

Extending the HITS Algorithm over Dioids

Francisco J. Valverde-Albacete^{1*} and Carmen Peláez-Moreno²

¹ Departamento de Lenguajes y Sistemas Informáticos
Universidad Nacional de Educación a Distancia, c/ Juan del Rosal, 16. 28040 Madrid,
Spain fva@lsi.uned.es, franciscojvalverde@gmail.com

² Departamento de Teoría de la Señal y de las Comunicaciones
Universidad Carlos III de Madrid, 28911 Leganés, Spain
carmen@tsc.uc3m.es

Abstract. In this paper we introduce extensions of Kleinberg’s Hubs & Authorities (HITS) algorithm to calculate the influence of nodes in a network whose adjacency matrix takes values over dioids, zerosumfree semirings with a natural order.

We relate these extensions to both the Singular Value Problem and the Eigen Problem of matrices in these semirings. We show the original HITS algorithm to be a particular instance of the generic construction, but also the advantages of working in idempotent semifields.

We also make some connections with extended \mathcal{K} -Formal Concept Analysis, where the particular kind of dioid is an idempotent semifield, and conclude that the type of knowledge extracted from a matrix by one procedure and the other are different.

1 Introduction

The Singular Value Decomposition (SVD) [1] is a set of results about the decomposition of a rectangular matrix with applications in data processing, signal theory, machine learning and computer science at large.

In particular, it is a crucial tool in the dual-projection approach to the analysis of bipartite networks—also known as 2-mode, affiliation or membership networks [2]. The dual projection approach postulates we can provide measures of centrality, core-vs.-periphery and structural equivalence for each of the projection networks with limited loss of global information, in terms of the left and right singular vectors.

On the other hand, one of the first well-known approaches to link analysis on the Web, the Hubs and Authorities algorithm or Hyperlink-Induced Topic Search (HITS) [3, 4] also stems from the SVD. HITS was designed to solve the problem of ranking the nodes of a dynamic, directed 1-mode network of nodes obtained from a query against a search engine. It postulates the existence of two qualities in nodes: their “authoritativeness”—their quality of being authorities

* CPM has been supported by the Spanish Government-Comisión Interministerial de Ciencia y Tecnología project TEC2011-26807.

with respect to a pervasive topic in the nodes—and their “hubness”—their quality of being good pointers to authorities—and solves this in terms of the left and right eigenvalues, again.

1.1 The original HITS formulation

Consider a directed graph $D = (V, E)$, where $V = \{v_i\}_{i=1}^n$ is a set of nodes and $E \subseteq V \times V$ is a set of edges. It can alternatively be defined by an adjacency matrix A_D with 0–1 weights in its edges, $(A_D)_{ij} = 1$ if $(v_i, v_j) \in E$ and $(A_D)_{ij} = 0$ otherwise. Note that this procedure can be reversed to obtain the digraph *induced* by a matrix A as $D_A = (\bar{n}, E_A)$ where $\bar{n} = \{1, \dots, n\}$ and $E_A = \{(i, j) \mid A_{ij} = 1\}$.

Also note that the vertices and edges of the digraph model the nodes and links in any social network. Precisely based in this model, HITS finds an authority score $a(v_i)$ and a hub score $h(v_i)$ for each node aggregated as vectors, based in the following iterative procedure:

- Start with initial vector estimates $h^{<0>}$ and $a^{<0>}$ of the hub and authority scores.
- Upgrade the scores with:

$$h \leftarrow Aa \qquad a \leftarrow A^T h \qquad (1)$$

so that in general, for $k \geq 1$:

$$\begin{aligned} h^{<k>} &= (AA^T)^k h^{<0>} & a^{<k>} &= (A^T A)^k a^{<0>} \\ h^{<k>} &= (AA^T)^{k-1} Aa^{<0>} & a^{<k>} &= (A^T A)^{k-1} A^T h^{<0>} \end{aligned} \qquad (2)$$

- Since matrix A is non-negative, in general the sequences $\{h^{<k>}\}_k$ and $\{a^{<k>}\}_k$ would diverge, so the next step is to prove that the limits:

$$\lim_{k \rightarrow \infty} \frac{h^{<k>}}{c^k} = h^{<*>} \qquad \lim_{k \rightarrow \infty} \frac{a^{<k>}}{d^k} = a^{<*>} \qquad (3)$$

exist, in which case they are eigenvectors of their respective matrices for seemingly arbitrary c and d ,

$$(AA^T)h^{<*>} = ch^{<*>} \qquad (A^T A)a^{<*>} = da^{<*>} \quad . \qquad (4)$$

- As long as the initial estimates do not inhabit the null space of these matrices—making them orthogonal to $h^{<*>}$ and $a^{<*>}$, respectively—the iterative process will end up finding the principal eigenvectors. The proof of this fact entails that the initial estimates $h^{<0>}$ and $a^{<0>}$ should be non-negative.

1.2 The generalisation over complete dioids

In this paper we generalise the setting above slightly to close the gap with the FCA problem:

1. First, as suggested by the form of (2), we consider directed weighted *bipartite* graphs between two sets G and M , because we want to study the quality of being a hub and being an authority separately.
2. Second, we consider edge weights $R \in D^{G \times M}$ in a *naturally ordered semiring* or *dioid* \mathcal{D} (cfr. §2.1). This implies passing from directed graphs to *networks*, that is weighted directed graphs. Then \mathcal{D} -formal contexts, denoted as (G, M, R) , are a natural encoding for this type of bipartite graphs.

Note that the original HITS setting can be recovered by using $G = M = V$ and $A = R$ and working in the $\mathcal{S} \cong \mathbb{R}_0^+$ semifield with $R_{ij} = 1$ if $(v_i, v_j) \in E$ and $R_{ij} = 0$ otherwise. Similarly, the original dual-projection approach deals only with the case where R is actually binary but is considered to be embedded in $\mathcal{S} \cong \mathbb{R}_0^+$.

