Linguistic Logics with Hedges

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Abstract. Human knowledge is commonly expressed linguistically, i.e., truth of vague sentences is given in linguistic terms, and many hedges are often used simultaneously to state different levels of emphasis. In this paper, we propose an axiomatization of mathematical fuzzy logic with many hedges where each hedge does not have any dual one. Then, we present linguistic logics built based on the proposed axiomatization and the one in a previous work in order to make it easier to represent and reason with linguistically-expressed human knowledge.

1 Introduction

Extending logical systems of mathematical fuzzy logic (MFL) with hedges is axiomatized by Hájek [1], Vychodil [2], Esteva *et al.* [3], and Le *et al.* [4]. Hedges are called *truth-stressing* or *truth-depressing* if they, respectively, strengthen or weaken the meaning of the applied proposition. Intuitively, on a chain of truth values, the truth function of a truth-depressing (resp., truth-stressing) hedge (connective) is a superdiagonal (resp., subdiagonal) non-decreasing function preserving 0 and 1. In [1–3], logical systems of MFL are extended by a truth-stressing hedge and/or a truth-depressing one. However, in the real world, we often use many hedges, e.g., *very*, *rather*, and *slightly*, simultaneously to express different levels of emphasis. Moreover, human knowledge is commonly expressed linguistically, i.e., truth of vague sentences is given in linguistic terms. Therefore, it is possibly worth extending MFL with many truth-stressing and truth-depressing hedges and using a linguistic truth domain (LTD) in order to make it easier to represent and reason with linguistically-expressed human knowledge.

In this paper, first, we propose a new axiomatization for MFL with many hedges which is simpler, but more general than the one in [4] since in this axiomatization, hedges are not required to be pairwise dual. Then, we present linguistic logics built based on the two axiomatizations. The linguistic logics use an LTD having values such as *true*, *very true* and *very slightly false*. Thus, domains and codomains of the hedge functions are the LTD. Since Gödel logic (G) and Łukasiewicz logic (Ł) are two of the three basic t-norm based ones (the other is product logic (Π)) which have received increasing attention in the last fifteen years, we also define Gödel and Łukasiewicz operations on an LTD. Therefore, we can have particular linguistic logics based on G or Ł.

The remainder of the paper is organized as follows. Section 2 gives an overview of the notions and results of MFL. Section 3 presents LTDs, operations of G-algebras and MV-algebras, and truth functions of hedges. Section 4 proposes an axiomatization for many hedges where each hedge does not have any dual one. Section 5 presents linguistic logics built based on the proposed axiomatization and the one in [4]. Section 5 concludes the paper and outlines our future work.

2 Preliminaries on Mathematical Fuzzy Logic

Let L be a propositional logic in a language $\mathcal{L}$, a set of connectives with finite arity. A truth constant $\overline{r}$ is a special formula whose truth value under every evaluation is r. Formulae are built from variables and truth constants using connectives in $\mathcal{L}$. Each evaluation e of variables by truth values uniquely extends to an evaluation $e(\varphi)$ of all formulae φ using truth functions of connectives. A formula φ is called an *1-tautology* if $e(\varphi) = 1$ for all evaluations e. *Axioms* of the logic are taken from 1-tautology formulae. A *theory* is a set of formulae. An evaluation e is called a *model* of a theory T if $e(\varphi) = 1$, $\forall \varphi \in T$. A *proof* in T is a sequence $\varphi_1, \ldots, \varphi_n$ of formulae whose each member is either an axiom of the logic or a member of T or follows from some preceding members of the sequence using the deduction rule(s) of the logic. A formula φ is called *provable*, denoted $T \vdash_L \varphi$, if φ is the last member of a proof in T. If $T = \emptyset$, φ is said to be *provable* in the logic [5, 6]. L is a *Rasiowa-implicative logic* if there is a binary connective $\rightarrow$ satisfying reflexivity, weakening, and the following [6]:

(MP) $\varphi, \varphi \rightarrow \psi \vdash_L \psi$

(T) $\varphi \rightarrow \psi, \psi \rightarrow \chi \vdash_L \varphi \rightarrow \chi$

(sCng) $\varphi \rightarrow \psi, \psi \rightarrow \varphi \vdash_L c(\chi_1, \ldots \chi_i, \varphi, \ldots, \chi_n) \rightarrow c(\chi_1, \ldots \chi_i, \psi, \ldots, \chi_n)$,
for each n-ary $c \in \mathcal{L}$ and each $i < n$.

Every finitary Rasiowa-implicative logic L is algebraizable. Its equivalent algebraic semantics, a class of L-algebras, denoted $\mathbb{L}$, is a quasivariety. The algebraic semantics enjoys the following strong completeness [6].

Theorem 1. *For every set $\Gamma \cup \{\varphi\}$ of formulae, $\Gamma \vdash_L \varphi$ iff for every $\mathbf{A} \in \mathbb{L}$ and every $\mathbf{A}$-model e of Γ, we have $e(\varphi) = 1$.*

Each L-algebra $\mathbf{A}$ is endowed with a relation $\leq$ (called *preorder*) by setting, $\forall a, b \in A$, $a \leq b$ iff $a \Rightarrow b = 1$, where $\Rightarrow$ is the truth function of $\rightarrow$. If $\leq$ is a total order, $\mathbf{A}$ is called an L-*chain*. L is called a *semilinear* logic iff it is strongly complete w.r.t. the class of L-chains [6]. Most logical systems called *fuzzy logics* are a finitary Rasiowa-implicative semilinear logic. They belong to a large class of systems that are axiomatic expansions of MTL satisfying (sCng) for any new connective [6]. The systems are called *core fuzzy logics*. Well-known examples are basic logic (BL), G, Ł, Π [5], SBL, NM, MTL, and SMTL [6].

