Category of isotone bonds between L-fuzzy contexts over different structures of truth degrees

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Abstract. We describe properties of compositions of isotone bonds between **L**-fuzzy contexts over different complete residuated lattices and we show that **L**-fuzzy contexts as objects and isotone bonds as arrows form a category.

1 Introduction

In Formal Concept Analysis, bonds represent relationships between formal contexts. One of the motivations for introducing this notion is to provide a tool for studying mappings between formal contexts, corresponding to the behavior of Galois connections between their corresponding concept lattices. The notions of bonds, scale measures and informorphisms were studied by [14] aiming at a thorough study of the theory of morphisms in FCA.

In our previous works, we studied generalizations of bonds into an L-fuzzy setting in [12, 11]. In [13] we also provided a study of bonds between formal fuzzy contexts over different structures of truth degrees. The bonds were based on mappings between complete residuated lattices, called residuation-preserving Galois connections. These mappings were too strict and in [9] we proposed to replace them by residuation-preserving (l, k)-connections or residuation-preserving dual (l, k)-connections between complete residuated lattices.

In the present paper we continue our study [12] of properties of bonds between formal contexts over different structures of truth degrees; this time we concern with bonds mimicking isotone Galois connections between concept lattices formed by isotone concept-forming operators. Particularly, we describe the category of formal fuzzy contexts and isotone bonds between them. The paper also extends [13, 9] as we consider a setting with fuzzy formal contexts over different complete residuated lattices.

The structure of the paper is as follows. First, in Section 2 we recall basic notions required in the rest of the paper. Section 3.1 considers weak homogeneous **L**-bonds w.r.t. isotone concept-forming operators and their compositions. Section 3.2 then generalizes the results to the setting of formal fuzzy contexts over different structure of truth degrees. Finally, we summarize our results and outline our future research in this area in Section 4.

2 Preliminaries

2.1 Residuated lattices, fuzzy sets, and fuzzy relations

We use complete residuated lattices as basic structures of truth degrees. A complete residuated lattice is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist;
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;
(iii) $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \rightarrow c$.

0 and 1 denote the least and greatest elements. The partial order of **L** is denoted by $\leqslant$. Throughout this work, **L** denotes an arbitrary complete residuated lattice.

Elements a of L are called truth degrees. Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of (truth functions of) "fuzzy conjunction" and "fuzzy implication".

An **L**-set (or **L**-fuzzy set) A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$. The set of all **L**-sets in a universe X is denoted L^X.

The operations with **L**-sets are defined componentwise. For instance, the intersection of **L**-sets $A, B \in L^X$ is an **L**-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$, etc.

An **L**-set $A \in L^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp **L**-sets can be identified with ordinary sets. For a crisp A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$. An **L**-set $A \in L^X$ is called empty (denoted by $\emptyset$) if $A(x) = 0$ for each $x \in X$. For $a \in L$ and $A \in L^X$, the a-multiplication $a \otimes A$ and a-shift $a \rightarrow A$ are **L**-sets defined by

$$(a \otimes A)(x) = a \otimes A(x),$$
$$(a \rightarrow A)(x) = a \rightarrow A(x).$$

Binary **L**-relations (binary **L**-fuzzy relations) between X and Y can be thought of as **L**-sets in the universe $X \times Y$. That is, a binary **L**-relation $I \in L^{X \times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I).

For an **L**-relation $I \in L^{X \times Y}$ we define its transpose as the **L**-relation $I^{\mathrm{T}} \in L^{Y \times X}$ given by $I^{\mathrm{T}}(y, x) = I(x, y)$ for each $x \in X, y \in Y$.

Various composition operators for binary **L**-relations were extensively studied by [6]; we will use the following composition operators, defined for relations $A \in L^{X\times F}$ and $B \in L^{F\times Y}$:

$$(A \circ B)(x,y) = \bigvee_{f\in F} A(x,f) \otimes B(f,y), \tag{1}$$

$$(A \triangleright B)(x,y) = \bigwedge_{f\in F} B(f,y) \to A(x,f). \tag{2}$$

Note also that for $L = \{0,1\}$, $A \circ B$ coincides with the well-known composition of binary relations.

We will occasionally use some of the following properties concerning the associativity of several composition operators, see [2].

Theorem 1. *The operator* $\circ$ *from above has the following properties concerning composition.*

– *Associativity:*

$$R \circ (S \circ T) = (R \circ S) \circ T. \tag{3}$$

– *Distributivity:*

$$(\bigcup_i R_i) \circ S = \bigcup_i (R_i \circ S), \quad \textit{and} \quad R \circ (\bigcup_i S_i) = \bigcup_i (R \circ S_i). \tag{4}$$

2.2 Formal fuzzy concept analysis

An **L**-context is a triplet $\langle X, Y, I\rangle$ where X and Y are (ordinary nonempty) sets and $I \in L^{X\times Y}$ is an **L**-relation between X and Y. Elements of X are called objects, elements of Y are called attributes, I is called an incidence relation. $I(x,y) = a$ is read: "The object x has the attribute y to degree a."

Consider the following pair $\langle \cap, \cup\rangle$ of operators $\cap : L^X \to L^Y$ and $\cup : L^Y \to L^X$ induced by an **L**-context $\langle X, Y, I\rangle$:

$$A^{\cap}(y) = \bigvee_{x\in X} A(x) \otimes I(x,y), \quad B^{\cup}(x) = \bigwedge_{y\in Y} I(x,y) \to B(y). \tag{5}$$

for all $A \in L^X$ and $B \in \mathbf{L}^Y$. When we consider concept-forming operators induced by multiple **L**-relations, we write the inducing **L**-relation as the subscript of the symbols of the operators. For example, the pair of concept-forming operators induced by **L**-relation I are written as $\langle \cap_I, \cup_I\rangle$.

