# The Fuzzy Description Logic $\mathcal{ALC}_{FLH}$

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**Abstract**

We present the fuzzy description logic $\mathcal{ALC}_{FLH}$. $\mathcal{ALC}_{FLH}$ is based on $\mathcal{ALC}_{FH}$, but linear hedges are used instead of exponential ones. This allows to solve the entailment and the subsumption problem in a fuzzy description logic, where arbitrary concepts and roles may be modified.

## 1 Motivation

In most applications of description logics that we are aware of, except [7] and [8], concepts are crisp unary relations, i.e., an object may or may not be an element of a particular concept. On the other hand, in many real-world applications like, for example, intelligent e-commerce information is often vague and imprecise. Fuzzy set theory introduced by Zadeh (see e.g. [9]) provides an ability to denote non-crisp concepts, i.e., an object may belong to a certain degree (typically a real number from the interval $[0, 1]$) to a particular relation.

Humans typically use linguistic adverbs like *very*, *more or less*, etc. to distinguish, for example, between a customer who is interested in technical details and one who is very interested in these details. In [10] Zadeh introduces so-called linguistic hedges modifying the shape of a fuzzy set by transforming it into another. Hedge algebras were considered in [5, 4] to give an algebraic charactrization of linguistic hedges. They have been applied to fuzzy logic in various ways (see e.g. [3]).

In $\mathcal{ALC}_{FH}$ [1, 2] hedges were used to modify concepts in the fuzzy description logic $\mathcal{ALC}_F$[7], which itself is an extension of $\mathcal{ALC}$. There was, however, a main restriction, viz. that modifiers had to be restricted to primitive concepts in order to solve the subsumption problem. In this paper we overcome this restriction by introducing so-called *linear* hedges in Section 3. $\mathcal{ALC}_{FLH}$ is defined as an extension of $\mathcal{ALC}_{FH}$ allowing to modify arbitray concepts and roles in Section 4. We specify two normal forms and show that each concept can be normalized in Section 5. The entailment and subsumption problems are solved in Sections 6 and 7, respectively. A brief disussion concludes the paper in Section 8.

## 2 Preliminaries

The approach presented in this paper is based on $\mathcal{ALC}_{FH}$ [1, 2], which is an extension of $\mathcal{ALC}$ and $\mathcal{ALC}_F$ [7]. *Concepts* $C$ and $D$ are constructed by the rule $C, D \rightarrow A \mid \top \mid \bot \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid MC \mid \exists R.C \mid \forall R.C$ where $A$ denotes primite concepts, $R$ roles, and $M$ modifiers. We interprete formulas as usual in a fuzzy setting by mapping concepts and roles onto membership functions. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation. Then,

$$
\begin{aligned}
A^{\mathcal{I}} &: \Delta^{\mathcal{I}} \rightarrow [0,1] \\
R^{\mathcal{I}} &: \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0,1] \\
\top^{\mathcal{I}}(d) &= 1 \text{ for all } d \in \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}}(d) &= 0 \text{ for all } d \in \Delta^{\mathcal{I}} \\
(\neg C)^{\mathcal{I}}(d) &= 1 - C^{\mathcal{I}}(d) \\
(C \sqcap D)^{\mathcal{I}}(d) &= \min\{C^{\mathcal{I}}(d), C^{\mathcal{I}}(d)\} \\
(C \sqcup D)^{\mathcal{I}}(d) &= \max\{C^{\mathcal{I}}(d), C^{\mathcal{I}}(d)\} \\
(MC)^{\mathcal{I}}(d) &= \eta_M(C^{\mathcal{I}}(d)) \\
(\forall R.C)^{\mathcal{I}}(d) &= \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{1 - R^{\mathcal{I}}(d,d'), C^{\mathcal{I}}(d)\}\} \\
(\exists R.C)^{\mathcal{I}}(d) &= \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{R^{\mathcal{I}}(d,d'), C^{\mathcal{I}}(d)\}\}
\end{aligned}
$$

where $\eta_M$ is used to modify a membership function and $d \in \Delta^{\mathcal{I}}$. Modifiers or *linguistic hedges* were introduced by Zadeh in [10], where he also proposed to use exponent functions as hedges. In [1, 2] a function *exponent* has been specified, which applied to a modifier $M$ computes an exponent $\beta$ such that $\eta_M(x) = x^{exponent\,(M)} = x^{\beta}$, where $x \in [0,1]$.