Definition 1. *[6] Let L be a core fuzzy logic and $\mathbb{K}$ a class of L-chains. L has the (finite) strong $\mathbb{K}$-completeness property, (F)S$\mathbb{K}$C, if for every (finite) set of formulae Γ and every formula φ, it holds that $\Gamma \vdash_L \varphi$ iff $e(\varphi) = 1$ for every L-algebra $\mathbf{A} \in \mathbb{K}$ and each $\mathbf{A}$-model e of Γ. L has the $\mathbb{K}$-completeness property, $\mathbb{K}$C, when the equivalence is true for $\Gamma = \emptyset$.*

When $\mathbb{K}$ is the class of all chains whose support is the unit interval $[0, 1]$ with the usual ordering, the (F)S$\mathbb{K}$C can be called the (finite) strong *standard* completeness, (F)SSC.

Theorem 2. *[6] Let L be a core fuzzy logic and $\mathbb{K}$ a class of L-chains. Then, (i) L has the S$\mathbb{K}$C iff every countable L-chain is embeddable into some member of $\mathbb{K}$, and (ii) if the language of L is finite, L has the FS$\mathbb{K}$C iff every countable L-chain is* partially *embeddable into some member of $\mathbb{K}$, i.e., for every finite partial of a countable L-chain, there is a one-to-one mapping preserving the operations into some member of $\mathbb{K}$.*

The language of BL [5, 6] consists of the primitive connectives $\&, \to$ and the truth constant $\overline{0}$. Further definable connectives are as follows: $\varphi \wedge \psi \equiv \varphi\&(\varphi \to \psi)$, $\varphi \vee \psi \equiv ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$, $\varphi \leftrightarrow \psi \equiv (\varphi \to \psi)\&(\psi \to \varphi)$, $\neg\varphi \equiv \varphi \to \overline{0}$, and $\overline{1} \equiv \neg\overline{0}$. Axioms of BL [6] are the following:

$$\begin{array}{ll}
(BL1) & (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \\
(BL4) & \varphi\&(\varphi \to \psi) \to \psi\&(\psi \to \varphi) \\
(BL5a) & (\varphi\&\psi \to \chi) \to (\varphi \to (\psi \to \chi)) \\
(BL5b) & (\varphi \to (\psi \to \chi)) \to (\varphi\&\psi \to \chi) \\
(BL6) & ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi) \\
(BL7) & \overline{0} \to \varphi
\end{array}$$

The only deduction rule of BL is modus ponens (MP).

The axiom system of G (resp., Ł) is an extension of that of BL by the axiom: (G) $\varphi \to \varphi\&\varphi$ (resp., (Ł) $\neg\neg\varphi \to \varphi$). G proves that $\varphi\&\psi$ is $\varphi \wedge \psi$. The following formulae are provable in MTL, BL, G, and Ł[6]:

$$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi), \tag{1}$$

$$(\varphi \to \psi) \vee (\psi \to \varphi). \tag{2}$$

Let $*, \Rightarrow, \cap, \cup, -$ denote truth functions of connectives $\&, \to, \wedge, \vee, \neg$, respectively. For BL, G and Ł, the truth functions $\cap$ and $\cup$ are min and max, respectively.

A *residuated lattice* is an algebra $\langle A, *, \Rightarrow, \cap, \cup, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ such that: (i) $\langle A, \cap, \cup, 0, 1\rangle$ is a lattice with the largest element 1 and least element 0 (w.r.t. the lattice ordering $\leq$); (ii) $\langle A, *, 1\rangle$ is a commutative semigroup with the unit element 1, i.e., * is commutative, associative, $1 * x = x$ for all x; (iii) $\Rightarrow$ is the *residuum* of $*$, i.e., for each $x, y, z \in A$ holds: $x * y \leq z$ iff $x \leq y \Rightarrow z$.

A residuated lattice $\langle A, *, \Rightarrow, \cap, \cup, 0, 1\rangle$ is a *BL-algebra* iff the following identities hold for all $x, y \in A$: (i) *Prelinear*: $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$; and (ii)

Divisible: $x \cap y = x * (x \Rightarrow y)$. BL-algebras satisfying $x * x = x$ are called *G-algebras*, while BL-algebras satisfying $- - x = x$ are called *MV-algebras*. The class of BL-algebras, G-algebras, and MV-algebras are the equivalent algebraic semantics of BL, G, and Ł, respectively.

3 Linguistic Truth Domains and Operations

3.1 Linguistic Truth Domains

Let *V, H, R, S, T*, and *F* stand for *Very, Highly, Rather, Slightly, True*, and *False*, respectively. In hedge algebra (HA) theory [7, 8], values of the linguistic variable *Truth*, e.g., *VT* and *VSF*, can be regarded as being generated from a set of primary terms $\mathcal{G} = \{F, T\}$ using hedges from a set $\mathcal{H} = \{V, S, \dots\}$ as unary operations. There exists a natural ordering among them, e.g., $ST < T$. Thus, a term domain of *Truth* is a partially ordered set (poset) and can be characterised by an HA $\underline{X} = (\mathcal{X}, \mathcal{G}, \mathcal{H}, \leq)$, where $\mathcal{X}$ is a term set, $\mathcal{G}$ is a set of primary terms, $\mathcal{H}$ is a set of hedges, and $\leq$ is a *semantic order relation* (SOR) on $\mathcal{X}$.