Remark 1. Notice that the pair of concept-forming operators can be interpreted as instances of the composition operators between relations. Applying the isomorphisms $\mathbf{L}^{1\times X} \cong \mathbf{L}^X$ and $\mathbf{L}^{Y\times 1} \cong \mathbf{L}^Y$ whenever necessary, one could write them, alternatively, as

$$A^{\cap} = A \circ I \quad \text{and} \quad B^{\cup} = I \triangleleft B \quad (= B \triangleright I^{\mathrm{T}}).$$

Furthermore, denote the set of fixed points of $\langle{}^{\cap},{}^{\cup}\rangle$ by $\mathcal{B}^{\cap\cup}(X,Y,I)$, i.e.

$$\mathcal{B}^{\cap\cup}(X,Y,I)=\{\langle A,B\rangle\in L^X\times L^Y\mid A^{\cap}=B,\,B^{\cup}=A\}. \tag{6}$$

The set of fixed points endowed with $\leqslant$, defined by

$$\langle A_1,B_1\rangle\leqslant\langle A_2,B_2\rangle\quad\text{if } A_1\subseteq A_2 \text{ (equivalently } B_2\subseteq B_1)$$

is a complete lattice [5], called an *attribute-oriented* **L***-concept lattice* associated with I, and its elements are called (*attribute-oriented*) *formal* **L***-concepts* (or just **L**-concepts). For thorough studies of attribute-oriented concept lattices, see [5, 7, 15]. In a formal concept $\langle A,B\rangle$, the A is called an *extent*, and B is called an *intent*. The set of all extents and the set of all intents are denoted by $\mathrm{Ext}^{\cap\cup}$ and $\mathrm{Int}^{\cap\cup}$, respectively. That is,

$$\begin{aligned}\mathrm{Ext}^{\cap\cup}(X,Y,I)&=\{A\in L^X\mid\langle A,B\rangle\in\mathcal{B}^{\cap\cup}(X,Y,I)\text{ for some }B\},\\ \mathrm{Int}^{\cap\cup}(X,Y,I)&=\{B\in L^Y\mid\langle A,B\rangle\in\mathcal{B}^{\cap\cup}(X,Y,I)\text{ for some }A\}.\end{aligned}\tag{7}$$

Equivalently, we can characterize $\mathrm{Ext}^{\cap\cup}(X,Y,I)$ and $\mathrm{Int}^{\cap\cup}(X,Y,I)$ as follows

$$\begin{aligned}\mathrm{Ext}^{\cap\cup}(X,Y,I)&=\{B^{\cup}\mid B\in L^Y\},\\ \mathrm{Int}^{\cap\cup}(X,Y,I)&=\{A^{\cap}\mid A\in L^X\}.\end{aligned}\tag{8}$$

We will need the following lemma from [4].

Lemma 1. *Consider* **L***-contexts* $\langle X,Y,I\rangle$*,* $\langle X,F,A\rangle$*, and* $\langle F,Y,B\rangle$*.*

(a) $\mathrm{Int}^{\cap\cup}(X,Y,I)\subseteq\mathrm{Int}^{\cap\cup}(F,Y,B)$ *if and only if there exists* $A'\in L^{X\times F}$ *such that* $I=A'\circ B$*,*
(b) $\mathrm{Ext}^{\cap\cup}(X,Y,A\circ B)\subseteq\mathrm{Ext}^{\cap\cup}(X,F,A)$*.*

Definition 1. *An* **L***-relation* $\beta\in L^{X_1\times Y_2}$ *is called a* homogeneous weak **L**-bond[3] *from* **L***-context* $\langle X_1,Y_1,I_1\rangle$ *to* **L***-context* $\langle X_2,Y_2,I_2\rangle$ *if*

$$\begin{aligned}\mathrm{Ext}^{\cap\cup}(X_1,Y_2,\beta)&\subseteq\mathrm{Ext}^{\cap\cup}(X_1,Y_1,I_1),\\ \mathrm{Int}^{\cap\cup}(X_1,Y_2,\beta)&\subseteq\mathrm{Int}^{\cap\cup}(X_2,Y_2,I_2).\end{aligned}\tag{9}$$

In this paper we assume only weak homogeneous **L**-bonds w.r.t. $\langle\cap,\cup\rangle$. In what follows, we omit the words 'weak homogeneous' and the pair of concept-forming operators and call them just '**L**-bonds'.

We will utilize the following characterization of **L**-bonds.

Lemma 2 ([7]). *An* **L***-relation* $\beta\in L^{X_1\times Y_2}$ *is an* **L***-bond from* $\langle X_1,Y_1,I_1\rangle$ *to* $\langle X_2,Y_2,I_2\rangle$ *iff there is such* **L***-relation* S_{e} *that* $\beta=S_{\mathrm{e}}\circ I_2$ *and* $\cup_{S_{\mathrm{e}}}$ *maps extents of* $\mathcal{B}^{\cap\cup}(X_2,Y_2,I_2)$ *to extents of* $\mathcal{B}^{\cap\cup}(X_2,Y_2,I_2)$*.*

Remark 2. Note that due to results on fuzzy relational equations we have that the **L**-relation S_{e} from Lemma 2 is equal to $\beta\triangleright I_2^{\mathrm{T}}$ (see [2]).

[3] The notion of **L**-bond was introduced in [12]; however we adapt its definition the same way as in [8, 10] w.r.t. $\langle\cap,\cup\rangle$

3 Results

Firstly, we describe compositions of **L**-bonds and show that they form a category. Later we generalize the results to setting of isotone bonds between fuzzy contexts over different complete residuated lattices.

3.1 Setting with uniform structures of truth degrees

We start with the notion of composition of **L**-bonds.