*Fuzzy assertions* are expressions of the form $\langle \alpha \circ n \rangle$, where $\circ \in \{>, \geq, \leq, <\}$, $\alpha$ is of type $a : C$ or $(a,b) : R$, and $n \in [0,1]$. *Fuzzy terminological axioms* as well as the semantics for fuzzy assertions and terminological axioms are defined as usual.

## 3 Linear Hedges

The main idea proposed in this paper is the use of linear instead of exponential hedges. Linear hedges were first introduced in [6], where it has been shown that they have better algebraic and computational properties than existing approaches for a parametric representation of linguistic truth-values. Here we consider the following linear hedges: Given a modifier $M$ and let $\beta = \mathit{exponent}(M)$ and $x \in [0,1]$, then

$$\eta_M(x) = \begin{cases} \frac{1}{\beta}x & \text{if } x \leq \frac{\beta}{\beta+1}, \\ 1 + \beta(x-1) & \text{otherwise.} \end{cases}$$

One should observe that its inverse function is

$$\eta_M^{-1}(x) = \begin{cases} \beta x & \text{if } x \leq \frac{1}{\beta+1}, \\ 1 + \frac{1}{\beta}(x-1) & \text{otherwise.} \end{cases}$$

Hence, $\eta_M^{-1}$ is obtained from $\eta_M$ by replacing $\beta$ by $\frac{1}{\beta}$. By abuse of notation we sometimes will use $\eta_\beta$ and $\eta_{\frac{1}{\beta}}$ to denote $\eta_M$ and $\eta_M^{-1}$, respectively.

# 4 $\mathcal{ALC}_{FLH}$

We can now specify $\mathcal{ALC}_{FLH}$ as follows: *Concepts* are defined as in Section 2. *Roles* $R$ are defined as $R \rightarrow Q \mid MR$, where $Q$ are primitive roles and $M$ are modifiers. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation. Then,

$$\begin{array}{rcl} Q^{\mathcal{I}} & : & \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0,1] \\ (MR)^{\mathcal{I}}(d,d') & = & \eta_M(R^{\mathcal{I}}(d,d')) \end{array}$$

where $\eta_M$ is a linear linguistic hedge as defined in Section 3.

For each modifier $M$ we assume that there is an *inverse modifier* $M^{-1}$ with $\eta_{M^{-1}} = \eta_M^{-1}$. In the view of semantics, this assumption will be useful for equivalently converting a concept in $\mathcal{ALC}_{FLH}$ into a normal form. It is easy to see that concepts can be expressed without using inverse modfiers. For instance, the concept $M^{-1}C$ is semanctically equal to $\neg M(\neg C)$.

# 5 Normalizing Concepts in $\mathcal{ALC}_{FLH}$

**Proposition 1** *The following semantic equivalences hold in $\mathcal{ALC}_{FLH}$:*

$$\begin{array}{lcl} M(\neg C) & \equiv & \neg M^{-1}C \\ M(C \sqcap D) & \equiv & M(C) \sqcap M(D) \\ M(C \sqcup D) & \equiv & M(C) \sqcup M(D) \\ M(\forall R.C) & \equiv & \forall M^{-1}R.MC \\ M(\exists R.C) & \equiv & \exists MR.MC \end{array}$$

The *set of simple concepts* is the smallest set satisfying the following conditions: (i) each primitive concept is a simple concept; (ii) if $X$ is a simple concept and $M$ is a modifier, then $MX$ is a simple concept. In other words, simple concepts are obtained from primitive concepts by prefixing the latter with a string of modifiers. Likewise, we define the *set of simple roles*.

A concept $C$ is said to be in *modifier normal form* (*MNF*) iff all modifiers occurring in $C$ act only on simple concepts and roles. The following result is an immediate consequence of Proposition 1. It is obtained by pushing modifiers into the concepts as much as possible.

**Proposition 2** *For each concept in $\mathcal{ALC}_{FLH}$ there is a semantically equivalent one in MNF.*

A concept $C$ is said to be in *negative modifier normal form* (*NMNF*) iff $C$ is in $MNF$ and all negation signs occurring in $C$ act only on simple concepts or roles.

**Proposition 3** *For each concept in $\mathcal{ALC}_{FLH}$ there is a semantically equivalent one in NMNF.*

This is result is shown by first transforming a given concept $C$ into MNF $C'$ and, thereafter, to transform $C'$ into NMNF as usual while treating simple concepts as atoms. The complexity of the whole normalization process is polynomial wrt the length of input concept.