There are natural semantic properties of hedges and terms. Hedges either increase or decrease the meaning of terms, i.e., $\forall h \in \mathcal{H}, \forall x \in \mathcal{X}$, either $hx \geq x$ or $hx \leq x$. It is denoted by $h \geq k$ if a hedge h modifies terms more than or equal to another hedge k, i.e., $\forall x \in \mathcal{X}$, $hx \leq kx \leq x$ or $x \leq kx \leq hx$. Since $\mathcal{H}$ and $\mathcal{X}$ are disjoint, the same notation $\leq$ can be used for different order relations on $\mathcal{H}$ and $\mathcal{X}$ without confusion. A hedge has a semantic effect on others. If h strengthens the degree of modification of k, i.e., $\forall x \in \mathcal{X}$, $hkx \leq kx \leq x$ or $x \leq kx \leq hkx$, then h is *positive* w.r.t. k. If h weakens the degree of modification of k, i.e., $\forall x \in \mathcal{X}$, $kx \leq hkx \leq x$ or $x \leq hkx \leq kx$, then h is *negative* w.r.t. k. An important semantic property of hedges, called *semantic heredity*, is that hedges change the meaning of a term, but somewhat preserve its original meaning. Thus, if $hx \leq kx$, where $x \in \mathcal{X}$, then $\mathcal{H}(hx) \leq \mathcal{H}(kx)$, where $\mathcal{H}(u)$ denotes the set of all terms generated from u by means of hedges, i.e., $\mathcal{H}(u) = \{\sigma u | \sigma \in \mathcal{H}^*\}$, where $\mathcal{H}^*$ is the set of all strings of symbols in $\mathcal{H}$ including the empty one.

For *Truth*, the primary terms $F \leq T$ are denoted by c^- and c^+, respectively. $\mathcal{H}$ can be divided into two disjoint subsets $\mathcal{H}^+$ and $\mathcal{H}^-$ by $\mathcal{H}^+ = \{h | hc^+ > c^+\} = \{h | hc^- < c^-\}$ and $\mathcal{H}^- = \{h | hc^+ < c^+\} = \{h | hc^- > c^-\}$. For example, $\mathcal{H} = \{V, H, R, S\}$ is decomposed into $\mathcal{H}^+ = \{V, H\}$ and $\mathcal{H}^- = \{R, S\}$. Hedges in each of $\mathcal{H}^+$ and $\mathcal{H}^-$ may be comparable. So, $\mathcal{H}^+$ and $\mathcal{H}^-$ become posets. Let $I \notin \mathcal{H}$ be an artificial hedge, called the *identity*, defined by $\forall x \in \mathcal{X}$, $Ix = x$. I is the least element in each of $\mathcal{H}^+ \cup \{I\}$ and $\mathcal{H}^- \cup \{I\}$. An HA is said to be *linear* if both $\mathcal{H}^+$ and $\mathcal{H}^-$ are linearly ordered. The term domain $\mathcal{X}$ of a linear HA is also linearly ordered. In this paper, we restrict ourselves to linear HAs.

Example 1. The HA $\underline{X} = (\mathcal{X}, \{c^-, c^+\}, \mathcal{H} = \{V, H, R, S\}, \leq)$ is a linear HA since in $\mathcal{H}^+$, we have $H < V$, and in $\mathcal{H}^-$, we have $R < S$.

A *linguistic truth domain* (LTD) $\overline{X}$ taken from a linear HA $\underline{X} = (\mathcal{X}, \{c^-, c^+\}, \mathcal{H}, \leq$) is the linearly ordered set $\overline{X} = \mathcal{X} \cup \{0, W, 1\}$, where 0 (*AbsolutelyFalse*), W (the *middle truth value*), and 1 (*AbsolutelyTrue*) are the least, neutral and greatest elements of $\overline{X}$, respectively [9], and $\forall x \in \{0, W, 1\}$ and $\forall h \in \mathcal{H}$, $hx = x$.

An *extended order relation* $\leq_e$ on $\mathcal{H} \cup \{I\}$ is defined as follows [9]: $\forall h, k \in \mathcal{H} \cup \{I\}$, $h \leq_e k$ if: (i) $h \in \mathcal{H}^-, k \in \mathcal{H}^+$; or (ii) $h, k \in \mathcal{H}^+ \cup \{I\}$ and $h \leq k$; or (iii) $h, k \in \mathcal{H}^- \cup \{I\}$ and $h \geq k$. It is denoted by $h <_e k$ if $h \leq_e k$ and $h \neq k$. For the HA in Example 1, we have $S <_e R <_e I <_e H <_e V$.

3.2 Gödel Operations

Gödel t-norm, its residuum, and negation can be defined on an LTD $\overline{X}$ as:

$$x * y = \min(x, y), \quad x \Rightarrow y = \begin{cases} 1 \text{ if } y \geq x \\ y \text{ otherwise} \end{cases}, \quad -x = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ otherwise} \end{cases}.$$

3.3 Łukasiewicz Operations

To have well-defined operations, we consider only finitely many truth values. An *l-limit* HA, where l is a positive integer, is a linear HA in which all terms have a length of at most $l + 1$. An LTD taken from an l-limit HA is finite [9]. Łukasiewicz t-norm and its residuum can be defined on an LTD $\overline{X} = \{v_0, \ldots, v_n\}$ with $v_0 \leq v_1 \leq \cdots \leq v_n$ as follows:

$$v_i * v_j = \begin{cases} v_{i+j-n} & \text{if } i + j - n > 0 \\ v_0 & \text{otherwise} \end{cases}, \quad v_i \Rightarrow v_j = \begin{cases} v_n & \text{if } i \leq j \\ v_{n+j-i} & \text{otherwise} \end{cases}.$$

The negation is defined by: given $x = \sigma c$, where $\sigma \in \mathcal{H}^*$ and $c \in \{c^+, c^-\}$, we have $y = -x$, if $y = \sigma c'$ and $\{c, c'\} = \{c^+, c^-\}$, e.g., $Vc^+ = -Vc^-, Vc^- = -Vc^+$.