Definition 2. *Let β_1 be an* **L***-bond from $\langle X_1, Y_1, I_1 \rangle$ to $\langle X_2, Y_2, I_2 \rangle$ and β_2 be an* **L***-bond from $\langle X_2, Y_2, I_2 \rangle$ to $\langle X_3, Y_3, I_3 \rangle$. Define composition of β_1 and β_2 as the* **L***-relation $(\beta_1 \triangleright I_2^{\mathrm{T}}) \circ \beta_2 \in L^{X_1 \times Y_3}$ and denote it $\beta_1 \bullet \beta_2$.*

Theorem 2. *The composition of* **L***-bonds is an* **L***-bond.*

Proof. Let β_1 be an **L**-bond from $\langle X_1, Y_1, I_1 \rangle$ to $\langle X_2, Y_2, I_2 \rangle$ and β_2 be an **L**-bond from $\langle X_2, Y_2, I_2 \rangle$ to $\langle X_3, Y_3, I_3 \rangle$. By Lemma 2 there are $S_{\mathrm{e}} \in L^{X_1 \times X_2}, {S_{\mathrm{e}}}' \in L^{X_2 \times X_3}$ such that $\beta_1 = S_{\mathrm{e}} \circ I_2$, $\beta_2 = {S_{\mathrm{e}}}' \circ I_3$. By Definition 2 and Remark 2 we have

$$\begin{aligned} \beta_1 \bullet \beta_2 &= (\beta_1 \triangleright I_2^{\mathrm{T}}) \circ \beta_2 \\ &= S_{\mathrm{e}} \circ {S_{\mathrm{e}}}' \circ I_3. \end{aligned}$$

Hence we have

$$\mathrm{Int}^{\cap\cup}(X_1, Y_3, \beta_1 \bullet \beta_2) \subseteq \mathrm{Int}^{\cap\cup}(X_3, Y_3, I_3) \tag{10}$$

by Lemma 1 (a). Note that the mapping $\cup_{S_{\mathrm{e}}}$ maps extents of I_2 to extents of I_1 by Lemma 2 and that $B^{\cup_{\beta_2}}$ is extent of I_2 for any $B \in \mathrm{Int}^{\cap\cup}(X_3, Y_3, I_3)$ by (8). Thus we have

$$B^{\cup_{\beta_1 \bullet \beta_2}} = B^{\cup_{\beta_2} \cup_{S_{\mathrm{e}}}} \in \mathrm{Ext}^{\cap\cup}(X_1, Y_1, I_1),$$

hence

$$\mathrm{Ext}^{\cap\cup}(X_1, Y_3, \beta_1 \bullet \beta_2) \subseteq \mathrm{Ext}^{\cap\cup}(X_1, Y_1, I_1). \tag{11}$$

The equalities (10) and (11) imply that $\beta_1 \bullet \beta_2$ is an **L**-bond. □

Lemma 3. *Let β be an* **L***-bond from* **L***-context $\langle X_1, Y_1, I_1 \rangle$ to* **L***-context $\langle X_2, Y_2, I_2 \rangle$. For any* **L***-set $A \in L^{X_1}$ we have that $A^{\cap_{I_1} \cup_{I_1} \cap_\beta} = A^{\cap_\beta}$.*

Proof. Let A be an arbitrary **L**-set from L^{X_1}. Then

$$\begin{aligned} A^{\cap_{I_1} \cup_{I_1} \cap_\beta} &\supseteq A^{\cap_\beta} \text{ since } (-)^{\cap_\beta} \text{ is isotone and } A^{\cap_{I_1} \cup_{I_1}} \supseteq A \\ &= A^{\cap_\beta \cup_\beta \cap_\beta} \\ &= A^{\cap_\beta \cup_\beta \cap_{I_2} \cup_{I_2} \cap_\beta} \text{ due to definition of } \mathbf{L}\text{-bond} \\ &\supseteq A^{\cap_{I_1} \cup_{I_1} \cap_\beta} \text{ since the mapping } (-)^{\cap_{I_1} \cup_{I_1} \cap_\beta} \text{ is isotone} \end{aligned}$$

Hence $A^{\cap_{I_1} \cup_{I_1} \cap_\beta} = A^{\cap_\beta}$. □

The equality from Lemma 3 written in relational form is $A \circ \beta = (A \circ I_1) \triangleright I_1^{\mathrm{T}}) \circ \beta$; we use that to prove the following theorem.

Theorem 3. *Composition of* $\mathbf{L}$*-bonds is associative.*

Proof. Let β_1 be an $\mathbf{L}$-bond from $\langle X_1, Y_1, I_1 \rangle$ to $\langle X_2, Y_2, I_2 \rangle$, β_2 be an $\mathbf{L}$-bond from $\langle X_2, Y_2, I_2 \rangle$ to $\langle X_3, Y_3, I_3 \rangle$, and β_3 be an $\mathbf{L}$-bond from $\langle X_3, Y_3, I_3 \rangle$ to $\langle X_4, Y_4, I_4 \rangle$. We have

$$\begin{aligned}
(\beta_1 \bullet \beta_2) \bullet \beta_3 &= (((\beta_1 \triangleright I_2^{\mathrm{T}}) \circ \beta_2) \triangleright I_3^{\mathrm{T}}) \circ \beta_3 && \text{by Definition 2} \\
&= ((S_{\mathrm{e}} \circ \beta_2) \triangleright I_3^{\mathrm{T}}) \circ \beta_3 && \text{by Remark 2} \\
&= ((S_{\mathrm{e}} \circ (S'_e \circ I_3)) \triangleright I_3^{\mathrm{T}}) \circ \beta_3 && \text{by Lemma 2} \\
&= (((S_{\mathrm{e}} \circ S'_e) \circ I_3) \triangleright I_3^{\mathrm{T}}) \circ \beta_3 && \text{by (3)} \\
&= (S_{\mathrm{e}} \circ S'_e) \circ \beta_3 && \text{by Lemma 3} \\
&= S_{\mathrm{e}} \circ (S'_e \circ \beta_3) && \text{by (3)} \\
&= S_{\mathrm{e}} \circ (\beta_2 \bullet \beta_3) = \beta_1 \bullet (\beta_2 \bullet \beta_3) && \text{by Remark 2 and Definition 2.}
\end{aligned}$$

□

We obtain a category of $\mathbf{L}$-contexts and $\mathbf{L}$-bonds.