Let $\mathcal{K} = \langle K, \oplus, \otimes, \perp \rangle$ be a dioid. Then the space of hub scores is $\mathcal{X} = K^G$ and that of authorities is $\mathcal{Y} = K^M$ and they get mutually transformed by the actions of two linear functions:

$$\begin{array}{ll} R^\top \otimes \cdot : K^G \mapsto K^M & R \otimes \cdot : K^M \mapsto K^G \\ x \mapsto R^\top \otimes x & y \mapsto R \otimes y \end{array} \quad (5)$$

But we want to emphasize their mutual dependence, so after [5] we write (1) in matrix form

$$\begin{bmatrix} h \\ a \end{bmatrix} \leftarrow \begin{bmatrix} \cdot & R \\ R^\top & \cdot \end{bmatrix} \otimes \begin{bmatrix} h \\ a \end{bmatrix}$$

The previous Section suggests that the solution to the HITS problem entails the solution of the following eigenproblem in the variable $z^\top = [x^\top y^\top]^\top$,

$$B \otimes z = z \otimes \sigma \Leftrightarrow \begin{bmatrix} \cdot & R \\ R^\top & \cdot \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \otimes \sigma \quad (6)$$

where we have substituted zero matrices for dots, to lessen the visual clutter. The solutions to this problem are of the form $w^\top = [h^\top a^\top]^\top$ that is, pairs of hub and authority vectors. To see this, we expand the system (6) into two equations called by Lanczos the “shifted eigenvalue problem” [5],

$$R \otimes a = h \otimes \sigma \quad R^\top \otimes h = a \otimes \sigma \quad (7)$$

Equation (7) is the proof that HITS is actually trying to solve the *singular value-singular vector problem* [1, 5], where h has the role of a left singular vector and a that of a right singular vector.

To relate this problem back to the original one, we premultiply (7) by R^T , respectively R , to obtain

$$(R \otimes R^T) \otimes h = h \otimes \sigma^{\otimes 2} \quad (R^T \otimes R) \otimes a = a \otimes \sigma^{\otimes 2} \quad (8)$$

This proves that in order to solve the HITS problem in a dioid we have to solve the Singular Value Problem (7) which amounts to solving both decoupled Eigenvalue Problems (8).

In this paper we develop a similar tool for bipartite networks with edges with weights in a dioid, but we develop it as if it were an instance of the better understood, HITS problem.

To develop our program we first introduce in Section 2.1 some definitions and notation about semirings, in general, and about dioids and semifields, in particular. In Section 2.2 we introduce the eigenproblem over dioids as a step to solving the singular value problem in dioids, and in Section 2.2 a very general technique to do so. Section 3 presents the weight of our results and we conclude with a short Example and a Discussion.

2 Theory and Methods

2.1 Semiring and semimodules over semirings

A *semiring* is an algebra $\mathcal{S} = \langle S, \oplus, \otimes, \epsilon, e \rangle$ whose additive structure, $\langle S, \oplus, \epsilon \rangle$, is a commutative monoid and whose multiplicative structure, $\langle S \setminus \{\epsilon\}, \otimes, e \rangle$, is a monoid with multiplication distributing over addition from right and left and an additive neutral element absorbing for \otimes , i.e. $\forall a \in S, \epsilon \otimes a = \epsilon$. A semiring is *commutative* if its multiplication is commutative.

A semiring is *zerosumfree* if it does not have non-null additive factors of zero, $a \oplus b = \epsilon \Rightarrow a = \epsilon$ and $b = \epsilon, \forall a, b \in S$. Likewise, it is *entire* if $a \otimes b = \epsilon \Rightarrow a = \epsilon$ or $b = \epsilon, \forall a, b \in S$. Entire zerosumfree semirings are called sometimes *information algebras*, and have abundant applications [6]. Their importance stems from the fact that they model *positive quantities*.

Importantly, every commutative semiring accepts a canonical preorder, as $a \leq b$ if and only if there exists $c \in D$ with $a \oplus c = b$ which is *compatible with addition*. A *dioid* is a commutative semiring \mathcal{D} where this relation is actually an order. Dioids are zerosumfree. A dioid that is also entire is a *positive dioid*.

An *idempotent semiring* is a dioid whose addition is idempotent, and a *selective semiring* one where the arguments attaining the value of the additive operation can be identified.

A *complete semiring* \mathcal{S} [7] is a semiring where for every (possibly infinite) family of elements $\{a_i\}_{i \in I} \subseteq S$ we can define an element $\sum_{i \in I} a_i \in S$ such that

1. if $I = \emptyset$, then $\sum_{i \in I} a_i = \epsilon$,
2. if $I = \{1 \dots n\}$, then $\sum_{i \in I} a_i = a_1 \oplus \dots \oplus a_n$,
3. if $b \in S$, then $b \otimes (\sum_{i \in I} a_i) = \sum_{i \in I} b \otimes a_i$ and $(\sum_{i \in I} a_i) \otimes b = \sum_{i \in I} a_i \otimes b$, and
4. if $\{I_j\}_{j \in J}$ is a partition of I , then $\sum_{i \in I} a_i = \sum_{j \in J} \left(\sum_{i \in I_j} a_i \right)$.

If I is countable in the definitions above, then \mathcal{S} is *countably complete* and already zerosumfree [8, Prop. 22.28]. The importance for us is that in complete semirings, the existence of the transitive closures is guaranteed (see below). Commutative *complete* dioids are already complete residuated lattices.

A semiring whose commutative multiplicative structure is a group will be called a *semifield* and *radicable* if the equation $a^b = c$ can be solved for a . This term is not standard: for instance, [9] prefer to use “semiring with a multiplicative group structure”, but we prefer *semifield* to shorten out statements. Semifields are all entire. We will use \mathcal{K} to refer to them.

A semifield which is also a dioid is both entire and naturally-ordered. These are sometimes called *positive semifields*, examples of which are the positive rationals, the positive reals or the max-plus and min-plus semifields. Semifields are all incomplete except for the booleans, but they can be completed [10], and we will not differentiate between *complete or completed* structures

A semimodule over a semiring is an analogue of a vector space over a field. Semimodules inherit from their defining semirings the qualities of being zerosumfree, complete or having a natural order. In fact, semimodules over complete commutative dioids are also complete lattices. Rectangular matrices over a semiring form a semimodule $\mathcal{M}_{m \times n}(\mathcal{S})$, and in particular, row- and column-spaces $\mathcal{S}^{1 \times n}$ and $\mathcal{S}^{n \times 1}$. The set of square matrices $\mathcal{M}_n(\mathcal{S})$ is also a semiring (but non-commutative unless $n = 1$).