# 6 The Entailment Problem in $\mathcal{ALC}_{FLH}$

Let $\Sigma$ be a set of fuzzy assertions. As in the case of $\mathcal{ALC}_{FH}$, an entailment problem $\Sigma \models \langle \alpha \circ n \rangle$ is converted into the problem of $\Sigma \cup \{\langle \alpha \bullet n \rangle\}$ is unsatisfiable, where $\bullet$ is the inverse of $\circ$. The latter problem is solved by a tableau algorithm. As usual, we use propagation rules on a set of assertions (or constraints) to convert constraints into simpler ones. This process will terminate and result in a *completion set* to which no propagation rule can be applied. As we will show in Proposition 4 a set of assertions is unsatisfiable iff the corresponding completion set contains a clash, where clashes are defined as in [7].

Due to lack of space we cannot present the 16 propagation rules, but show only some of them. In the following let $\beta = \mathit{exponent}(M)$. We start with the rules for modified concepts:

$$(M_{\geq}) \quad \langle w : MC \geq n \rangle \quad \rightarrow \quad \langle w : C \geq \eta_{\frac{1}{\beta}}(n) \rangle$$

The rules $(M_{>})$, $(M_{\leq})$ and $(M_{<})$ can be defined similarly. Because $\mathcal{ALC}_{FLH}$ allows negated and modified roles we need 8 new propagation rules to cope with these extensions.

$$(\neg^R_{\geq}) \quad \langle (w,w') : \neg Q \geq n \rangle \quad \rightarrow \quad \langle (w,w') : Q < 1-n \rangle$$

$$(M^R_{\geq}) \quad \langle (w,w') : MQ \geq n \rangle \quad \rightarrow \quad \langle (w,w') : Q \geq \eta_{\frac{1}{\beta}}(n) \rangle$$

The rules for $(\neg^R_{>})$, $(\neg^R_{\leq})$, $(\neg^R_{<})$, $M^R_{>}$, $M^R_{\leq}$ and $M^R_{<}$ can be defined similarly. The existence of complex roles in $\mathcal{ALC}_{FLH}$ forces us to change the propagation rules $(\forall_{\geq})$, $(\forall_{>})$, $(\exists_{\leq})$ and $(\exists_{<})$ used in $\mathcal{ALC}_{FH}$ slightly. We show one of these modified propagation rules; the other rules can be modified in a similar way.

$$(\forall_{\geq}) \quad \langle w_1 : \forall R.C \geq n \rangle, \psi^c \quad \rightarrow \quad \langle w_2 : C \geq n \rangle,$$

where $\psi = \langle (w_1, w_2) : Q \leq \eta_{\frac{1}{\beta_k}}(...\eta_{\frac{1}{\beta_2}}(\eta_{\frac{1}{\beta_1}}(1-n))...) \rangle$, $R = M_1(M_2(...M_k(Q)..))$, $\mathit{exponent}(M_i) = \beta_i$ for $i = 1..k$, and $\psi^c$ is the conjugated constraint of $\psi$. This condition is similar to the one of the corresponding propagation rule in $\mathcal{ALC}_{FH}$.

**Proposition 4** *A finite set of constraints in $\mathcal{ALC}_{FLH}$ is unsatisfiable iff it contains a clash.*

The proof of this result is analogous to the case of $\mathcal{ALC}_{FH}$ proved in [1]. Because we use the same way as [7, 1] to deal with the entailment problem, our given decision procedure is also PSPACE-hard. As in [1] the propagation rules for $\forall$ and $\exists$ would lead to an exponential explosion. In [7, 1] this problem was solved by using so-called *trace rules*. We have not yet investigated the use of trace rules in $\mathcal{ALC}_{FLH}$, but, due to the similarity of the propagation rules we believe it also works for our case.

# 7 The Subsumption Problem in $\mathcal{ALC}_{FLH}$

We will approach the subsumption problem as in the case of $\mathcal{ALC}_{FH}$ [2] or $\mathcal{ALC}_{F}$ [7]. In a first step, all complex concepts are expanded, which leads to a subsumption problem over an empty set of terminological axioms, and, thereafter, the subsumption problem is reduced to an entailment problem.

**Proposition 5** *Let $C$ and $D$ be two concepts in $\mathcal{ALC}_{FLH}$. Then $C \sqsubseteq_{\emptyset} D$ iff $\langle a : C \geq n \rangle \models \langle a : D \geq n \rangle$ for all $n \in (0,1]$, where $a$ is an arbitrary individual.*

The proof of this proposition is similar to the corresponding one in [1]. The entailment problems considered in Proposition 5 will be reduced to equivalent unsatisfiability problems of the form $S = \{\langle a : C \geq n \rangle, \langle a : D < n \rangle\}$ for all $n \in (0,1]$, to which we can apply the tableau algorithm presented in Section 6. Wlog we assume that all concepts mentioned in this section are in MNF.