Theorem 4. *The structure of* $\mathbf{L}$*-contexts and* $\mathbf{L}$*-bonds forms a category:*

Objects *are* $\mathbf{L}$*-contexts,*
Arrows *are* $\mathbf{L}$*-bonds where*
 identity arrow *of any formal* $\mathbf{L}$*-context* $\langle X, Y, I \rangle$ *is its incidence relation* I*,*[4]
 composition of arrows $\beta_1 \bullet \beta_2$ *is given by Definition 2.*

Remark 3. The category is equivalent to category of attribute-oriented concept lattices and isotone Galois connections. That is analogous to results in [12]. We will bring more about is in full version of the paper.

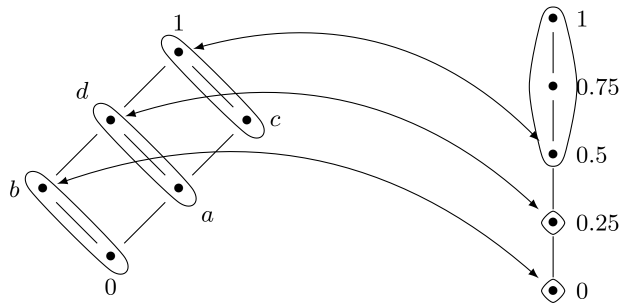
3.2 Setting with different structures of truth degrees

In this section we generalize the previous results into a setting in which fuzzy contexts are defined over different complete residuated lattices. To do that we need to explore compositions of underlying morphisms called residuation-preserving (l, k)-connections between complete residuated lattices.

(l, k)-connections and their compositions
Firstly, let us recall definition and basic properties of the (l, k)-connections introduced in [9].

Definition 3 ([9]). *Let* $\mathbf{L}_1, \mathbf{L}_2$ *be complete residuated lattices, let* $l \in L_1, k \in L_2$ *and let* $\lambda : L_1 \to L_2, \kappa : L_2 \to L_1$ *be mappings, such that*

[4] Clearly, I is an $\mathbf{L}$-bond from $\langle X, Y, I \rangle$ to $\langle X, Y, I \rangle$.



$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	a	0	a
b	0	0	b	0	b	b
c	0	a	0	c	a	c
d	0	0	b	a	b	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	c	c	1	c	1	1
c	b	d	b	1	d	1
d	a	c	d	c	1	1
1	0	a	b	c	d	1

Fig. 1. Six-element residuated lattice, with $\otimes$ and $\rightarrow$ as showed in the bottom part (`011010:00A0B0BCAB` in [3]), (top left), five-element Łukasiewicz chain (`111:000AB` in [3]), (top right), and $(c, 0.5)$-connection between them.

- $\langle \lambda, \kappa \rangle$ *is an isotone Galois connection between* $\mathbf{L}_1$ *and* $\mathbf{L}_2$,
- $\kappa\lambda(a_1) = l \rightarrow_1 (l \otimes_1 a_1)$ *for each* $a_1 \in L_1$,
- $\lambda\kappa(a_2) = k \otimes_2 (k \rightarrow_2 a_2)$ *for each* $a_2 \in L_2$.

We call $\langle \lambda, \kappa \rangle$ *an* (l, k)-connection *from* $\mathbf{L}_1$ *to* $\mathbf{L}_2$. *An* (l, k)-connection *from* $\mathbf{L}_1$ *to* $\mathbf{L}_2$ *is called* residuation-preserving *if*

$$\kappa(k \otimes_2 (\lambda(a) \rightarrow_2 \lambda(b))) = \kappa\lambda(a) \rightarrow_1 \kappa\lambda(b) \tag{12}$$

holds true for any $a, b \in L_2$.

Theorem 5 ([9]). *Let* $\langle \lambda, \kappa \rangle$ *be a residuation-preserving* (l, k)*-connection from* $\mathbf{L}_1$ *to* $\mathbf{L}_2$. *The algebra* $\langle \mathrm{fix}(\lambda, \kappa), \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ *where* $\wedge$ *and* $\vee$ *are given by the order*

$$\begin{aligned} \langle a_1, a_2 \rangle \leqslant \langle b_1, b_2 \rangle \quad & \text{if } a_1 \leqslant_1 b_1, \\ & \text{(equivalently, if } a_2 \leqslant_2 b_2\text{)} \end{aligned} \tag{13}$$

and the adjoint pair is given by

$$\langle a_1, a_2 \rangle \rightarrow \langle b_1, b_2 \rangle = \langle a_1 \rightarrow_1 b_1, k \otimes_2 (a_2 \rightarrow_2 b_2) \rangle \tag{14}$$

$$= \langle a_1 \rightarrow_1 b_1, k \otimes_2 ((k \rightarrow_2 a_2) \rightarrow_2 (k \rightarrow_2 b_2)) \rangle, \tag{15}$$

$$\langle a_1, a_2 \rangle \otimes \langle b_1, b_2 \rangle = \langle l \rightarrow_1 (l \otimes_1 a_1 \otimes_1 b_1), a_2 \otimes_2 (k \rightarrow_2 b_2) \rangle \tag{16}$$

$$= \langle l \rightarrow_1 (l \otimes_1 a_1 \otimes_1 b_1), (k \rightarrow_2 a_2) \otimes_2 b_2 \rangle \tag{17}$$

is a complete residuated lattice.

Figure 1 shows an example of (l, k)-connection. We refer the reader to [9] for ideas behind (l, k)-connections, examples and further details.

