 1$ where it follows by definition. □

Proposition 5. *Let $i > 1$. If i is even then for each $y \in Y$ it holds*

$$\{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}} = \{y\}^{\downarrow_{K_i}\uparrow_{K_i}} = \{y\}^{\downarrow_{K_i}\uparrow_I}.$$

If i is odd then for each $x \in X$ we have

$$\{x\}^{\uparrow_{K_{i-1}}\downarrow_{K_{i-1}}} = \{x\}^{\uparrow_{K_i}\downarrow_{K_i}} = \{x\}^{\uparrow_{K_i}\downarrow_I}.$$

Proof. We will prove the assertion for even i. By Prop. 4, $\{y\}^{\downarrow_{K_i}}$ is an extent of I. The corresponding intent is

$$\{y\}^{\downarrow_{K_i}\uparrow_I} = \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I\uparrow_I} = \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}} \tag{5}$$

(by Prop. 4, $\{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}$ is an intent of I). Moreover, as $K_i \subseteq I$ (Prop. 2), we have

$$\{y\}^{\downarrow_{K_i}\uparrow_{K_i}} \subseteq \{y\}^{\downarrow_{K_i}\uparrow_I}. \tag{6}$$

We prove $\{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}} \subseteq \{y\}^{\downarrow_{K_i}\uparrow_{K_i}}$. Let $y' \in \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}$. It holds

$$\{y'\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}} \subseteq \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}$$

($^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}$ is a closure operator). Thus, $\{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I} \subseteq \{y'\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I}$ and so by 2. of Prop. 3, $\{y\}^{\downarrow_{K_i}} \subseteq \{y'\}^{\downarrow_{K_i}}$. Applying $^{\uparrow_{K_i}}$ to both sides we obtain $\{y'\}^{\downarrow_{K_i}\uparrow_{K_i}} \subseteq \{y\}^{\downarrow_{K_i}\uparrow_{K_i}}$ proving $y' \in \{y\}^{\downarrow_{K_i}\uparrow_{K_i}}$.

This, together with (5) and (6), proves the proposition. □

Proposition 6. *Let $i > 1$ be even. Then for each intent B of K_{i-1} it holds $B^{\downarrow_{K_i}} = B^{\downarrow_I}$. Moreover, if B is an attribute intent (i.e. there is $y \in Y$ such that $B = \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}$) then $\langle B^{\downarrow_{K_i}}, B\rangle$ is a concept of I.*

If $i > 1$ is odd then for each extent A of K_{i-1} it holds $A^{\uparrow_{K_i}} = A^{\uparrow_I}$. If A is an object extent (i.e. there is $x \in X$ such that $A = \{x\}^{\uparrow_{K_{i-1}}\downarrow_{K_{i-1}}}$) then $\langle A, A^{\uparrow_{K_i}}\rangle$ is a concept of I.

Proof. We will prove the assertion for even i. Let B be an intent of K_{i-1}. It holds $B = \bigcup_{y\in B}\{y\}$ (obviously) and hence $B = \bigcup_{y\in B}\{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}$ (since ${}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}$ is a closure operator). Therefore (2. of Prop. 3),

$$
\begin{aligned}
B^{\downarrow_{K_i}} &= \left(\bigcup_{y\in B}\{y\}\right)^{\downarrow_{K_i}} = \bigcap_{y\in B}\{y\}^{\downarrow_{K_i}} = \bigcap_{y\in B}\{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I} \\
&= \left(\bigcup_{y\in B}\{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}\right)^{\downarrow_I} = B^{\downarrow_I},
\end{aligned}
$$

proving the first part.

Now let B be an attribute intent of K_{i-1}, $B = \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}$. By 2. of Prop. 3 it holds $B^{\downarrow_I} = \{y\}^{\downarrow_{K_i}}$. By Prop. 5, $B^{\downarrow_I\uparrow_I} = \{y\}^{\downarrow_{K_i}\uparrow_I} = \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}} = B$. $\square$

Now we turn to complete lattices V_i defined above. We have already shown in Prop. 1 that the complete lattice V_1 and the concept lattice $\mathcal{B}(P, Y, K_1)$ are isomorphic. Now we give a general result for $i > 0$.