**Proposition 6** *Let $C$ and $D$ be two concepts in $\mathcal{ALC}_{FLH}$ and $S = \{\langle a : C \geq n\rangle, \langle a : D < n\rangle\}$. The following holds:*

1. *There is a finite set of completion sets of $S$, to each of which an interval $(a, b] \subseteq (0, 1]$ is associated.*

2. *Let $\tilde{S}$ be a completion set of $S$, $(a, b]$ the interval associated with $\tilde{S}$, $x \in (a, b]$, and $\tilde{S}'$ be obtained from $\tilde{S}$ by replacing $n$ by $x$. If $\tilde{S}'$ is unsatisfiable then $\tilde{S}$ is unsatisfiable for all $y \in (a, b]$.*

3. *$C \sqsubseteq_\emptyset D$ iff all completion sets of $S$ are unsatisfiable.*

This proposition specifies an obvious method to solve the subsumption problem. Due to lack of space we can only sketch the proof here. The proof is based on the following property:

**Proposition 7** *Let $C$ and $D$ be two $\mathcal{ALC}_{FLH}$ concepts, $S = \{\langle a : C \geq n\rangle, \langle a : D < n\rangle\}$, and $S'$ be obtained from $S$ by applying some propagation rules. Then, every constraint in $S'$ is in one of the following forms:*

$$\langle w \geq \eta_{\beta_l}(\eta_{\beta_{l-1}}(...\eta_{\beta_1}(n)...))\rangle,\ \langle w \leq \eta_{\beta_l}(\eta_{\beta_{l-1}}(...\eta_{\beta_1}(1-n)...))\rangle,$$
$$\langle w < \eta_{\beta_l}(\eta_{\beta_{l-1}}(...\eta_{\beta_1}(n)...))\rangle,\ \textit{or}\ \langle w > \eta_{\beta_l}(\eta_{\beta_{l-1}}(...\eta_{\beta_1}(1-n)...))\rangle,$$

*where $\beta_1, \beta_2, ..., \beta_l > 0$, $l \geq 0$ and $w$ is the form of $a : C$ or $(x, y) : R$.*

The proof is by induction on the length of the derivation generated by the propagation rules. Returning to the proof of Proposition 6, we introduce $\langle S, (a, b]\rangle$ to be a pair where all constraints occurring in $S$ are in the form listed in Proposition 7 and $(a, b]$ stands for the value restriction of the symbol "$n$" in $S$. In order to determine whether a propagation rule is applicable to a pair $\langle S, (a, b]\rangle$, we classify the propagation rules into three types: (a) non-deterministic rules consisting of $(\sqcup_\geq)$, $(\sqcup_>)$, $(\sqcap_\leq)$ and $(\sqcap_<)$, (b) condition rules consisting of $(\forall_\geq)$, $(\forall_>)$, $(\exists_\leq)$ and $(\exists_<)$, and (c) deterministic rules consisting of all remaining rules.

Now, a propagation rule is said to be applicable to $\langle S, (a, b]\rangle$ iff it is applicable to any set obtained by replacing "$n$" with an arbitrary $x \in (a, b]$. The following case analysis shows a way to check whether a propagation rule is applicable to this pair or not.

**(a)**: A non-deterministic rule $\Phi \rightarrow \Psi_1|\Psi_2$ is applicable to $\langle S, (a, b]\rangle$ iff $S \supseteq \Phi$. In this case we obtain two new pairs $\langle S_1, (a, b]\rangle$ and $\langle S_2, (a, b]\rangle$ where $S_1 = S \cup \Psi_1$ and $S_2 = S \cup \Psi_2$.

**(b)**: A condition rule $\Phi \rightarrow \Psi$ if $\Gamma$ is applicable to $\langle S, (a, b]\rangle$ iff $S \supseteq \Phi$ and $\Gamma$ is satisfied. In some cases $\Gamma$ will force us to devide $\langle S, (a, b]\rangle$ into $m \geq 1$ pairs $\langle S, (a_i, a_{i+1}]\rangle$ with $i = 1..m$ where $(a, b] = (a_1, a_2]\dot{\cup}(a_2, a_3]\dot{\cup}...\dot{\cup}(a_m, a_{m+1}]$. In order to complete this part of the proof, one needs to to apply the following properties of linear functions:

1. $1 - \eta_\beta(n) = \eta_{\frac{1}{\beta}}(1-n)$
2. $f(n) = \eta_{\beta_l}(...\eta_{\beta_1}(n)...)$ is an increasing, continuous function, which it increases from 0 to 1 when $n$ runs from 0 to 1.