Now we define composition of (l, k)-connections and show that it is an (l, k)-connection as well. In addition, the composition preserves residuation-preservation, that means that composition of residuation-preserving (l, k)-connections is a residuation-preserving (l, k)-connection as well.

Theorem 6. *Let $\langle \lambda_1, \kappa_1 \rangle$ be an (l_1, k_2)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$ and $\langle \lambda_2, \kappa_2 \rangle$ be an (k_2, j_3)-connection from $\mathbf{L}_2$ to $\mathbf{L}_3$. Then the pair of mappings $\lambda \colon \mathbf{L}_1 \to \mathbf{L}_3$, $\kappa \colon \mathbf{L}_3 \to \mathbf{L}_1$, defined by*

$$\begin{aligned} \lambda(a_1) &= \lambda_2(k_2 \to_2 \lambda_1(a_1)), \\ \kappa(a_3) &= \kappa_1(k_2 \otimes_2 \kappa_2(a_3)) \end{aligned} \tag{18}$$

for each $a_1 \in L_1$ and $a_2 \in L_2$, is an (l_1, j_3)-connection from $\mathbf{L}_1$ to $\mathbf{L}_3$.

Proof. First, we prove that $\kappa\lambda(a_1) = l_1 \to_1 (l_1 \otimes_1 a_1)$ for each $a_1 \in L_1$ and $\lambda\kappa(a_2) = j_3 \otimes_3 (j_3 \to_3 a_3)$ for each $a_3 \in L_3$. For each $a_1 \in \mathbf{L}_1$, we have

$$\begin{aligned} \kappa\lambda(a_1) &= \kappa_1(k_2 \otimes_2 \kappa_2(\lambda_2(k_2 \to_2 \lambda_1(a_1)))) \\ &= \kappa_1(k_2 \otimes_2 (k_2 \to_2 (k_2 \otimes (k_2 \to_2 \lambda_1(a_1))))) \\ &= \kappa_1(k_2 \otimes_2 (k_2 \to_2 \lambda_1(a_1))) \\ &= \kappa_1(\lambda_1(\kappa_1(\lambda_1(a_1)))) \\ &= \kappa_1(\lambda_1(a_1)) \\ &= l_1 \to_1 (l_1 \otimes_1 a_1). \end{aligned}$$

Similarly, we have for each $a_3 \in L_3$

$$\begin{aligned} \lambda\kappa(a_3) &= \lambda_2(k_2 \to_2 \lambda_1(\kappa_1(k_2 \otimes_2 \kappa_2(a_3)))) \\ &= \lambda_2(k_2 \to_2 (k_2 \otimes_2 (k_2 \to_2 (k_2 \otimes_2 \kappa_2(a_3))))) \\ &= \lambda_2(k_2 \to_2 (k_2 \otimes_2 \kappa_2(a_3))))) \\ &= \lambda_2(\kappa_2(\lambda_2(\kappa_2(a_3)))) \\ &= \lambda_2(\kappa_2(a_3)) \\ &= j_3 \otimes_3 (j_3 \to_3 a_3). \end{aligned}$$

Since $\kappa\lambda(a_1) = l_1 \to_1 (l_1 \otimes_1 a_1) \geqslant_1 a_1$ and $\lambda\kappa(a_3) = j_3 \otimes_3 (j_3 \otimes_3 a_3) \leqslant_3 a_3$ we only need to show monotony to prove that $\langle \lambda, \kappa \rangle$ is an isotone Galois connection: For each $a_1, b_1 \in \mathbf{L}_1$ we have

$a_1 \leqslant_1 b_1$ implies $\lambda_1(a_1) \leqslant_2 \lambda_1(b_1)$ since λ_1 is monotone,
implies $k_2 \to_2 \lambda_1(a_1) \leqslant_2 k_2 \to_2 \lambda_1(b_1)$ since $\to_2$ is monotone in its second argument,
implies $\lambda_2(k_2 \to_2 \lambda_1(a_1)) \leqslant_3 \lambda_2(k_2 \to_2 \lambda_1(b_1))$ since λ_2 is monotone.

Thus $a_1 \leqslant_1 b_1$ implies $\lambda(a_1) \leqslant_3 \lambda(b_1)$ for each $a_1, b_1 \in \mathbf{L}_1$. Similarly, one can show that $a_3 \leqslant_3 b_3$ implies $\kappa(a_3) \leqslant_1 \kappa(b_3)$.

□

Theorem 7. *Let $\langle\lambda_1,\kappa_1\rangle$ be a residuation-preserving (l_1,k_2)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$ and $\langle\lambda_2,\kappa_2\rangle$ be a residuation-preserving (k_2,j_3)-connection from $\mathbf{L}_2$ to $\mathbf{L}_3$. Then the pair of mappings $\lambda\colon \mathbf{L}_1\to\mathbf{L}_3$, $\kappa\colon\mathbf{L}_3\to\mathbf{L}_1$, defined by* (18)*, is a residuation-preserving (l_1,j_3)-connection from $\mathbf{L}_1$ to $\mathbf{L}_3$.*