3.4 Truth Functions of Hedge Connectives

Definition 2. *[10] Let $\underline{X} = (\mathcal{X}, \{c^+, c^-\}, \mathcal{H}, \leq)$ be a linear HA.* Truth functions *$h^\bullet : \overline{X} \to \overline{X}$ of all hedges $h \in \mathcal{H} \cup \{I\}$ satisfy the following conditions:*

$$\text{for all } x \in \{0, W, 1\}, h^\bullet(x) = x \tag{3}$$

$$\text{for all } x \in \overline{X}, I^\bullet(x) = x \tag{4}$$

$$h^\bullet(hc^+) = c^+ \tag{5}$$

$$\text{if } x \geq y, h^\bullet(x) \geq h^\bullet(y) \tag{6}$$

$$\text{for all } k \in \mathcal{H} \cup \{I\} \text{ such that } h \leq_e k, h^\bullet(x) \geq k^\bullet(x) \tag{7}$$

By (7), given the hedges in Example 1, we have $\forall x \in \overline{X}$, $V^\bullet(x) \leq H^\bullet(x) \leq x \leq R^\bullet(x) \leq S^\bullet(x)$. This is in accordance with fuzzy-set-based interpretations of hedges [11], which satisfy the *semantic entailment* [12]: x is *veryA* $\Rightarrow$ x is *highlyA* $\Rightarrow$ x is A $\Rightarrow$ x is *ratherA* $\Rightarrow$ x is *slightlyA*, where A is a fuzzy predicate. Truth functions of hedges always exist [9].

Example 2. Consider a 2-limit HA $\underline{X} = (\mathcal{X}, \{c^+, c^-\}, \{V, H, R, S\}, \leq)$. Table 1 gives an example of truth functions of hedges, where the value of a truth function, in the first row, of a value x, in the first column, is in the corresponding cell, e.g., $V^\bullet(VRc^+) = VSc^+$. Truth values in the first column are in ascending order.

Table 1. Truth functions of hedge connectives

	$V^\bullet$	$H^\bullet$	$R^\bullet$	$S^\bullet$	
0	0	0	0	0	
kVc^-	VVc^-	VVc^-	kHc^-	c^-	[a]
kHc^-	VVc^-	kVc^-	c^-	kRc^-	[a]
c^-	Vc^-	Hc^-	Rc^-	Sc^-	
VRc^-	VHc^-	RHc^-	SSc^-	VSc^-	
HRc^-	HHc^-	SHc^-	RSc^-	VSc^-	
Rc^-	Hc^-	c^-	Sc^-	VSc^-	
RRc^-	RHc^-	VRc^-	HSc^-	VSc^-	
SRc^-	SHc^-	VRc^-	VSc^-	VSc^-	
SSc^-	SHc^-	VRc^-	VSc^-	VSc^-	
RSc^-	SHc^-	HRc^-	VSc^-	VSc^-	
Sc^-	c^-	Rc^-	VSc^-	VSc^-	
HSc^-	VRc^-	RRc^-	VSc^-	VSc^-	
VSc^-	RRc^-	SRc^-	VSc^-	VSc^-	
W	W	W	W	W	
VSc^+	VSc^+	VSc^+	SRc^+	RRc^+	
HSc^+	VSc^+	VSc^+	RRc^+	VRc^+	
Sc^+	VSc^+	VSc^+	Rc^+	c^+	
RSc^+	VSc^+	VSc^+	HRc^+	SHc^+	
SSc^+	VSc^+	VSc^+	VRc^+	SHc^+	
SRc^+	VSc^+	VSc^+	VRc^+	SHc^+	
RRc^+	VSc^+	HSc^+	VRc^+	RHc^+	
Rc^+	VSc^+	Sc^+	c^+	Hc^+	
HRc^+	VSc^+	RSc^+	SHc^+	HHc^+	
VRc^+	VSc^+	SSc^+	RHc^+	VHc^+	
c^+	Sc^+	Rc^+	Hc^+	Vc^+	
kHc^+	kRc^+	c^+	kVc^+	VVc^+	[a]
kVc^+	c^+	kHc^+	VVc^+	VVc^+	[a]
1	1	1	1	1	

[a] k is any of the hedges, including the identity I.

4 An Axiomatization for Many Hedges

A hedge may modify truth more than another [11, 7, 8]. For example, S (resp., V) modify truth more than R (resp., H) since $ST < RT < T$ (resp., $T < HT < VT$). To ease the presentation, let s_0, d_0 denote the *identity* connective, i.e., for all φ, $\varphi \equiv s_0\varphi \equiv d_0\varphi$, and their truth functions $s_0^\bullet$ and $d_0^\bullet$ are the identity.