2.2 The eigenvalue problem and transitive closures over dioids

Given (4), understanding the HITS iteration is easier once understood the eigenvalue problem in a semiring. So let $\mathcal{M}_n(\mathcal{S})$ be the semiring of square matrices over a semiring \mathcal{S} with the usual operations. Given $A \in \mathcal{M}_n(\mathcal{S})$ the *right (left) eigenproblem* is the task of finding the *right eigenvectors* $v \in \mathcal{S}^{n \times 1}$ and *right eigenvalues* $\rho \in \mathcal{S}$ (respectively *left eigenvectors* $u \in \mathcal{S}^{1 \times n}$ and *left eigenvalues* $\lambda \in \mathcal{S}$) satisfying:

$$u \otimes A = \lambda \otimes u \qquad A \otimes v = v \otimes \rho \qquad (9)$$

The left and right eigenspaces and spectra are the sets of these solutions:

$$\begin{aligned} \Lambda(A) &= \{\lambda \in \mathcal{S} \mid \mathcal{U}_\lambda(A) \neq \{\epsilon^n\}\} & \text{P}(A) &= \{\rho \in \mathcal{S} \mid \mathcal{V}_\rho(A) \neq \{\epsilon^n\}\} \\ \mathcal{U}_\lambda(A) &= \{u \in \mathcal{S}^{1 \times n} \mid u \otimes A = \lambda \otimes u\} & \mathcal{V}_\rho(A) &= \{v \in \mathcal{S}^{n \times 1} \mid A \otimes v = v \otimes \rho\} \\ \mathcal{U}(A) &= \bigcup_{\lambda \in \Lambda(A)} \mathcal{U}_\lambda(A) & \mathcal{V}(A) &= \bigcup_{\rho \in \text{P}(A)} \mathcal{V}_\rho(A) \end{aligned} \qquad (10)$$

Since $\Lambda(A) = \text{P}(A^T)$ and $\mathcal{U}_\lambda(A) = \mathcal{V}_\lambda(A^T)$, from now on we will omit references to left eigenvalues, eigenvectors and spectra, unless we want to emphasize differences.

In order to solve (6) in dioids we have to use the following theorem [11, 9]:

Theorem 1 (Gondran and Minoux, [11, Theorem 1]). *Let $A \in \mathcal{S}^{n \times n}$. If A^* exists, the following two conditions are equivalent:*

1. $A_i^+ \otimes \mu = A_i^* \otimes \mu$ for some $i \in \{1 \dots n\}$, and $\mu \in S$.
2. $A_i^+ \otimes \mu$ (and $A_i^* \otimes \mu$) is an eigenvector of A for e , $A_i^+ \otimes \mu \in \mathcal{V}_e(A)$.

where we define the *transitive closure* $A^+ = \sum_{k=1}^{\infty} A^k$ and the *transitive reflexive closure* $A^* = \sum_{k=0}^{\infty} A^k$ of A (also called *Kleene's plus and star operators*).

In [12, 13, 14] this result was made more specific in two directions: on the one hand, by focusing on particular types of completed idempotent semirings—semirings with a natural order where infinite additions of elements exist so transitive closures are guaranteed to exist and sets of generators can be found for the eigenspaces—and, on the other hand, by considering more easily visualizable subsemimodules than the whole eigenspace—a better choice for exploratory data analysis.

From Theorem 1 it is clear that we need efficient methods to obtain the closures. The following account is a summary of results in this respect, and we refer the reader to [12, 13] for proofs.

Lemma 1. *Let $A \in \mathcal{M}_n(A)$ be a square matrix over a commutative semiring S . A^* exists if and only if A^+ exists and then:*

$$A^+ = A \otimes A^* = A^* \otimes A \qquad A^* = I \oplus A^+$$

But since in incomplete semirings the existence of the closures is not warranted, our natural environment should be that of *complete semirings*. In dioids we have,

Lemma 2. *Let $A \in \mathcal{M}_n(A)$ be a square matrix over a dioid \mathcal{S} . For partition $\bar{n} = \alpha \cup \beta$ call $\text{PER}(A) = A_{\beta\alpha} A_{\alpha\alpha}^* A_{\alpha\beta} \oplus A_{\beta\beta}$. Then*

$$\begin{pmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{pmatrix}^+ = \begin{pmatrix} A_{\alpha\alpha}^+ \oplus A_{\alpha\alpha}^* A_{\alpha\beta} \text{PER}(A)^* A_{\beta\alpha} A_{\alpha\alpha}^* & A_{\alpha\alpha}^* A_{\alpha\beta} \text{PER}(A)^* \\ \text{PER}(A)^* A_{\beta\alpha} A_{\alpha\alpha}^* & \text{PER}(A)^+ \end{pmatrix}$$

Proof. Adapted from [15, Lemma 4.101] □

Closures and simultaneous row and column permutations commute:

Lemma 3. *Let $A, B \in \mathcal{M}_n(S)$ and let P be a permutation such that $B = P^T A P$. Then $B^+ = P^T A^+ P$ and $B^* = P^T A^* P$.*

Given the importance of the transitive closure of a matrix in the calculations of eigenvalues and eigenvectors highlighted by Theorem 1, we use the inductive structure of reducible matrices over dioids to calculate them. First a basic case: A square matrix is *irreducible* if it cannot be simultaneously permuted into a triangular upper (or lower) form. Otherwise we say it is *reducible*. Irreducibility expresses itself as a graph property on the induced digraph D_A of Section 1.1.

Lemma 4. *If $A \in \mathcal{M}_n(S)$ is irreducible, then:*

- The induced digraph D_A has a single strongly connected component.
- All nodes in its induced digraph D_A are connected by cycles.

A structure based in irreducible matrices is described next.

Lemma 5 (Recursive Upper Frobenius Normal Form, UFNF). *Let $A \in \mathcal{M}_n(\mathcal{S})$ be a matrix over a semiring and \overline{G}_A its condensation digraph. Then,*

1. (UFNF₃) *If A has zero lines it can be transformed by a simultaneous row and column permutation of V_A into the following form:*

$$P_3^T \otimes A \otimes P_3 = \begin{bmatrix} \mathcal{E}_{\iota\iota} & \cdot & \cdot & \cdot \\ \cdot & \mathcal{E}_{\alpha\alpha} & A_{\alpha\beta} & A_{\alpha\omega} \\ \cdot & \cdot & A_{\beta\beta} & A_{\beta\omega} \\ \cdot & \cdot & \cdot & \mathcal{E}_{\omega\omega} \end{bmatrix} \quad (11)$$

where either $A_{\alpha\beta}$ or $A_{\alpha\omega}$ or both are non-zero, and either $A_{\alpha\omega}$ or $A_{\beta\omega}$ or both are non-zero. Furthermore, P_3 is obtained concatenating permutations for the indices of simultaneously zero columns and rows V_ι , the indices of zero columns but non-zero rows V_α , the indices of zero rows but non-zero columns V_ω and the rest V_β as $P_3 = P(V_\iota)P(V_\alpha)P(V_\beta)P(V_\omega)$.

2. (UFNF₂) *If A has no zero lines it can be transformed by a simultaneous row and column permutation $P_2 = P(A_1) \dots P(A_k)$ into block diagonal UFNF:*

$$P_2^T \otimes A \otimes P_2 = \begin{bmatrix} A_1 & \cdot & \dots & \cdot \\ \cdot & A_2 & \dots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \dots & A_K \end{bmatrix} \quad (12)$$

where $\{A_k\}_{k=1}^K, K \geq 1$ are the matrices of connected components of \overline{G}_A .