Proposition 7. *For each $i > 0$, the concept lattice $\mathcal{B}(P, Y, K_i)$ (for $i = 1$) resp. $\mathcal{B}(X, Y, K_i)$ (for $i > 1$) and the complete lattice V_i are isomorphic. The isomorphism is given by $\langle B^{\downarrow_{K_i}}, B\rangle \mapsto \langle B^{\downarrow_I}, B\rangle$ if i is odd and by $\langle A, A^{\uparrow_{K_i}}\rangle \mapsto \langle A, A^{\uparrow_I}\rangle$ if i is even.*

Proof. We will proceed by induction on i. The base step $i = 1$ has been already proved in Prop. 1. We will do the induction step for even i, the other case is dual.

As $V_i = \mathrm{C}_\bigwedge V_{i-1}$, we have to

1. show that the set $W = \{\langle A, A^{\uparrow_I}\rangle \mid A \text{ is an extent of } K_i\}$ is a subset of $\mathcal{B}(X, Y, I)$, containing V_{i-1} and
2. find for each $\langle A, A^{\uparrow_{K_i}}\rangle \in \mathcal{B}(X, Y, K_i)$ a set of concepts from V_{i-1} whose infimum in $\mathcal{B}(X, Y, I)$ has extent equal to A.

1. By Prop. 4, each extent of K_i is also an extent of I. Thus, $W \subseteq \mathcal{B}(X, Y, I)$. If $\langle A, B\rangle \in V_{i-1}$ then by the induction hypothesis B is an intent of K_{i-1} ($i-1$ is odd). By Prop. 6, $B^{\downarrow_{K_i}} = B^{\downarrow_I} = A$ is an extent of K_i and so $\langle A, B\rangle \in W$.

2. Denote $B = A^{\uparrow_{K_i}}$. For each $y \in Y$, $\{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}$ is an intent of K_{i-1}. By Prop. 3 and the induction hypothesis,

$$\langle\{y\}^{\downarrow_{K_i}}, \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}\rangle = \langle\{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}\downarrow_I}, \{y\}^{\downarrow_{K_{i-1}}\uparrow_{K_{i-1}}}\rangle \in V_{i-1}.$$

Now, the extent of the infimum (taken in $\mathcal{B}(X, Y, I)$) of these concepts for $y \in B$ is equal to $\bigcap_{y\in B}\{y\}^{\downarrow_{K_i}} = B^{\downarrow_{K_i}} = A$. $\square$

If X and Y are finite then 2. of Prop. 2 implies there is a number $n > 1$ such that $K_{n+1} = K_n$. Denote this relation by J. According to Prop. 7, there are two isomorphisms of the concept lattice $\mathcal{B}(X, Y, J)$ and $V_n = V_{n+1} = V$. We will show that these two isomorphisms coincide and $\mathcal{B}(X, Y, J)$ is actually equal to V. This will also imply J is a closed subrelation of I.

Proposition 8. $\mathcal{B}(X, Y, J) = V$.

Proof. Let $\langle A, B\rangle \in \mathcal{B}(X, Y, J)$. It suffices to show that $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$. As $J = K_{n+1} = K_n$ we have $J = K_i$ for some even i and also $J = K_i$ for some odd i. We can therefore apply both parts of Prop. 6 to J obtaining $A = B^{\downarrow_J} = B^{\downarrow_I}$ and $B = A^{\uparrow_J} = A^{\uparrow_I}$. □

Algorithm 1 uses our results to compute the subrelation J for given $\langle X, Y, I\rangle$ and P.

Algorithm 1 Computing the closed subrelation J.