Proof. For each $a_1,b_1\in\mathbf{L}_1$ we have

$$\begin{aligned}
&\kappa\lambda(a_1)\to_1\kappa\lambda(b_1)=\\
&=(l_1\to_1(l_1\otimes_1 a_1))\to_1(l_1\to_1(l_1\otimes_1 b_1))\\
&=\kappa_1\lambda_1(a_1)\to_1\kappa_1\lambda_1(b_1)\\
&=\kappa_1(k_2\otimes_2(\lambda_1(a_1)\to_2\lambda_1(b_1)))\\
&=\kappa_1(k_2\otimes_2(\lambda_1\kappa_1\lambda_1(a_1)\to_2\lambda_1\kappa_1\lambda_1(b_1)))\\
&=\kappa_1(k_2\otimes_2((k_2\otimes_2(k_2\to_2\lambda_1(a_1)))\to_2(k_2\otimes_2(k_2\to_2\lambda_1(b_1)))))\\
&=\kappa_1(k_2\otimes_2((k_2\otimes_2(k_2\to_2(k_2\otimes_2(k_2\to_2\lambda_1(a_1)))))\to_2(k_2\otimes_2(k_2\to_2\lambda_1(b_1)))))\\
&=\kappa_1(k_2\otimes_2((k_2\to_2(k_2\otimes_2(k_2\to_2\lambda_1(a_1))))\to_2(k_2\to_2(k_2\otimes_2(k_2\to_2\lambda_1(b_1)))))\\
&=\kappa_1(k_2\otimes_2(\kappa_2\lambda_2(k_2\to_2\lambda_1(a_1))\to_2\kappa_2\lambda_2(k_2\to_2\lambda_1(b_1))))\\
&=\kappa_1(k_2\otimes_2\kappa_2(j_3\otimes_3(\lambda_2(k_2\to_2\lambda_1(a_1))\to_3\lambda_2(k_2\to_2\lambda_1(b_1)))))\\
&=\kappa_1(k_2\otimes_2\kappa_2(j_3\otimes_3(\lambda(a_1)\to_3\lambda(b_1))))\\
&=\kappa(j_3\otimes_3(\lambda(a_1)\to_3\lambda(b_1))).
\end{aligned}$$

□

We call $\langle\lambda,\kappa\rangle$ from (18) a composition of $\langle\lambda_1,\kappa_1\rangle$ and $\langle\lambda_2,\kappa_2\rangle$ and we denote it as $\langle\lambda_1,\kappa_1\rangle\bullet\langle\lambda_2,\kappa_2\rangle=\langle\lambda_1\bullet\lambda_2,\kappa_1\bullet\kappa_2\rangle$. Now we show, that the composition of (l,k)-connections is associative.

Theorem 8. *Let $\langle\lambda_1,\kappa_1\rangle$ be an (l_1,k_2)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$, $\langle\lambda_2,\kappa_2\rangle$ be a (k_2,j_3)-connection from $\mathbf{L}_2$ to $\mathbf{L}_3$, and $\langle\lambda_3,\kappa_3\rangle$ be a (j_3,m_4)-connection from $\mathbf{L}_3$ to $\mathbf{L}_4$. Then*

$$\langle\lambda_1,\kappa_1\rangle\bullet(\langle\lambda_2,\kappa_2\rangle\bullet\langle\lambda_3,\kappa_3\rangle)=(\langle\lambda_1,\kappa_1\rangle\bullet\langle\lambda_2,\kappa_2\rangle)\bullet\langle\lambda_3,\kappa_3\rangle.$$

Proof. We have for each $a\in L_1$

$$\begin{aligned}
(\lambda_1\bullet(\lambda_2\bullet\lambda_3))(a_1)&=(\lambda_2\bullet\lambda_3)(k_2\to_2\lambda_1(a_1))\\
&=\lambda_3(j_3\to\lambda_2(k_2\to_2\lambda_1(a_1)))\\
&=\lambda_3(j_3\to(\lambda_1\bullet\lambda_2)(a_1))\\
&=((\lambda_1\bullet\lambda_2)\bullet\lambda_3)(a_1)
\end{aligned}$$

and similarly for the κ-part. □

Theorem 9. *The following structure forms a category.*

Objects *are pairs $\langle\mathbf{L},e\rangle$, where $\mathbf{L}$ is a complete residuated lattices and $e\in L$.*
Arrows *from $\langle\mathbf{L}_1,l\rangle$ to $\langle\mathbf{L}_2,k\rangle$ are (l,k)-connections from $\mathbf{L}_1$ to $\mathbf{L}_2$, where*

identity arrow *on any* $\langle \mathbf{L}, e\rangle$ *is* (e,e)*-connection* $\langle \lambda, \kappa\rangle$ *where* $\lambda(a) = e \otimes a$ *and* $\kappa(a) = e \to a$ *for each* $a \in L$.

composition of arrows *is as defined in* (18).

If we use just residuation-preserving (l,k)*-connections we obtain a sub-category.*

Now, we can explore bonds based on residuation-preserving (l,k)-connections.

Definition 4. *Let* $\mathbf{L}_1, \mathbf{L}_2$ *be complete residuated lattices,* $\langle \lambda, \kappa\rangle$ *be residuation-preserving* (l,k)*-connection from* $\mathbf{L}_1$ *to* $\mathbf{L}_2$*, and let* $\langle X_1, Y_1, I_1\rangle$ *and* $\langle X_2, Y_2, I_2\rangle$ *be* $\mathbf{L}_1$*-context and* $\mathbf{L}_2$*-context, respectively. We call* $\beta \in L^{X_1\times Y_2}_{\langle\lambda,\kappa\rangle}$ *a* $\langle \lambda, \kappa\rangle$-bond *from* $\langle X_1, Y_1, I_1\rangle$ *to* $\langle X_2, Y_2, I_2\rangle$ *if the following inclusions hold.*

$$\mathrm{Ext}^{\vartriangle\triangledown}(X_1, Y_2, \beta) \subseteq \mathrm{Ext}^{\cap\cup}(X_1, Y_1, \kappa\lambda(I_1)), \tag{19}$$

$$\mathrm{Int}^{\vartriangle\triangledown}(X_1, Y_2, \beta) \subseteq \mathrm{Int}^{\cap\cup}(X_2, Y_2, \lambda\kappa(I_2)). \tag{20}$$