3. (UFNF₁) *If A is reducible with no zero lines and a single connected component it can be simultaneously row- and column-permuted by P_1 to*

$$P_1^T \otimes A \otimes P_1 = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1R} \\ \cdot & A_{22} & \dots & A_{2R} \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \dots & A_{RR} \end{bmatrix} \quad (13)$$

where A_{rr} are the matrices associated to each of its R strongly connected components (sorted in a topological ordering), and $P_1 = P(A_{11}) \dots P(A_{RR})$.

Note that irreducible blocks are the base case of UFNF₁, so we sometimes refer to irreducible matrices as being in UFNF₀.

2.3 Graphs, eigenvalues and eigenvectors of matrices over complete dioids

Graph induced by a matrix. It is interesting to consider matrices under a different cryptomorphism: For a matrix $A \in \mathcal{M}_n(\mathcal{S})$, the *network or weighted digraph induced by A* , $N_A = (V_A, E_A, w_A)$, consists of a set of vertices V_A , a set

of arcs , $E_A = \{(i, j) \mid A_{ij} \neq \epsilon_S\}$, and a weight $w_A : V_A \times V_A \rightarrow S$, $(i, j) \mapsto w_A(i, j) = a_{ij}$.

This allows us to apply intuitively all notions from networks to matrices and vice versa, like the underlying graph $G_A = (V_A, E_A)$, the set of paths $\Pi_A^+(i, j)$ between nodes i and j or the set of cycles C_A^+ .

Orthogonality of eigenvectors. In spectral decomposition, orthogonality of the eigenvectors plays an important role. In zerosumfree semimodules orthogonality cannot be as prevalent as in standard vectors spaces. To see this, first call the *support* of a vector, the set of indices of non-null coordinates $\text{supp}(v) = \{i \in \bar{n} \mid v_i \neq \epsilon\}$, and consider a simple lemma:

Lemma 6. *In semimodules over entire, zerosumfree semirings, only vectors with empty intersection of supports are orthogonal.*

Proof. Suppose $v \perp u$, then $\sum_{i=1}^n v_i \otimes u_i = \epsilon$. If any $v_i = \epsilon$ or $u_i = \epsilon$ then their product is null, so we need only consider a non-empty $\text{supp}(v) \cap \text{supp}(u)$. In this case, $v^T \otimes u = \sum_{i \in \text{supp}(v) \cap \text{supp}(u)} v_i \otimes u_i$. But if \mathcal{S} is zerosumfree, for the sum to be null every factor has to be null. And for a factor to be null, since \mathcal{S} is entire, either v_i is null, or u_i is null, and then i would not belong in the common support. \square

The null eigenspaces. If any, the eigenvectors of the null eigenvalue are interesting in that they define the null eigenspace. Also, the particular eigenvalue \perp can only appear in UFNF₃. The following proposition describes the null eigenvalue and eigenspace:

Proposition 1. *Let $\bar{\mathcal{S}}$ be a semiring and $A \in \mathcal{M}_n(\bar{\mathcal{S}})$. Then:*

1. *If the i -th column is zero then the characteristic vector e_i is a fundamental eigenvector of ϵ for A and $\epsilon \in \mathbb{P}^{\mathcal{P}}(A)$.*
2. *Non-collinear eigenvectors of ϵ are orthogonal, so the order of multiplicity of $\perp \in \mathbb{P}^{\mathcal{P}}(A)$ is the number of empty columns of A .*
3. *If \mathcal{S} is entire, then G_A has no cycles if and only if $\mathbb{P}^{\mathcal{P}}(A) = \{\epsilon\}$.*
4. *If \mathcal{S} is entire and zerosumfree, the null eigenspace is generated by the fundamental eigenvectors of ϵ for A , $\mathcal{V}_{\epsilon}(A) = \langle \text{FEV}_{\rho}(\epsilon) A \rangle$.*

Proof. See [12, 3.6 & 3.7]. Claim 2 is a consequence of claim 1 and Lemma 6. \square

Note that these are important in as much as they generate \perp coordinates in the eigenvectors, that is, in the hubs and authorities vectors.

Eigenvalues and eigenvectors of matrices over positive semifields.

When \mathcal{S} has more structure we can improve on the results in the previous section. The first proposition advises us to concentrate on the irreducible blocks:

Proposition 2. *If \mathcal{K} is a positive semifield, $A \in \mathcal{M}_n(\mathcal{K})$ is irreducible, and $v \in \mathcal{V}_{\rho}(A)$ then $\rho > \epsilon$ and $\forall i \in \bar{n}, v_i > \epsilon$.*

Proof. See [9, Lemma 4.1.2]. \square

Note that these results apply to \mathbb{R}_0^+ , but not to $\mathbb{R}_{+, \times}$ or to $\mathbb{C}_{+, \times}$, since the latter are *not* dioids.

For a finite $\rho \in K$ in a semifield, let $(\tilde{A}^\rho)^+ = (\rho^{-1} \otimes A)^+$ be the *normalized transitive closure* of A . The lemma below allows us to change the focus from the transitive closures to the circuit structure of G_A and vice-versa.

Lemma 7. *If \mathcal{K} is a semifield and $A \in \mathcal{M}_n(\mathcal{K})$, then if $(\rho^{-1} \otimes A)^*$ exists and if either $\sum_{c \in C_i} w(c) \otimes (\rho^{-1})^{l(c)} \oplus e = \sum_{c \in C_i} w(c) \otimes (\rho^{-1})^{l(c)}$ where C_i denotes the set of circuits in C_A^+ containing node v_i , or*

$$(\rho^{-1} \otimes A)_{\cdot, i}^* = (\rho^{-1} \otimes A)_{\cdot, i}^+ \quad (14)$$

then $(\rho^{-1} \otimes A)_{\cdot, i}^*$ is an eigenvector of A for eigenvalue ρ .

Proof. See [9, Chapter 6, Corollary 2.4]. \square

When \mathcal{K} is a radicable semifield, the *mean of cycle c* is $\mu_\oplus(c) = \sqrt[l(c)]{w(c)}$. If the semifield is (additively) idempotent the *aggregate cycle mean* of A is $\mu_\oplus(A) = \sum \{\mu_\oplus(c) \mid c \in C_A^+\}$. If the semiring is idempotent and selective, the nodes in the circuits that attain this mean are called the *critical nodes of A* , $V_A^c = \{i \in c \mid \mu_\oplus(c) = \mu_\oplus(A)\}$.