**(c)**: A deterministic rule $\Phi \to \Psi$ is applicable to $\langle S, (a,b]\rangle$ iff $S \supseteq \Phi$. In this case we obtain the the new pair $\langle S', (a,b]\rangle$ where $S' = S \cup \Psi$.

By this case analysis, we have a way to apply the propagation rule to an arbitrary pair. To continue, we consider $\langle \tilde{S}, (a,b]\rangle$ to be a completion pair, in which no propagation rules can be applied to $\tilde{S}$ over $(a,b]$. We continue to check the existence of a clash in $\tilde{S}$ over $(a,b]$. If $\tilde{S}$ contains an unsatisfiable constraint, it is trivially true that $\tilde{S}$ is unsatisfiable over $(a,b]$. If $\tilde{S}$ contains a conjugated pair of constraints, then there are at most four kinds of clashing pairs according to Proposition 7:

**Case 1:** The pair $\langle w \geq \eta_{\beta_k}(...\eta_{\beta_1}(n)...)\rangle$ and $\langle w \leq \eta_{\alpha_l}(...\eta_{\alpha_1}(1-n)...)\rangle$ clashes iff $\eta_{\beta_k}(...\eta_{\beta_1}(n)...) > \eta_{\alpha_l}(...\eta_{\alpha_1}(1-n)...)$ or $\eta_{\frac{1}{\alpha_1}}(...\eta_{\frac{1}{\alpha_l}}(\eta_{\beta_k}(...\eta_{\beta_1}(n)...)...)) > 1-n$. It is easy to see that the equation $\eta_{\frac{1}{\alpha_1}}(...\eta_{\frac{1}{\alpha_l}}(\eta_{\beta_k}(...\eta_{\beta_1}(n)...)...)) = 1-n$ has only one solution $n_m \in (0,1]$. If $n_m \leq a$, i.e., the inequation holds for $n \in (a,b]$, then $\tilde{S}$ is unsatifiable for all $n \in (a,b]$. If $n_m > b$, i.e., the inequation does not hold for $n \in (a,b]$ then $\tilde{S}$ is satifiable for all $n \in (a,b]$. If $n_m \in (a,b]$, i.e., the inequation does not hold for $n \in (a, n_m]$ but holds for $n \in (n_m, b]$, then $\langle S, (a,b]\rangle$ is divided into $\langle S, (a, n_m]\rangle$ which is satisfiable over $(a, n_m]$ and $\langle S, (n_m, b]\rangle$ which is unsatisfiable over $(n_m, b]$.

**Case 2, 3 and 4:** The pairs $(\langle w \geq \eta_{\beta_k}(...\eta_{\beta_1}(n)...)\rangle, \langle w < \eta_{\alpha_l}(...\eta_{\alpha_1}(n)...)\rangle)$ and $(\langle w > \eta_{\beta_k}(...\eta_{\beta_1}(1-n)...)\rangle, \langle w \leq \eta_{\alpha_l}(...\eta_{\alpha_1}(1-n)...)\rangle)$ and $(\langle w > \eta_{\beta_k}(...\eta_{\beta_1}(1-n)...)\rangle, \langle w < \eta_{\alpha_l}(...\eta_{\alpha_1}(n)...)\rangle)$ can be treated similarly as in case 1.

It follows that eventually we obtain a set of completion sets with corresponding intervals. Let us start with the initial pair $\langle S, (0,1]\rangle$. We will check a propagation rule can be applied. After applying a rule, we get at least one and at most finitely many new pairs. For each new pair, we repeat this process until no rule can be applied any longer. This process will terminate because the size of $S$ is finite. Finally, we get a finite set of completion sets, which is checked for a clash. This proves the two first parts of Proposition 6. Part 3 follows immediately because if all completion sets are unsatisfiable then the initial set $S$ is unsatisfiable for all $n \in (0,1]$ and vice versa. That finishes our proof of Proposition 6. Furthermore, because the way used to solve the problem is similar to the case of the entailment problem, the decision procedure of subsumption problem has the same complexity.

# 8 Conclusion

In this paper we have introduced the fuzzy description logic $\mathcal{ALC}_{FLH}$, where concepts and roles are modified using linear hedges, and have solved the entailment as well as the subsumption problem. This extends previous work [1, 2], where modifiers were restricted to primitive concepts in order to solve the subsumption problem. We are currently in the process of building in trace rules in order to avoid an exponential explosion when applying the propagation rules for the quantifiers. We are also working on a prototypical implementation, which is the basis for running real-world examples.

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