Definition 3. *Let L be a core fuzzy logic. A logic $L_{s,d}^{p,q}$, where p, q are positive integers, is an expansion of L with new unary connectives $s_1, ..., s_p$ (for truth-stressers) and $d_1, ..., d_q$ (for truth-depressers) by the following additional axioms, for $i = 1, ..., p$ and $j = 1, ..., q$:*

$$(S_i) \quad s_i\varphi \to s_{i-1}\varphi \qquad (S_{p+1}) \quad s_p\overline{1} \qquad (D_j) \quad d_{j-1}\varphi \to d_j\varphi \qquad (D_{q+1}) \quad \neg d_q\overline{0}$$

and the following additional deduction rule:

(DR_h) *from* $(\varphi \to \psi) \vee \chi$ *infer* $(h\varphi \to h\psi) \vee \chi$, *for each* $h \in \{s_1, ..., s_p, d_1, ..., d_q\}$.

Axiom (S_i) (resp., axiom (D_j)) expresses that s_i (resp., d_j) modifies truth more than s_{i-1} (resp., d_{j-1}), for $i = 2, ..., p$ (resp., $j = 2, ..., q$).

Lemma 1. *The following deductions are valid, for $i = 1, ..., p$ and $j = 1, ..., q$:*

$$\vdash_{L_{s,d}^{p,q}} s_i\varphi \to \varphi \tag{8}$$

$$\vdash_{L_{s,d}^{p,q}} \varphi \to d_j\varphi \tag{9}$$

$$\vdash_{L_{s,d}^{p,q}} \neg s_i\overline{0} \tag{10}$$

$$\vdash_{L^{p,q}_{s,d}} s_i\overline{1} \tag{11}$$

$$\vdash_{L^{p,q}_{s,d}} d_j\overline{1} \tag{12}$$

$$\vdash_{L^{p,q}_{s,d}} \neg d_j\overline{0} \tag{13}$$

$$\varphi \to \psi \vdash_{L^{p,q}_{s,d}} s_i\varphi \to s_i\psi \tag{14}$$

$$\psi \vdash_{L^{p,q}_{s,d}} s_i\psi \tag{15}$$

$$\varphi \to \psi \vdash_{L^{p,q}_{s,d}} d_j\varphi \to d_j\psi \tag{16}$$

$$d_j\varphi, \varphi \to \psi \vdash_{L^{p,q}_{s,d}} d_j\psi \tag{17}$$

$$s_i\varphi, \varphi \to \psi \vdash_{L^{p,q}_{s,d}} s_i\psi \tag{18}$$

$$s_i\varphi, s_i(\varphi \to \psi) \vdash_{L^{p,q}_{s,d}} s_i\psi \tag{19}$$

Proof. (8) follows from (S_i),...,(S_1), and (T).
(9) follows from (D_1),...,(D_j), and (T).
(10) follows immediately from (8) taking $\varphi = \overline{0}$.
(11) $\vdash_{L^{p,q}_{s,d}} s_p\varphi \to s_i\varphi$ (for $i < p$, by $(S_p), ..., (S_{i+1})$ and (T)),
$\vdash_{L^{p,q}_{s,d}} s_p\overline{1} \to s_i\overline{1}$ (by taking $\varphi = \overline{1}$), $\vdash_{L^{p,q}_{s,d}} s_i\overline{1}$ (by (S_{p+1}) and (MP)).
(12) follows immediately from (9) taking $\varphi = \overline{1}$.
(13) $\vdash_{L^{p,q}_{s,d}} d_j\varphi \to d_q\varphi$ (for $j < q$, by $(D_{j+1}), ..., (D_q)$, and (T)),
$\vdash_{L^{p,q}_{s,d}} \neg d_q\varphi \to \neg d_j\varphi$ (by (1) and (MP)),
$\vdash_{L^{p,q}_{s,d}} \neg d_q\overline{0} \to \neg d_j\overline{0}$ (by taking $\varphi = \overline{0}$), $\vdash_{L^{p,q}_{s,d}} \neg d_j\overline{0}$ (by (D_{q+1}) and (MP)).
(14) follows directly from Rule (DR$_h$) taking $\chi = \overline{0}$, $h = s_i$.
(15) follows from (14) taking $\varphi = \overline{1}$ and using (11).
(16) follows directly from Rule (DR$_h$) taking $\chi = \overline{0}$, $h = d_j$.
(17) follows immediately from (16) and (MP).
(18) follows immediately from (14) and (MP).
(19) follows from (8), (MP) and (18).

Properties (10)-(13) imply that truth functions of s_i and d_j preserve 0 and 1. Property (15) is the necessitation deduction rule of Hájek's and Vychodil's axiomatizations [1, 2]. Properties (17) and (18) are a stronger version of modus ponens: if φ implies ψ, then *very* (resp., *slightly*) φ implies *very* (resp., *slightly*) ψ. Property (19) is a deduction version of the K-like axiom used in the previous axiomatizations of logics with hedges [1, 2]. Since (14) and (16) express that Property (sCng) is satisfied for s_i and d_j, $L^{p,q}_{s,d}$ is a finitary Rasiowa-implicative logic, and its equivalent algebraic semantics is the class of $L^{p,q}_{s,d}$-algebras.