Then the critical nodes are $V_A^c = \{i \in V_A \mid (\tilde{A})_{ii}^+ = e\}$, and we define the set of (right) *fundamental eigenvectors of A for ρ* as

$$\text{FEV}_\rho(A) = \{(\tilde{A})_{\cdot, i}^+ \mid i \in V_A^c\} = \{(\tilde{A})_{\cdot, i}^+ \mid (\tilde{A})_{ii}^+ = e\}.$$

The basic building block is the spectrum of irreducible matrices:

Theorem 2 ((Right) spectral theory for irreducible matrices, [12]). *Let $A \in \mathcal{M}_n(\overline{\mathcal{K}})$ be an irreducible matrix over a complete commutative selective radicable semifield. Then:*

1. *The right spectrum of the matrix includes the whole semiring but the zero:*

$$\text{P}(A) = \overline{\mathcal{K}} \setminus \{\perp\}$$

2. *The right proper spectrum only comprises the aggregate cycle mean:*

$$\text{P}^p(A) = \{\mu_\oplus(A)\}$$

3. *If an eigenvalue is improper $\rho \in \text{P}(A) \setminus \text{P}^p(A)$, then its eigenspace (and eigenlattice) is reduced to the two vectors:*

$$\mathcal{V}_\rho(A) = \{\perp^n, \top^n\} = \mathcal{L}_\rho(A)$$

4. *The eigenspace for a finite proper eigenvalue $\rho = \mu_\oplus(A) < \top$ is generated from its fundamental eigenvectors over the whole semifield, while the eigenlattice is generated by the semifield $\mathfrak{B} = \langle \{\perp, e, \top\}, \oplus, \otimes, \perp, e, \top \rangle$.*

$$\mathcal{V}_\rho(A) = \langle \text{FEV}_\rho(A) \rangle_{\overline{\mathcal{K}}} \supset \mathcal{L}_\rho(A) = \langle \text{FEV}_\rho(A) \rangle_{\mathfrak{B}}$$

Note how this theorem introduces the notion of eigenlattices to finitely represent an eigenspace over an idempotent semifield. Refer to [12] for further details.

We will see in our results that the only other UFNF type we need be concerned about is UFNF₂: Let the partition of V_A generating the permutation that renders A in UFNF₂, block diagonal form, be $V_A = \{V_k\}_{k=1}^K$, and write $A = \bigsqcup_{k=1}^K A_k$, $A_k = A(V_k, V_k)$.

Lemma 8. *Let $A = \bigsqcup_{k=1}^K A_k \in \mathcal{M}_n(\mathcal{S})$ be a matrix in UFNF₂, over a semiring, and $\mathcal{V}_\rho(A_k)$ ($\mathcal{U}_\lambda(A_k)$) a right (left) eigenspace of A_k for ρ (λ). Then,*

$$\mathcal{U}_\lambda(A) \cong \times_{k=1}^K \mathcal{U}_\lambda(A_k) \quad \mathcal{V}_\rho(A) \cong \times_{k=1}^K \mathcal{V}_\rho(A_k). \quad (15)$$

Note that this procedure is constructive and how the combinatorial nature of the proof in [13] makes the claim hold in any semiring. Clearly, if $\rho \in \mathbb{P}^{\mathbb{P}(A_k)}$ for any k , then $\rho \in \mathbb{P}^{\mathbb{P}(A)}$. Since $\mathbb{P}^{\mathbb{P}(A_k)} = \Lambda^{\mathbb{P}(A_k)}$ for matrices admitting an UFNF₂, $\mathbb{P}^{\mathbb{P}(A)} = \Lambda^{\mathbb{P}(A)} = \bigcup_{k=1}^K \mathbb{P}^{\mathbb{P}(A_k)}$.

Corollary 1. *Let $A \in \mathcal{M}_n(\mathcal{S})$ be a matrix in UFNF₂ over a semiring. Then the (left) right eigenspace of A for $\rho \in \mathbb{P}(A)$ is the product of the (left) right eigenspaces for the blocks, $\mathcal{U}_\lambda(A) = \times_{k=1}^K \mathcal{U}_\lambda(A_k)$ and $\mathcal{V}_\rho(A) = \times_{k=1}^K \mathcal{V}_\rho(A_k)$.*

In complete semirings, looking for generators for the eigenspaces with $\delta_k(k) = e$ and $\delta_k(i) = \perp$ for $k \neq i$, we define the right fundamental eigenvectors as

$$\text{FEV}_\rho^2(A) = \bigcup_{k=1}^K \left[\times_{i=1}^K \delta_k(i) \otimes \text{FEV}_\rho^1(A_i) \right]. \quad (16)$$

Lemma 8 proves that $\text{FEV}_\rho^2(A) \subset \mathcal{V}_\rho(A)$, but we also have,

Lemma 9. *Let $A \in \mathcal{M}_n(\overline{\mathcal{D}})$ be a matrix in UFNF₂ over a complete idempotent semiring with $\rho \in \mathbb{P}(A)$. Then,*

1. *If $\rho \in \mathbb{P}^{\mathbb{P}(A)}$, then $\text{FEV}_\rho^{2,\mathbb{F}}(A) = \bigcup_{k|\rho \in \mathbb{P}^{\mathbb{P}(A_k)}} \left[\times_{i=1}^K \delta_k(i) \otimes \text{FEV}_\rho^{1,\mathbb{F}}(A_i) \right]$.*
2. *If $\rho \in \mathbb{P}(A) \setminus \mathbb{P}^{\mathbb{P}(A)}$ then $\text{FEV}_\rho^2(A) = \text{FEV}^{2,\top}(A)$.*
3. *If $\rho \in \mathbb{P}^{\mathbb{P}(A)}$ then $\text{FEV}_\rho(A) = \text{FEV}_\rho^{2,\mathbb{F}}(A) \cup \text{FEV}^{2,\top}(A) \setminus \top \otimes \text{FEV}_\rho^{2,\mathbb{F}}(A)$.*
4. *$\text{FEV}^{2,\top}(A) = \top \otimes \text{FEV}_\rho^2(A)$.*

So call $\text{FEV}^{2,\top}(A)$ the *saturated fundamental eigenvectors* of A , and define the (right) *saturated eigenspace* as $\mathcal{V}^\top(A) = \langle \text{FEV}^{2,\top}(A) \rangle_{\overline{\mathcal{D}}}$.

Corollary 2. *Let $A \in \mathcal{M}_n(\mathcal{S})$ be a matrix in UFNF₂ over a complete selective radicable idempotent semifield. Then*

1. *For $\rho \in \mathbb{P}^{\mathbb{P}(A)}$, $\mathcal{V}_\rho(A) \supseteq \mathcal{V}^\top(A)$.*
2. *For $\rho \in \mathbb{P}(A) \setminus \mathbb{P}^{\mathbb{P}(A)}$, $\mathcal{V}_\rho(A) = \mathcal{V}^\top(A)$.*

Notice that the very general proposition below is for *all* complete dioids.