Input: formal context $\langle X, Y, I\rangle$, subset $P \subseteq \mathcal{B}(X, Y, I)$
Output: the closed subrelation of $J \subseteq I$ whose concept lattice is equal to $\mathrm{C}_{\bigvee\bigwedge} P$

```
J ← relation K_1 (3)
i ← 1
repeat
    L ← J
    i ← i + 1
    if i is even then
        J ← {⟨x, y⟩ ∈ X × Y | x ∈ {y}^{↓L ↑L ↓I}}
    else
        J ← {⟨x, y⟩ ∈ X × Y | y ∈ {x}^{↑L ↓L ↑I}}
    end if
until i > 2 & J = L
return J
```

Proposition 9. *Algorithm 1 is correct and terminates after at most* $\max(|I| + 1, 2)$ *iterations.*

Proof. Correctness follows from Prop. 8. The terminating condition ensures we compare J and L only when they are both subrelations of the context $\langle X, Y, I\rangle$ (after the first iteration, L is a subrelation of $\langle P, Y, K_1\rangle$ and the comparison would not make sense).

After each iteration, L holds the relation K_{i-1} and J holds K_i (4). Thus, except for the first iteration, we have $L \subseteq J$ before the algorithm enters the terminating condition (Prop. 2). As J is always a subset of I (Prop. 2), the number of iterations will not be greater than $|I| + 1$. The only exception is $I = \emptyset$. In this case, the algorithm will terminate after 2 steps due to the first part of the terminating condition. □

4 Examples and experiments

Let $\langle X, Y, I\rangle$ be the formal context from Fig. 1 (left). The associated concept lattice $\mathcal{B}(X, Y, I)$ is depicted in Fig. 1 (right). Let $P = \{c_1, c_2, c_3\}$ where

I	y_1	y_2	y_3	y_4	y_5
x_1	×			×	
x_2	×	×	×		
x_3			×		×
x_4			×		
x_5		×			

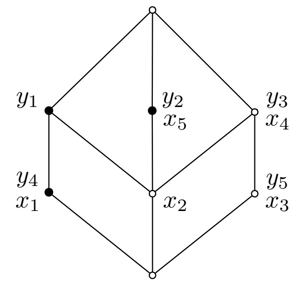

Fig. 1: Formal context $\langle X, Y, I\rangle$ (left) and concept lattice $\mathcal{B}(X, Y, I)$, together with a subset $P \subseteq \mathcal{B}(X, Y, I)$, depicted by filled dots (right).

$c_1 = \langle\{x_1\}, \{y_1, y_4\}\rangle$, $c_2 = \langle\{x_1, x_2\}, \{y_1\}\rangle$, $c_3 = \langle\{x_2, x_5\}, \{y_2\}\rangle$ are concepts from $\mathcal{B}(X, Y, I)$. These concept are depicted in Fig. 1 by filled dots.

First, we construct the context $\langle P, Y, K_1\rangle$ (3). Rows in this context are intents of concepts from P (see Fig. 2, left). The concept lattice $\mathcal{B}(P, Y, K_1)$ (Fig. 2, center) is isomorphic to the $\bigvee$-subsemilattice $V_1 = \mathrm{C}_{\bigvee} P \subseteq \mathcal{B}(X, Y, I)$ (Fig. 2, right). It is easy to see that elements of $\mathcal{B}(P, Y, K_1)$ and corresponding elements

K_1	y_1	y_2	y_3	y_4	y_5
c_1	×			×	
c_2	×				
c_3		×			

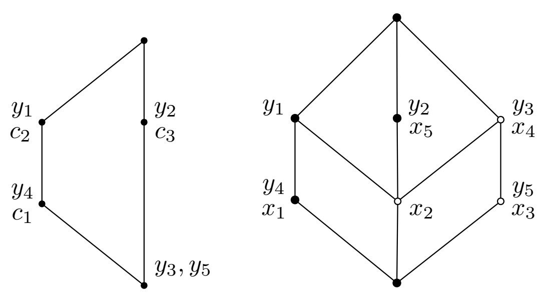

Fig. 2: Formal context $\langle P, Y, K_1\rangle$ (left), the concept lattice $\mathcal{B}(P, Y, K_1)$ (center) and the $\bigvee$-subsemilattice $\mathrm{C}_{\bigvee} P \subseteq \mathcal{B}(X, Y, I)$, isomorphic to $\mathcal{B}(P, Y, K_1)$, depicted by filled dots (right).

of V_1 have the same intents.