Definition 4. *An algebra* $\boldsymbol{A}$=$\langle A, *, \Rightarrow, \cap, \cup, 0, 1, s^\bullet_1, ..., s^\bullet_p, d^\bullet_1, ..., d^\bullet_q\rangle$ *of type* $\langle 2, 2, 2, 2, 0, 0, 1, ..., 1\rangle$ *is an* $L^{p,q}_{s,d}$*-algebra if it is an L-algebra expanded by unary operators* $s^\bullet_i, d^\bullet_j : A \to A$ *that satisfy, for all* $x, y, z \in A$, $i = \overline{1,p}$ *and* $j = \overline{1,q}$,

$$s^\bullet_i(x) \le s^\bullet_{i-1}(x) \tag{20}$$

$$s^\bullet_p(1) = 1 \tag{21}$$

$$d^\bullet_j(x) \ge d^\bullet_{j-1}(x) \tag{22}$$

$$d_q^\bullet(0) = 0 \tag{23}$$

$$\textit{if } (x \Rightarrow y) \cup z = 1 \textit{ then } (s_i^\bullet(x) \Rightarrow s_i^\bullet(y)) \cup z = 1 \tag{24}$$

$$\textit{if } (x \Rightarrow y) \cup z = 1 \textit{ then } (d_j^\bullet(x) \Rightarrow d_j^\bullet(y)) \cup z = 1 \tag{25}$$

where $s_i^\bullet$ *and* $d_j^\bullet$ *are truth functions of connectives* s_i *and* d_j*, for all* $i = 1, ..., p$ *and* $j = 1, ..., q$*, respectively.*

By (20), $s_i^\bullet(x) \leq x$, i.e., $s_i^\bullet$ is subdiagonal, for all $i = \overline{1,p}$. By (22), $d_j^\bullet$ is superdiagonal, for all $j = \overline{1,q}$. In a chain of truth values, the quasiequations (24) and (25) turn out to be equivalently expressed by: if $x \Rightarrow y = 1$, then $s_i^\bullet(x) \Rightarrow s_i^\bullet(y) = 1$ and $d_j^\bullet(x) \Rightarrow d_j^\bullet(y) = 1$, respectively, i.e., $s_i^\bullet$ and $d_j^\bullet$ are non-decreasing.

If $\langle A, *, \Rightarrow, \cap, \cup, 0, 1\rangle$ is an L-chain, and $s_i^\bullet$ and $d_j^\bullet$ satisfy (20)-(25), the expanded structure $\left\langle A, *, \Rightarrow, \cap, \cup, 0, 1, s_1^\bullet, ..., s_p^\bullet, d_1^\bullet, ..., d_q^\bullet\right\rangle$ is an $\mathrm{L}_{s,d}^{p,q}$-chain.

Theorem 3. *Let L be a core fuzzy logic,* $\mathbb{K}$ *a class of L-chains, and* $\mathbb{K}_{s,d}^{p,q}$ *the class of the* $L_{s,d}^{p,q}$*-chains whose* $s_1, ..., s_p, d_1, ..., d_q$*-free reducts are in* $\mathbb{K}$*. Then: (i)* $L_{s,d}^{p,q}$ *is a conservative expansion of L; (ii)* $L_{s,d}^{p,q}$ *is strongly complete w.r.t. the class of all* $L_{s,d}^{p,q}$*-chains, i.e.,* $L_{s,d}^{p,q}$ *is semilinear; (iii) L has the FSSC, FS*$\mathbb{K}$*C, SSC, and S*$\mathbb{K}$*C iff* $L_{s,d}^{p,q}$ *has the FSSC, FS*$\mathbb{K}$*C, SSC, and S*$\mathbb{K}$*C, respectively.*

Proof. (i) Let $\mathcal{L}$ be the language of L. We show that, for every set $\Gamma \cup \{\varphi\}$ of $\mathcal{L}$-formulae, $\Gamma \vdash_{L_{s,d}^{p,q}} \varphi$ iff $\Gamma \vdash_L \varphi$. Obviously, if $\Gamma \vdash_L \varphi$ then $\Gamma \vdash_{L_{s,d}^{p,q}} \varphi$. If $\Gamma \nvdash_L \varphi$, there is an L-chain **A** and an **A**-evaluation e such that e is **A**-model of Γ and $e(\varphi) \neq 1$. **A** can be expanded to an $\mathrm{L}_{s,d}^{p,q}$-chain **A'** by defining $s_i(1) = 1$; $\forall a \in A \setminus \{1\}$, $s_i(a) = 0$; $d_j(0) = 0$; and $\forall a \in A \setminus \{0\}$, $d_j(a) = 1$, for all $i = \overline{1,p}$ and $j = \overline{1,q}$. Thus, in the expanded language, we have $\Gamma \nvdash_{L_{s,d}^{p,q}} \varphi$.
(ii) Since $\vee$ remains a disjunction in $\mathrm{L}_{s,d}^{p,q}$ and (2) is valid in L, $\mathrm{L}_{s,d}^{p,q}$ is semilinear.
(iii) We prove for the case of the SSC, and the others can be done analogously. Since $\mathrm{L}_{s,d}^{p,q}$ is a conservative expansion of L, if $\mathrm{L}_{s,d}^{p,q}$ has the SSC, so does L. Assume that L has the SSC. We show that any countable $\mathrm{L}_{s,d}^{p,q}$-chain **A** can be embedded into a standard $\mathrm{L}_{s,d}^{p,q}$-chain. By Theorem 2, the $s_1, ..., s_p, d_1, ..., d_q$-free reduct of **A** can be embedded into a standard L-chain $\mathbf{B} = \langle[0, 1], *, \Rightarrow, \cap, \cup, 0, 1\rangle$ by a mapping f. Since **A** is countable, for each $1 \leq k \leq p$, we may arrange all points $\{\langle f(x), f(s_k(x))\rangle | x \in A\}$ into a sequence $\{\langle f(x_i), f(s_k(x_i))\rangle | x_i \in A, i = 1, 2, \ldots\}$, where $0 = x_1 < x_2 < \ldots$ and $\lim_{i\to\infty} x_i = 1$. Let $s'_k : [0, 1] \to [0, 1]$ be the piecewise linear function connecting neighboured points from $\{\langle f(x_i), f(s_k(x_i))\rangle\}$. Similarly, for each $1 \leq l \leq q$, let d'_l be the piecewise linear function connecting neighboured points from $\{\langle f(x_i), f(d_l(x_i))\rangle\}$. It can be shown that all s'_k and d'_l satisfy (20)-(25). Hence, **B** expanded by all s'_k and d'_l is a standard $\mathrm{L}_{s,d}^{p,q}$-chain into which **A** is embedded.