Proposition 3. *Let $A \in \mathcal{M}_n(\overline{\mathcal{D}})$ be a matrix in $UFNF_2$ over a complete dioid. Then,*

1. *For $\rho \in P(A) \setminus P^P(A)$,*

$$\mathcal{U}^\top(A) = \langle \text{FEV}^{2,\top}(A^\top) \rangle_{\mathfrak{3}} \quad \mathcal{V}^\top(A) = \langle \text{FEV}^{2,\top}(A) \rangle_{\mathfrak{3}}.$$

2. *For $\rho \in P^P(A)$, $\rho < \top$,*

$$\mathcal{U}_\lambda(A) = \langle \text{FEV}_\rho^2(A^\top) \rangle_{\overline{\mathcal{D}}} \quad \mathcal{V}_\rho(A) = \langle \text{FEV}_\rho^2(A) \rangle_{\overline{\mathcal{D}}}.$$

To better represent eigenspaces, we define the *spectral lattices* of A ,

$$\mathcal{L}_\lambda(A) = \langle \text{FEV}_\rho^2(A^\top)^\top \rangle_{\mathfrak{3}} \quad \mathcal{L}_\rho(A) = \langle \text{FEV}_\rho^2(A) \rangle_{\mathfrak{3}}.$$

involving the product of the component lattices, $\mathcal{L}_\rho(A) = \times_{k=1}^K \mathcal{L}_\rho(A_k)$.

3 The Hubs & Authorities algorithm over dioids

Let (G, M, R) be an \mathcal{S} -formal context over a complete dioid \mathcal{S} , and $A \in \mathcal{M}_n(\overline{\mathcal{S}})$ as in (6). Equation (6) states the solution of the singular value-singular vector problem in dioids in terms of the eigenproblem of a symmetric matrix.

3.1 The eigenproblem in symmetric matrices

Since A is symmetric, we no longer have to worry about the distinction between the left and right eigenproblem:

Lemma 10. *If A is symmetric then $A^P(A) = P^P(A)$ and $(\mathcal{U}_\rho(A))^\top = \mathcal{V}_\rho(A)$.*

We can also refine the results in Proposition 1:

Proposition 4. *Let $\overline{\mathcal{S}}$ be a semiring and a symmetric $A \in \mathcal{M}_n(\overline{\mathcal{S}})$. Then:*

1. *The multiplicity of $\perp \in P^P(A)$ is the number of empty rows or columns of A .*
2. *If \mathcal{S} is entire, $P^P(A) = \{\epsilon\}$ if and only if $A = \mathcal{E}_n$.*

Proof. First, if $A = A^\top$, then the number of empty rows and empty columns is the same, and Proposition 1.2 provides the result. Second, after 1.3 $P^P(A) = \{\epsilon\}$ means G_A has no cycles. But if $A_{ij} \neq \epsilon$ then $c = i \rightarrow j \rightarrow i$ is a cycle with non-null weight $w(c) = A_{ij} \otimes A_{ji} \neq \epsilon$, which is a contradiction. \square

Note that empty rows of R generate left eigenvalues while empty columns generate right eigenvalues, so the multiplicity of the null singular value may change from left to right.

To use Lemma 7 and Theorem 2, we need the maximum cycle mean:

Proposition 5. *Let \mathcal{K} be a complete idempotent semifield and let $A \in S^n$ be symmetric. Then $\mu_\oplus(A) = \sup_{i,j} A_{ij}$, where the sup, taken in the natural order of the semifield is attained.*

Proof. Since A is symmetric, $c = i \rightarrow j \rightarrow i$ is a cycle whenever $A_{ij} = A_{ji} \neq \perp$. Then $\mu_{\oplus}(c) = A_{ij}$. Consider one c' such that $\mu_{\oplus}(c') = \sup_{i,j} A_{ij} = \max_{i,j} A_{i,j}$ in the order of the semiring. This must exist since i, j are finite. If we can extend any of these critical cycles with another node k such that $c'' = i \rightarrow j \rightarrow k \rightarrow i$ then $w(c'') = A_{ij} \otimes A_{jk} \otimes A_{ki} \leq A_{ij}^3 = \mu_{\oplus}(c')^{l(c'')}$, so in the aggregate mean $\mu_{\oplus}(c') \oplus \mu_{\oplus}(c'') = \mu_{\oplus}(c')$. So we induce on the length of any cycle that is an extension of c' that $\sum_{c \in C_A^+} \mu_{\oplus}(c) = \mu_{\oplus}(c') = \sup_{i,j} A_{i,j}$. \square

To find the cycle means easily, we use the UFNF form.

3.2 UFNF forms of symmetric matrices and their closures

For symmetric reducible matrices, the feasible UFNF types are simplified:

Proposition 6. *Let \mathcal{S} be a dioid, and a symmetric $A \in \mathcal{M}_n(\mathcal{S})$. Then:*

- (UFNF₃) A admits a proper symmetric UFNF₃ form if it has zero lines, and, in that case, the set of zero lines and zero rows are the same.

$$P_3^T \otimes A \otimes P_3 = \begin{bmatrix} A_{\beta\beta} & \cdot \\ \cdot & \mathcal{E}_{\iota\iota} \end{bmatrix} \quad (17)$$

- (UFNF₂) If a A has no zero lines it can be transformed by a simultaneous row and column permutation $P_2 = P(A_1) \dots P(A_k)$ into symmetric block diagonal UFNF:

$$P_2^T \otimes A \otimes P_2 = \begin{bmatrix} A_1 & \cdot & \dots & \cdot \\ \cdot & A_2 & \dots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \dots & A_K \end{bmatrix} = \bigoplus_{k=1}^K A_k \quad (18)$$

where $\{A_k\}_{k=1}^K, K \geq 1$ are the symmetric matrices of connected components of \overline{G}_A .

- (no UFNF₁) A cannot be permuted into a proper UFNF₁ form.

We will see that this is almost the only structure we need worry about. Consider R of the form:

$$R = \begin{bmatrix} R_1 & R_{12} \\ R_{21} & R_2 \end{bmatrix} \quad (19)$$

If R_{12} and R_{21} are null, then we can find P a permutation so that

$$P^T \otimes A \otimes P = P^T \otimes \begin{bmatrix} \cdot & \cdot & R_1 & \cdot \\ \cdot & \cdot & \cdot & R_2 \\ R_1^T & \cdot & \cdot & \cdot \\ \cdot & R_2^T & \cdot & \cdot \end{bmatrix} \otimes P = \begin{bmatrix} \cdot & R_1 & \cdot & \cdot \\ R_1^T & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & R_2 \\ \cdot & \cdot & R_2^T & \cdot \end{bmatrix} \quad (20)$$