The concept-forming operators $\langle \vartriangle, \triangledown\rangle$ *induced by* $\langle \lambda, \kappa\rangle$*-bond* β *from* $\langle X_1, Y_1, I_1\rangle$ *to* $\langle X_2, Y_2, I_2\rangle$ *are given by*[5]

$$\begin{aligned} A^{\vartriangle_\beta} &= \lambda(A)^{\cap_{\mathrm{proj}_2(\beta)}}, \\ B^{\triangledown_\beta} &= \kappa(B)^{\cup_{\mathrm{proj}_1(\beta)}}. \end{aligned} \tag{21}$$

Theorem 10. *Let* $\langle X_1, Y_1, I_1\rangle$ *be an* $\mathbf{L}_1$*-context,* $\langle X_2, Y_2, I_2\rangle$ *be an* $\mathbf{L}_2$*-context, and* $\langle \lambda, \kappa\rangle$ *an* (l,k)*-connection from* $\mathbf{L}_1$ *to* $\mathbf{L}_2$*. Then* $\beta \in L_{\langle\lambda,\kappa\rangle}$ *is a* $\langle \lambda, \kappa\rangle$*-bond from* $\langle X_1, Y_1, I_1\rangle$ *to* $\langle X_2, Y_2, I_2\rangle$ *if and only if it is a* $\mathbf{L}_{\langle\lambda,\kappa\rangle}$*-bond w.r.t.* $\langle \cap, \cup\rangle$ *from* $\langle X_1, Y_1, \langle \kappa\lambda(I_1), \lambda(I_1)\rangle\rangle$ *to* $\langle X_2, Y_2, \langle \kappa(I_2), \lambda\kappa(I_2)\rangle\rangle$.

Proof. Directly from the definition and (21). □

For what follows we will need the following product of fuzzy relations. Let $\langle \lambda_1, \kappa_1\rangle$ be (l_1, k_2)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$, $\langle \lambda_2, \kappa_2\rangle$ be (k_2, m_3)-connection from $\mathbf{L}_2$ to $\mathbf{L}_3$, and $I \in \mathbf{L}^{X\times Y}_{\langle\lambda_1,\kappa_1\rangle}$, $J \in \mathbf{L}^{Y\times Z}_{\langle\lambda_2,\kappa_2\rangle}$. Then $I \boxtimes J \in \mathbf{L}^{X\times Z}_{\langle\lambda_1\bullet\lambda_2,\kappa_1\bullet\kappa_2\rangle}$ is defined as

$$I \boxtimes J = \langle \kappa_1(K), \lambda_2(k_2 \to_2 K)\rangle \quad \text{where } K = \mathrm{proj}_2(I) \circ_2 \mathrm{proj}_1(J) \tag{22}$$

and $\circ_2$ is composition of $\mathbf{L}_2$-relations (1).

Lemma 4. *Let* $\langle X_1, Y_1, I_1\rangle$ *be an* $\mathbf{L}_1$*-context,* $\langle X_2, Y_2, I_2\rangle$ *be an* $\mathbf{L}_2$*-context, and* $\langle \lambda, \kappa\rangle$ *an* (l,k)*-connection from* $\mathbf{L}_1$ *to* $\mathbf{L}_2$.

(a) An $\mathbf{L}_{\langle\lambda,\kappa\rangle}$*-relation* β *for which exist* $\mathbf{L}_{\langle\lambda,\kappa\rangle}$*-relations* $S_\mathrm{e} \in L^{X_1\times X_2}_{\langle\lambda,\kappa\rangle}$ *and* $S_\mathrm{i} \in L^{Y_1\times Y_2}_{\langle\lambda,\kappa\rangle}$ *such that*

$$\beta = \langle \kappa\lambda(I_1), \lambda(I_1)\rangle \boxtimes S_\mathrm{i} = S_\mathrm{e} \boxtimes \langle \kappa(I_2), \lambda\kappa(I_2)\rangle$$

is a $\langle \lambda, \kappa\rangle$*-bond from* $\langle X_1, Y_1, I_1\rangle$ *to* $\langle X_2, Y_2, I_2\rangle$.

[5] proj_1, proj_2 denote projection of first and second component of a pair, respectively.

(b) Each $\langle\lambda,\kappa\rangle$-bond β from $\langle X_1,Y_1,I_1\rangle$ to $\langle X_2,Y_2,I_2\rangle$ satisfies that there is $S_{\mathrm{e}} \in L^{X_1\times X_2}_{\langle\lambda,\kappa\rangle}$ such that

$$\beta = S_{\mathrm{e}} \boxtimes \langle\kappa(I_2),\lambda\kappa(I_2)\rangle.$$

Proof. From Theorem 10 and Lemma 1. □

Theorem 11. *Let $\langle\lambda_1,\kappa_1\rangle$ be an (l_1,k_2)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$, $\langle\lambda_2,\kappa_2\rangle$ be an (k_2,j_3)-connection from $\mathbf{L}_2$ to $\mathbf{L}_3$, β_1 be $\langle\lambda_1,\kappa_1\rangle$-bond from $\langle X_1,Y_1,I_1\rangle$ to $\langle X_2,Y_2,I_2\rangle$, and β_2 be $\langle\lambda_2,\kappa_2\rangle$-bond from $\langle X_2,Y_2,I_2\rangle$ to $\langle X_3,Y_3,I_3\rangle$.*

$$\beta = S_{\mathrm{e}} \boxtimes \beta_2, \tag{23}$$

where $S_{\mathrm{e}} = \beta_1 \triangleright \langle\kappa_1(I_2^{\mathrm{T}}),\lambda_1\kappa_1(I_2^{\mathrm{T}})\rangle$, is a $\langle\lambda_1\bullet\lambda_2,\kappa_1\bullet\kappa_2\rangle$-bond from $\langle X_1,Y_1,I_1\rangle$ to $\langle X_3,Y_3,I_3\rangle$.

Let us denote β from (23) as $\beta=\beta_1\bullet\beta_2$ and call it a composition of isotone $\langle\lambda,\kappa\rangle$-bonds. Now we show associativity of this composition.