Next step is to construct the subrelation $K_2 \subseteq I$. By (4), K_2 consists of elements $\langle x, y\rangle \in X \times Y$ satisfying $x \in \{y\}^{\downarrow_{K_1}\uparrow_{K_1}\downarrow_I}$. The concept lattice $\mathcal{B}(X, Y, K_2)$ is isomorphic to the $\bigwedge$-subsemilattice $V_2 = \mathrm{C}_{\bigwedge} V_1 \subseteq \mathcal{B}(X, Y, I)$. K_2, $\mathcal{B}(X, Y, K_2)$, and V_2 are depicted in Fig. 3.

The subrelation $K_3 \subseteq I$ is computed again by (4). K_3 consists of elements $\langle x, y\rangle \in X \times Y$ satisfying $y \in \{x\}^{\uparrow_{K_2}\downarrow_{K_2}\uparrow_I}$. The result can be viewed in Fig. 4.

K_2	y_1	y_2	y_3	y_4	y_5
x_1	×			×	
x_2	×	×	·		
x_3			·		·
x_4			·		
x_5		×			

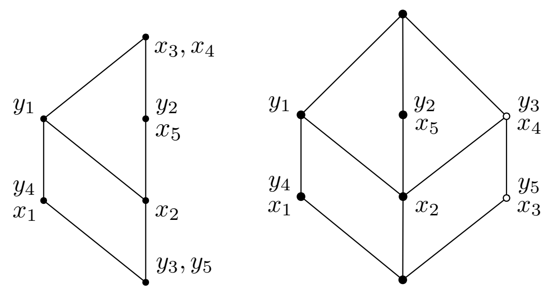

Fig. 3: Formal context $\langle X, Y, K_2 \rangle$ (left), the concept lattice $\mathcal{B}(X, Y, K_2)$ (center) and the $\bigwedge$-subsemilattice $V_2 = \mathrm{C}_{\bigwedge} V_1 \subseteq \mathcal{B}(X, Y, I)$, isomorphic to $\mathcal{B}(X, Y, K_2)$, depicted by filled dots (right). Elements of $I \setminus K_2$ are depicted by dots in the table.

K_3	y_1	y_2	y_3	y_4	y_5
x_1	×			×	
x_2	×	×	×		
x_3			·		·
x_4			·		
x_5		×			

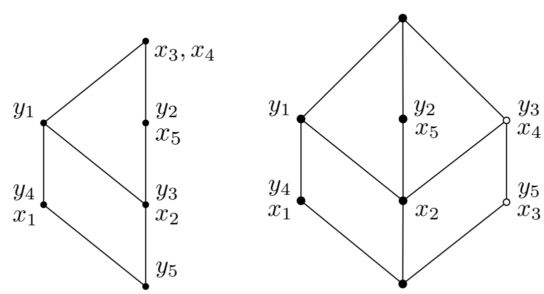

Fig. 4: Formal context $\langle X, Y, K_3 \rangle$ (left), the concept lattice $\mathcal{B}(X, Y, K_3)$ (center) and the $\bigvee$-subsemilattice $V_3 = \mathrm{C}_{\bigvee} V_2 \subseteq \mathcal{B}(X, Y, I)$, isomorphic to $\mathcal{B}(X, Y, K_3)$, depicted by filled dots (right). Elements of $I \setminus K_3$ are depicted by dots in the table. As $K_3 = K_4 = J$, it is a closed subrelation of I and $V_4 = \mathrm{C}_{\bigwedge} V_3 = V_3$ is a complete sublattice of $\mathcal{B}(X, Y, I)$.

Notice that already $V_3 = V_2$ but $K_3 \neq K_2$. We cannot stop and have to perform another step. After computing K_4 we can easily check that $K_4 = K_3$. We thus obtained the desired closed subrelation $J \subseteq I$ and $V_4 = V_3$ is equal to the desired complete sublattice $V \subseteq \mathcal{B}(X, Y, I)$.