Now, if $R_2 = \mathcal{E}_{G_2 M_2}$ is null, then (20) is in UFN_3 with $\mathcal{E}_{\iota\iota} = \mathcal{E}_{(G_2+M_2)(G_2+M_2)}$, while if R_1 and R_2 are both full, then (20) is in UFN_2 with blocks A_1 and A_2 respectively. Other other blocked forms of R simply generate an irreducible A , since the UFNF_1 form is not possible. So we can suppose that A can be simultaneously row and column permuted into a diagonal block form

$$P^T \otimes A \otimes P = \begin{bmatrix} A_1 & \cdots & \cdot & \cdot \\ \vdots & \ddots & \vdots & \vdots \\ \cdot & \cdots & A_K & \cdot \\ \cdot & \cdots & \cdot & \mathcal{E}_{\iota\iota} \end{bmatrix} = \left(\bigoplus_{k=1}^K A_k \right) \uplus \mathcal{E}_{\iota\iota} \quad A_k = \begin{bmatrix} \cdot & R_k \\ R_k^T & \cdot \end{bmatrix} \quad (21)$$

with the empty lines and rows permuted to the beginning $\mathcal{E}_{\iota\iota}$ and irreducible blocks A_k . Recall that closures and permutations commute, whence the closures of the matrices in the forms above are really simple: the closure of (21) is straightforward in terms of the closures of the blocks:

$$P^T \otimes A^+ \otimes P = \begin{bmatrix} A_1^+ & \cdots & \cdot & \cdot \\ \vdots & \ddots & \vdots & \vdots \\ \cdot & \cdots & A_K^+ & \cdot \\ \cdot & \cdots & \cdot & \mathcal{E}_{\iota\iota} \end{bmatrix} \quad (22)$$

The solution of this base case is highly dependent in the dioid in which the problem is stated. Since we will be solving the problem in idempotent semifields, for the irreducible base case we need only be concerned about matrix:

$$\tilde{A}^{\mu_{\oplus}(A)} = \begin{bmatrix} \cdot & \tilde{R}^{\mu_{\oplus}(A)} \\ (\tilde{R}^{\mu_{\oplus}(A)})^T & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & B \\ B^T & \cdot \end{bmatrix} \quad (23)$$

where we are using the shorthand $B = \tilde{R}^{\mu_{\oplus}(A)}$ to account for the normalization with the cycle means that we need to use to find the eigenvectors. To find the closures of such irreducible matrices apply Lemma 2 to (23):

$$\left(\tilde{A}^{\mu_{\oplus}(A)} \right)^+ = \begin{bmatrix} \cdot & \tilde{R}^{\mu_{\oplus}(A)} \\ (\tilde{R}^{\mu_{\oplus}(A)})^T & \cdot \end{bmatrix}^+ = \begin{bmatrix} (B \otimes B^T)^+ & B \otimes (B^T \otimes B)^* \\ B^T \otimes (B \otimes B^T)^* & (B^T \otimes B)^+ \end{bmatrix} \quad (24)$$

where we have used that $(B^T \otimes B)^* \otimes B^T = B^T \otimes (B \otimes B^T)^*$.

3.3 Pairs of singular vectors of symmetric matrices over idempotent semifields.

To extract the eigenvectors corresponding to left singular vectors $i \in G$ (respectively, right singular vectors $j \in M$) we need to check Theorem 1 against each block in its diagonal:

$$\begin{aligned} i \in G, \left(\tilde{A}^{\mu_{\oplus}(A)} \right)^*_{\cdot i} &= \left(\tilde{A}^{\mu_{\oplus}(A)} \right)^+_{\cdot i} \Leftrightarrow (B \otimes B^T)^*_{\cdot i} = (B \otimes B^T)^+_{\cdot i} \\ j \in M, \left(\tilde{A}^{\mu_{\oplus}(A)} \right)^*_{\cdot j} &= \left(\tilde{A}^{\mu_{\oplus}(A)} \right)^+_{\cdot j} \Leftrightarrow (B^T \otimes B)^*_{\cdot j} = (B^T \otimes B)^+_{\cdot j} \end{aligned}$$

So the existence of $(\tilde{A}^{\mu_{\oplus}(A)})^+$ requires the existence of the transitive closures $(B^T \otimes B)^+$ and $(B \otimes B^T)^+$ as we could have expected from (8). Note that after (4), the hub and authority scores are those columns such that:

$$(B \otimes B^T)_{.i}^+ = (B \otimes B^T)_{.i}^* \quad (B^T \otimes B)_{.j}^+ = (B^T \otimes B)_{.j}^*$$

but, importantly, (24) gives us the form of the authority score related to a particular hub score and vice-versa, which is a kind of formal-concept property:

$$h_i = (B \otimes B^T)_{.i}^* \Leftrightarrow a_i = B^T \otimes (B \otimes B^T)_{.i}^* \quad (25)$$

$$a_j = (B^T \otimes B)_{.j}^* \Leftrightarrow h_j = B \otimes (B^T \otimes B)_{.j}^* \quad (26)$$

This solves completely the description of the left and right singular vectors. To find the singular values, we note from (8) that they are the square roots of the cycle means of the independent blocks or, equally, the proper eigenvalues of A ,

$$\Sigma = \{\sqrt{\rho} \mid \rho = \mu_{\oplus}(A_k), A = \oplus_k A_k\} = \{\sqrt{\rho} \mid \rho = P^P(A)\}$$

This would include the bottom if and only if one of the blocks is empty.

4 Example

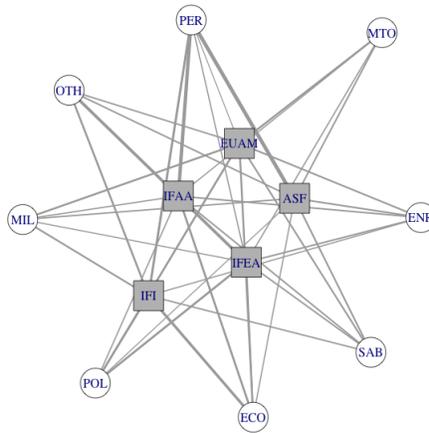
In this section we present a HITS analysis for a weighted two-mode network both using standard HITS and HITS over the max-min-plus idempotent semifield. The data being analysed is the example in [16, p. 31], the worries data, which is a two-mode network of the type of worry declared as most prevalent by 1554 adult Israeli depending on their living countries—and sometimes those of their parents. The graph of the network is depicted in Fig. 1.(a).

The “authority” and “hub” scores are differentiated for each of the modes: they return “type of worry” and “procedence” scores, respectively. We can see that HITS and i-HITS produce somehow different results: the actual meaning of these differences is data-dependent and a matter for future, more specialized analyses. The idempotent primitives were developed in-house and are available from the authors upon request.