Theorem 12. *Let $\langle\lambda_1,\kappa_1\rangle$ be an (l_1,k_2)-connection from $\mathbf{L}_1$ to $\mathbf{L}_2$, $\langle\lambda_2,\kappa_2\rangle$ be an (k_2,j_3)-connection from $\mathbf{L}_2$ to $\mathbf{L}_3$, $\langle\lambda_3,\kappa_3\rangle$ be an (j_3,m_4)-connection from $\mathbf{L}_3$ to $\mathbf{L}_4$, and β_i be $\langle\lambda_i,\kappa_i\rangle$-bond from $\langle X_i,Y_i,I_i\rangle$ to $\langle X_{i+1},Y_{i+1},I_{i+1}\rangle$. Then*

$$\beta_1\bullet(\beta_2\bullet\beta_3) = (\beta_1\bullet\beta_2)\bullet\beta_3.$$

Proof. Follows from Theorem 3, Theorem 8, and Theorem 10. □

Finally, we can state that **L**-contexts over different structures of truth degrees and bonds between them form a category.

Theorem 13. Objects *are pairs $\langle\mathbf{K},e\rangle$, where $\mathbf{K}$ is a $\mathbf{L}$-context and $e\in L$.*
Arrows *between $\langle\mathbf{K_1},l\rangle$ and $\langle\mathbf{K_2},k\rangle$, where $\mathbf{K}_1$ is an $\mathbf{L}_1$-context, $\mathbf{K}_2$ is an $\mathbf{L}_2$-context and $l\in L_1, k\in L_2$, are $\langle\lambda,\kappa\rangle$-bonds, where $\langle\lambda,\kappa\rangle$ is an (l,k)-connection.*
identity arrow *for a pair $\langle K,e\rangle$ of $\mathbf{L}$-context $\langle X,Y,I\rangle$ and e is $\langle\lambda,\kappa\rangle$-bond I with $\langle\lambda,\kappa\rangle$ are (e,e)-connections $\langle\lambda,\kappa\rangle$ where $\lambda(x)=e\rightarrow a$ and $\kappa(x)=e\otimes a$ for each $a\in L$.*
composition of arrows *$\beta_1\bullet\beta_2$ is given by (23).*

4 Future Research

Our future research in this area includes addressing the following issues:

- Antitone bonds between fuzzy contexts over different complete residuated lattices were studied in [9]; basics of Isotone bonds are presented in this paper. We want to extend this study to heterogeneous bonds[11]. We will bring results on them and their compositions in the full version of this paper.
- As block relations are a special case of bonds, they share many properties (see [11]). It can be fruitful to study the compositions described in this paper in context of block **L**-relations. In addition, the composition applied on block (crisp) relations correspond with multiplication used in calculus studied in [1]. This observation deserves deeper study; we believe that this can bring a new interesting insight to the calculus.

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References

1. Eduard Bartl and Michal Krupka. Residuated lattices of block relations: Size reduction of concept lattices. *to appear in IJGS*, 2015.
2. Radim Belohlavek. *Fuzzy Relational Systems: Foundations and Principles*. Kluwer Academic Publishers, Norwell, USA, 2002.
3. Radim Belohlavek and Vilem Vychodil. Residuated lattices of size $\leqslant 12$. *Order*, 27(2):147–161, 2010.
4. Radim Belohlavek and Jan Konecny. Row and Column Spaces of Matrices over Residuated Lattices. *Fundam. Inform.*, 115(4):279–295, 2012.
5. George Georgescu and Andrei Popescu. Non-dual fuzzy connections. *Arch. Math. Log.*, 43(8):1009–1039, 2004.
6. Ladislav J Kohout and Wyllis Bandler. Relational-product architectures for information processing. *Information Sciences*, 37(1-3):25–37, 1985.
7. Jan Konecny. Isotone fuzzy Galois connections with hedges. *Information Sciences*, 181(10):1804–1817, 2011.
8. Jan Konecny. Antitone L-bonds. In: Information Processing and Management of Uncertainty in Knowledge-Based Systems – 15th International Conference, IPMU 2014, pages 71–80, 2014.
9. Jan Konecny. Bonds between L-fuzzy contexts over different structures of truth-degrees, In: 13th International Conference, ICFCA 2015, Nerja, Spain, June 23-26, pages 81–96, 2015.
10. Jan Konecny and Manuel Ojeda-Aciego. Isotone L-bonds. In Manuel Ojeda-Aciego and Jan Outrata, editors, *CLA*, volume 1062 of *CEUR Workshop Proceedings*, pages 153–162. CEUR-WS.org, 2013.
11. Jan Konecny and Manuel Ojeda-Aciego. On homogeneous L-bonds and heterogeneous L-bonds. *to appear in IJGS*, 2015.
12. Ondrej Krídlo, Stanislav Krajči, and Manuel Ojeda-Aciego. The Category of L-Chu Correspondences and the Structure of L-Bonds. *Fundam. Inform.*, 115(4):297–325, 2012.
13. Ondrej Kridlo and Manuel Ojeda-Aciego. CRL-Chu Correspondences. In Manuel Ojeda-Aciego and Jan Outrata, editors, *CLA*, volume 1062 of *CEUR Workshop Proceedings*, pages 105–116. CEUR-WS.org, 2013.
14. Markus Krötzsch, Pascal Hitzler, and Guo-Qiang Zhang. Morphisms in context. In *Conceptual Structures: Common Semantics for Sharing Knowledge, 13th International Conference on Conceptual Structures, ICCS 2005, Kassel, Germany, July 17-22, 2005, Proceedings*, pages 223–237, 2005.
15. Jesús Medina. Multi-adjoint property-oriented and object-oriented concept lattices. *Inf. Sci.*, 190:95–106, 2012.