In [1], the authors present an algorithm for computing a sublattice of a given lattice generated by a given set of elements. Originally, we planned to include a comparison between their approach and our Alg. 1. Unfortunately, the algorithm in [1] turned out to be incorrect. It is based on the false claim that (using our notation) the smallest element of V, which is greater than or equal to an element $v \in \mathcal{B}(X, Y, I)$, is equal to $\bigwedge\{p \in P \mid p \geq v\}$. The algorithm from [1] fails e.g. on the input depicted in Fig. 5.

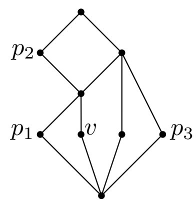

Fig. 5: An example showing that the algorithm from [1] is incorrect. A complete lattice with a selected subset $P = \{p_1, p_2, p_3\}$. The least element of the sublattice V generated by P which is greater than or equal to v is $p_1 \vee v$. The algorithm incorrectly chooses p_2 and "forgets" to add $p_1 \vee v$ to the output.

The time complexity of our algorithm is clearly polynomial w.r.t. $|X|$ and $|Y|$. In Prop. 9 we proved that the number of iterations is $\mathcal{O}(|I|)$. Our experiments indicate that this number might be much smaller in the practice. We used the *Mushroom* dataset from the UC Irvine Machine Learning Repository, which contains 8124 objects, 119 attributes and 238710 concepts. For 39 different sizes of the set P, we selected randomly its elements, 1000 times for each of the sizes. For each P, we ran our algorithm and measured the number n of iterations, after which the algorithm terminated. We can see in Tbl. 1 maximal and average values of n, separately for each size of P. From the results in Tbl. 1 we can see

$\lvert P\rvert(\%)$	Max n	Avg n	$\lvert P\rvert(\%)$	Max n	Avg n	$\lvert P\rvert(\%)$	Max n	Avg n
0.005	11	7	0.25	6	3	0.90	5	3
0.010	10	6	0.30	6	3	0.95	4	3
0.015	10	5	0.35	6	3	1	4	3
0.020	10	5	0.40	5	3	2	4	3
0.025	8	5	0.45	5	3	3	4	3
0.030	8	4	0.50	5	3	4	4	3
0.035	8	4	0.55	6	3	5	4	2
0.040	7	4	0.60	5	3	6	4	2
0.045	10	4	0.65	4	3	7	4	2
0.050	8	4	0.70	5	3	8	3	2
0.100	6	4	0.75	6	3	9	3	2
0.150	6	4	0.80	6	3	10	3	2
0.200	6	4	0.85	4	3	11	3	2

Table 1: Results of experiments on *Mushrooms* dataset. The size of P is given by the percentage of the size of the concept lattice.

that the number of iterations (both maximal and average values) is very small compared to the number of objects and attributes. There is also an apparent decreasing trend of number of iterations for increasing size of P.

5 Conclusion and open problems

An obvious advantage of our approach is that we avoid computing the whole concept lattice $\mathcal{B}(X, Y, I)$. This should lead to shorter computation time, especially if the generated sublattice V is substantially smaller than $\mathcal{B}(X, Y, I)$.

The following is an interesting observation and an open problem. It is mentioned in [2] that the system of all closed subrelations of I is not a closure system and, consequently, there does not exist a closure operator assigning to each subrelation of I a least greater (w.r.t. set inclusion) closed subrelation. This is indeed true as the intersection of closed subrelations need not be a closed subrelation. However, our method can be easily modified to compute for any subrelation $K \subseteq I$ a closed subrelation $J \supseteq K$, which seems to be minimal in some sense. Indeed, we can set $K_1 = K$ and compute a relation J as described by Alg. 1, regardless of the fact that K does not satisfy our requirements (intents of K need not be intents of I). The relation J will be a closed subrelation of I and it will contain K as a subset. Also note that the dual construction leads to a different closed subrelation.

Another open problem is whether it is possible to improve the estimation of the number of iterations of Alg. 1 from Prop. 9. In fact, we were not able to construct any example with the number of iterations greater than $\min(|X|, |Y|)$.

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