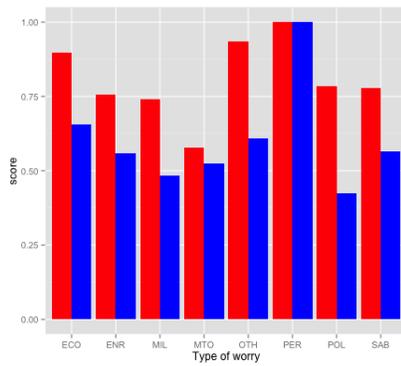
5 Discussion

We have generalized the HITS algorithms for semirings, then instantiated it in dioids, semifields (including the original semifield where it was defined), and finally in idempotent semifields.

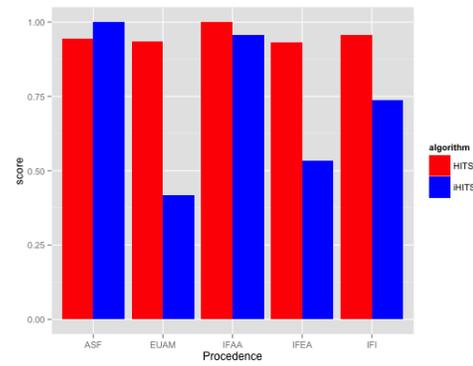
More importantly, solving these problems entails solving the Singular value-Singular vector problems in the semiring of choice. We have sketched the general solution for dioids and positive semifields, and provided a complete solution in idempotent semifields.



(a) Worries weighted bipartite network



(b) Principal worry scores



(c) Procidence scores

Fig. 1: Weighted, directed graph of the worries data [16, p. 31] and its weighted idempotent and standard hub and authority scores. There are clear differences in both approaches.

Note that the original HITS problem was set in a positive semifield that is not idempotent, hence our method of solution does not apply yet to that case, but does apply to the max-min-plus and max-min-times semifields, examples of which are given in terms of their normalized closures. In the case of the \mathbb{R}_0^+ the Perron-Frobenius theorem is invoked to solve HITS iteratively by means of the power method.

A reviewer of this paper requested a consideration of the solution of the HITS problem for the rest of dioids which are not semifields. The problem with further generalization of our scheme is the base case for the recursion of Frobenius normal forms. In the idempotent semifield case, the cycle means provide the eigenvalues

needed for the normalization of the matrices that allow the calculation of the closures in the irreducible case. But in the generic positive semifield case the cycle means and the possibility of choosing the critical circuits to select the eigenvectors in the closures are not granted. Also, in inclines—and in the fuzzy semirings included in them—the base, irreducible case is completely different to that of semifields [9]. But since the generic development on dioids for UFNF1 and UFNF2 is based in combinatorial considerations, we believe that a solution for the base case for other dioids could be plugged into this UFNF recursion to obtain analog results to those presented here. These extensions will be considered in future work.

On another note, the SVD in standard algebra makes a strong case about the orthogonality of the left and right singular vectors belonging to different singular values in order to guarantee certain properties of the bases of singular vectors in the reconstruction. But in entire zerosumfree semirings, and in entire dioids or positive semifields a fortiori, orthogonality is a rare phenomenon, after Lemma 6.

Indeed, irreducible matrices do not have any orthogonal, but rather collinear left or right singular vectors. Regarding reducible matrices, note that (21) factors in all the possible orthogonality between eigenvectors. In fact we have,

Corollary 3. *Let (G, M, R) be an \mathcal{S} -formal context over a dioid \mathcal{S} , and $A \in \mathcal{M}_n(\mathcal{S})$ as in (24). Then two of the eigenvectors for A or (left, right) singular vectors for R can only be orthogonal only if they arise from different blocks.*

and Proposition 3 proves that even in that case they might not be orthogonal.

Regarding the glimpse of a formal concept-like property of (25), a cursory analysis with the techniques in [10], shows that the analogue of the polars in FCA actually come from two different generalised Galois connections, so their order-properties are not the same. Yet, the isomorphism between the ranges of the operators in (5) defining the duality between hubs and authorities is clear and attested in a more generic context than HITS was initially conceived.

Bibliography

- [1] Strang, G.: The fundamental theorem of linear algebra. The American Mathematical Monthly **100** (1993) 848–855
- [2] Everett, M.G., Borgatti, S.P.: The dual-projection approach for two-mode networks. Social networks **35** (2013) 204–210
- [3] Kleinberg, J.M.: Authoritative sources in a hyperlinked environment. Journal of the ACM **46** (1999) 604–632
- [4] Easley, D.A., Kleinberg, J.M.: Networks, Crowds, and Markets - Reasoning About a Highly Connected World. Cambridge University Press 2010 (2010)
- [5] Lanczos, C.: Linear Differential Operators. Dover Publications (1997)

- [6] Pouly, M., Kohlas, J.: *Generic Inference. A Unifying Theory for Automated Reasoning*. John Wiley & Sons (2012)
- [7] Golan, J.S.: *Power Algebras over Semirings. With Applications in Mathematics and Computer Science*. Volume 488 of *Mathematics and its applications*. Kluwer Academic, Dordrecht, Boston, London (1999)
- [8] Golan, J.S.: *Semirings and their Applications*. Kluwer Academic (1999)
- [9] Gondran, M., Minoux, M.: *Graphs, Dioids and Semirings. New Models and Algorithms*. *Operations research/Computer Science Interfaces*. Springer (2008)
- [10] Valverde-Albacete, F.J., Peláez-Moreno, C.: Extending conceptualisation modes for generalised Formal Concept Analysis. *Information Sciences* **181** (2011) 1888–1909
- [11] Gondran, M., Minoux, M.: Valeurs propres et vecteurs propres dans les dioïdes et leur interprétation en théorie des graphes. EDF, Bulletin de la Direction des Etudes et Recherches, Serie C, *Mathématiques Informatique* **2** (1977) 25–41
- [12] Valverde-Albacete, F.J., Peláez-Moreno, C.: Spectral lattices of irreducible matrices over completed idempotent semifields. *Fuzzy Sets and Systems* (2014)
- [13] Valverde Albacete, F.J., Peláez-Moreno, C.: The spectra of reducible matrices over completed commutative idempotent semifields and their spectral lattices. *International Journal of General Systems* (Accepted for publication.)
- [14] Valverde-Albacete, F.J., Peláez-Moreno, C.: Spectral lattices of reducible matrices over completed idempotent semifields. In Ojeda-Aciego, M., Outrata, J., eds.: *Concept Lattices and Applications (CLA 2013)*, Université de la Rochelle, Laboratory L31, La Rochelle (2013) 211–224
- [15] Cohen, G., Gaubert, S., Quadrat, J.P.: Duality and separation theorems in idempotent semimodules. *Linear Algebra and Its Applications* **379** (2004) 395–422
- [16] Mirkin, B.: *Mathematical Classification and Clustering*. Volume 11 of *Non-convex Optimization and Its Applications*. Kluwer Academic Publishers (1996)