5 Linguistic Logics with Hedges

5.1 Linguistic Logic with Many Hedges

Let L be a core fuzzy logic. Given a linear HA, we can build a linguistic logic with many hedges based on L and the HA. For instance, given the HA in Example

1, $\underline{X} = (\mathcal{X}, \{c^-, c^+\}, \mathcal{H} = \{V, H, R, S\}, \leq)$, a linguistic logic, denoted L^{lh}, is an expansion of L with new unary connectives V, H, R, S by the following axioms:

$$(S_1^{lh}) \quad H\varphi \to \varphi \qquad (S_2^{lh}) \quad V\varphi \to H\varphi \qquad (S_3^{lh}) \quad V\overline{1}$$
$$(D_1^{lh}) \quad \varphi \to R\varphi \qquad (D_2^{lh}) \quad R\varphi \to S\varphi \qquad (D_3^{lh}) \quad \neg S\overline{0}$$

and the following additional deduction rule:

$$(DR^{lh}) \text{ from } (\varphi \to \psi) \vee \chi \text{ infer } (h\varphi \to h\psi) \vee \chi, \text{ for each } h \in \{V, H, R, S\}.$$

The equivalent algebraic semantics of L^{lh} is the class of L^{lh}-algebras, denoted $\mathbb{L}^{lh}$. L^{lh}-algebras utilize a linear linguistic domain $\overline{X}$ taken from the HA.

An *L^{lh}-algebra* is an L-algebra expanded by unary non-decreasing operators $V^\bullet, H^\bullet, R^\bullet, S^\bullet : \overline{X} \to \overline{X}$ satisfying, for all $x \in \overline{X}$,

$$H^\bullet(x) \leq x, \qquad V^\bullet(x) \leq H^\bullet(x), \qquad V^\bullet(1) = 1,$$
$$R^\bullet(x) \geq x, \qquad S^\bullet(x) \geq R^\bullet(x), \qquad S^\bullet(0) = 0.$$

Theorem 4 (Strong Completeness). *For every set $\Gamma \cup \{\varphi\}$ of formulae, $\Gamma \vdash_{L^{lh}} \varphi$ iff for every $\mathbf{A} \in \mathbb{L}^{lh}$ and every $\mathbf{A}$-model e of Γ, $e(\varphi) = 1$.*

In particular, given the Gödel and Łukasiewicz operations respectively defined in Subsections 3.2 and 3.3 and truth functions of hedges in Example 2, we can have linguistic logics based on G or Ł with the well-defined operators.

5.2 Mathematical Fuzzy logic with Many Dual Hedges

It can be observed that each hedge can have a dual one, e.g., *slightly* and *rather* can be seen as a dual hedge of *very* and *highly*, respectively. Thus, there might be axioms expressing dual relations of hedges in addition to axioms expressing their comparative truth modification strength.

Definition 5. *[4] Let L be a core fuzzy logic. A logic $L^{2n}_{s,d}$, where n is a positive integer, is an expansion of L with new unary connectives $s_1, ..., s_n$ (for truth-stressers) and $d_1, ..., d_n$ (for truth-depressers) by the following additional axioms, for $i = 1, ..., n$:*

$$(S_i^{dh})\ s_i\varphi \to s_{i-1}\varphi \quad (S_{n+1}^{dh})\ s_n\overline{1} \quad (D_i^{dh})\ d_{i-1}\varphi \to d_i\varphi \quad (SD_i^{dh})\ d_i\varphi \to \neg s_i \neg\varphi$$

and the following additional deduction rule:

$$(DR^{dh}) \text{ from } (\varphi \to \psi) \vee \chi \text{ infer } (h\varphi \to h\psi) \vee \chi, \text{ for } h \in \{s_1, ..., s_n, d_1, ..., d_n\}.$$

The logic $\mathrm{L}^{2n}_{s,d}$ is L expanded by $2n$ hedges, where hedges are divided into pairs of dual ones. Axiom (SD_i) expresses the dual relation between hedges s_i and d_i and coincides with Axiom (ST2) in Vychodil's axiomatization. For the case of *very*, *slightly*, and $\varphi =$ *young*, it means "*slightly young* implies not *very old*".

$L^{2n}_{s,d}$ is also a finitary Rasiowa-implicative logic, and its equivalent algebraic semantics is the class of $L^{2n}_{s,d}$-algebras.

Definition 6. *[4] An algebra* $\boldsymbol{A}$=$\langle A, *, \Rightarrow, \cap, \cup, 0, 1,\ s_1^\bullet, ..., s_n^\bullet, d_1^\bullet, ..., d_n^\bullet\rangle$ *of type* $\langle 2, 2, 2, 2, 0, 0, 1, ..., 1\rangle$ *is an* $L_{s,d}^{2n}$*-algebra if it is an L-algebra expanded by unary operators* $s_i^\bullet, d_i^\bullet : A \to A$ *that satisfy, for all* $x, y, z \in A$ *and* $i = 1, ..., n$,

$$\begin{array}{cc} s_i^\bullet(x) \leq s_{i-1}^\bullet(x), & s_n^\bullet(1) = 1, \\ d_i^\bullet(x) \geq d_{i-1}^\bullet(x), & d_i^\bullet(x) \leq -s_i^\bullet(-x), \end{array}$$
$$\text{if } (x \Rightarrow y) \cup z = 1 \text{ then } (s_i^\bullet(x) \Rightarrow s_i^\bullet(y)) \cup z = 1,$$
$$\text{if } (x \Rightarrow y) \cup z = 1 \text{ then } (d_i^\bullet(x) \Rightarrow d_i^\bullet(y)) \cup z = 1.$$

where $s_i^\bullet$ *and* $d_i^\bullet$ *are truth functions of connectives* s_i *and* d_i*, respectively.*

Theorem 5. *[4] Let L be a core fuzzy logic,* $\mathbb{K}$ *a class of L-chains, and* $\mathbb{K}_{s,d}^{2n}$ *the class of the* $L_{s,d}^{2n}$*-chains whose* $s_1, ..., s_n, d_1, ..., d_n$*-free reducts are in* $\mathbb{K}$*. Then: (i)* $L_{s,d}^{2n}$ *is a conservative expansion of L; (ii)* $L_{s,d}^{2n}$ *is strongly complete w.r.t. the class of all* $L_{s,d}^{2n}$*-chains, i.e.,* $L_{s,d}^{2n}$ *is semilinear; (iii) L has the FSSC, FS*$\mathbb{K}$*C, SSC, and S*$\mathbb{K}$*C iff* $L_{s,d}^{2n}$ *has the the FSSC, FS*$\mathbb{K}$*C, SSC, and S*$\mathbb{K}$*C, respectively.*

It can be seen that in a case when there is one truth-stressing (resp., truth-depressing) hedge without a dual one, we just add the axioms expressing its relations to the existing truth-stressing (resp., truth-depressing) hedges according to their comparative truth modification strength.

5.3 Linguistic Logic with Many Dual Hedges

Let L be a core fuzzy logic. Given a linear HA, we can build a linguistic logic with many dual hedges based on L. For example, given the HA in Example 1, $\underline{X} = (\mathcal{X}, \{c^-, c^+\}, \mathcal{H} = \{V,\ H, R, S\}, \leq)$, a linguistic logic, denoted L^{ldh}, is an expansion of L with new unary connectives V, H, R, S by the following axioms:

$(S_1^{ldh})\ H\varphi \to \varphi$ $\quad(S_2^{ldh})\ V\varphi \to H\varphi$ $\quad(S_3^{ldh})\ V\overline{1}$ $\quad(D_1^{ldh})\ \varphi \to R\varphi$
$(D_2^{ldh})\ R\varphi \to S\varphi$ $\quad(SD_1^{ldh})\ R\varphi \to \neg H\neg\varphi$ $\quad(SD_2^{ldh})\ S\varphi \to \neg V\neg\varphi$

and the following additional deduction rule:

(DR^{ldh}) from $(\varphi \to \psi) \vee \chi$ infer $(h\varphi \to h\psi) \vee \chi$, for each $h \in \{V, H, R, S\}$.

The equivalent algebraic semantics of L^{ldh} is the class of L^{ldh}-algebras, denoted $\mathbb{L}^{ldh}$. L^{ldh}-algebras utilize a linear linguistic domain $\overline{X}$ taken from the HA.

An L^{ldh}*-algebra* is an L-algebra expanded by unary non-decreasing operators $V^\bullet, H^\bullet, R^\bullet, S^\bullet : \overline{X} \to \overline{X}$ satisfying, for all $x \in \overline{X}$,

$$\begin{array}{llll} H^\bullet(x) \leq x, & V^\bullet(x) \leq H^\bullet(x), & V^\bullet(1) = 1, & R^\bullet(x) \geq x, \\ S^\bullet(x) \geq R^\bullet(x), & S^\bullet(x) \leq -V^\bullet(-x), & R^\bullet(x) \leq -H^\bullet(-x). & \end{array}$$

Taking into account that $\forall x \in \overline{X},\ R^\bullet(x) = -H^\bullet(-x),\ S^\bullet(x) = -V^\bullet(-x)$, we can see that truth functions of hedges in Example 2 satisfy the above conditions.

Theorem 6 (Strong Completeness). *For every set* $\Gamma \cup \{\varphi\}$ *of formulae,* $\Gamma \vdash_{L^{ldh}} \varphi$ *iff for every* $\mathbf{A} \in \mathbb{L}^{ldh}$ *and every* $\mathbf{A}$*-model* e *of* Γ, $e(\varphi) = 1$.

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7 Conclusion and Future Work

This paper proposes an axiomatization for mathematical fuzzy logic with many hedges, where each hedge does not have any dual one. Then, based on the proposed axiomatization and the one in a previous work, it proposes linguistic logics for representing and reasoning with linguistically-expressed human knowledge, where truth of vague sentences is given in linguistic terms, and many hedges are often used simultaneously to express different levels of emphasis. For future work, we will study first-order fuzzy logics with hedges.

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