Optimization and Discretization in 2D Problems of Electromagnetic Invisible Cloaking

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Abstract. We study control problems for the 2D electromagnetic field model describing scattering TM-polarized electromagnetic waves in unbounded homogeneous medium containing a penetrable inhomogeneous dielectric obstacle with the boundary partially coated for masking. These problems arise when developing the design technologies of electromagnetic cloaking devices using optimization method. Two constitutive parameters: variable refraction index of the obstacle and surface conductivity of the coated part of the boundary play the role of controls. Solvability of control problems is proved, the optimality system which describes the necessary conditions of extremum is derived, uniqueness and stability of optimal solutions are established. Two numerical algorithms are proposed and discussed. The first of them is based on strategy "optimize-then-discretize" and the second one is based on opposite strategy "discretize-then-optimize".

Keywords: TM-polarization · invisibility cloaking · design · regularization · discretization · control problem · optimality system · uniqueness · stability

1 Introduction. Statement of Direct Problem

In recent years significant research has focused on design of devices cloaking material objects from detection of radar system. Beginning with papers [8, 11, 13] the large number of papers (see, e.g., [5, 7, 10, 12, 15]) was devoted to developing different schemes of cloaking. These schemes include metamaterial cloaking based on transformation optics (TO) proposed by Pendry et al. [13], conformal method proposed by Leonhardt [11], plasmonic cloaking method based on scattering cancellation proposed by Alú and Engheta [5], mantle cloaking, impedance cloaking, etc.

Development of the above-mentioned approaches have opened up the opportunities for creation the invisibility cloaking design strategies. They obtained the name of direct design strategies as they were based on solving the forward electromagnetic (or acoustic) problems. It should be noted that the invisibility devices (hereafter, cloaks) designed on the basis of direct strategies possess serious drawbacks. The main one is the

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difficulty of their technical realization. For example, the design of the TO-based cloaks involves extreme values of constitutive parameters and spatially varying distributions of the permittivity and permeability tensors which are very difficult to implement [18].

That is why the another cloak design strategy began develop recently. It obtained the name of inverse design as it is related with solving inverse electromagnetic (or acoustic) problems. The optimization method forms the core of this inverse design methodology. This enables us to solve some substantial limitations of previous cloaking solutions. A growing number of papers is devoted to applying the inverse design methodology in various cloaking problems. Among them we mention [14, 16, 17] where numerical optimization algorithms are applied for finding the unknown material parameters of TO-based cloak. It was shown there that the optimized multi-layer cloak essentially outperforms the similarly sized metamaterial cloak designed by using the TO approach. In [18] the authors review the invisibility cloak design methodologies and discuss the recent transition from forward design to inverse design. We also mention papers [1–4] where the mathematical apparatus for solving impedance cloaking problem on the basis of optimization approach is developed.

This paper is devoted to theoretical analysis of control problems for 2D electromagnetic wave scattering model. These problems arise when optimization method is applied for solving cloaking problems for respective 2D electromagnetic scattering model.

We begin with formulation of the direct scattering problem. Let Ω be a bounded domain in $\mathbb{R}^2$ with a connected complement $\Omega^c := \mathbb{R}^2 \setminus \overline{\Omega}$ and Lipschitz boundary Γ consisting of two parts Γ_D and Γ_I. We consider the scattering problem for TM-polarized electromagnetic waves in homogeneous medium containing penetrable inhomogeneous dielectric obstacle Ω with coated partially (for masking) boundary. Mathematically this problem is reduced to finding functions w in Ω and $u = u^{inc} + u^s$ in Ω^c satisfying equations

$$\Delta w + k^2 n(x) w = 0 \text{ in } \Omega, \ \triangle u + k^2 u = 0 \text{ in } \Omega^c, \tag{1}$$

mixed transmission conditions on the boundary Γ

$$w - u|_\Gamma = 0, \frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu}|_{\Gamma_D} = 0, \ \frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu}|_{\Gamma_I} = i\eta(x)u \tag{2}$$

and the Sommerfeld radiation condition in $\mathbb{R}^2$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad r = |x|. \tag{3}$$

Here u^{inc} is the incident wave, u^s is the scattered wave, k is the wave number, $k^2 = \varepsilon_0\mu_0\omega^2$ where ω is an angular frequency, ε_0 and μ_0 are constant electric permittivity and magnetic permeability, $n(x) > 0$ is a variable index of refraction of dielectric obstacle Ω, η is the surface conductivity of the coated part Γ_I of Γ, i is an imaginary unit, ν is the outward (relative to Ω) unit normal on Γ.

One can find the formulation and brief analysis of problem (1)–(3) in [6]. Besides, in [6] the inverse scattering problem of recovering the shape and surface conductivity of a partially coated dielectric infinite cylinder from the far field data was also studied. The control problems considered in our paper consist of minimization of certain cost functionals dependent on the state (electromagnetic field) and unknown functions

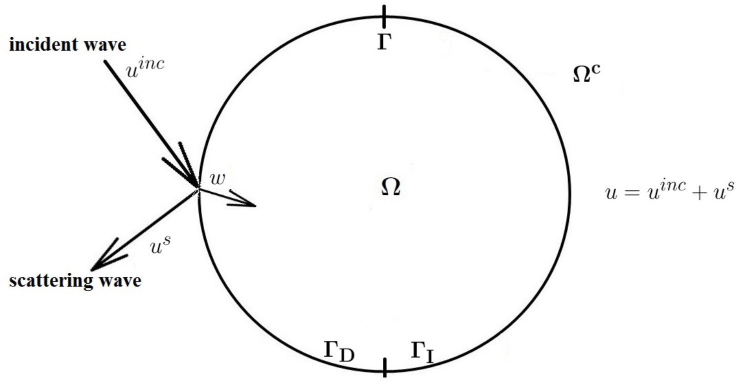

Fig. 1. Geometry of cloaking problem

(controls or design parameters) satisfying equations (1)–(3). As the cost functional we choose one of the following:

$$I_1(U) = \int_Q |U - u^d|^2 dx, \; I_2(U) = \int_{\Gamma_r} |U - u^d|^2 d\sigma. \tag{4}$$

Here U is the function equalled to w in Ω and to u in Ω^c, function $u^d \in L^2(Q)$ (or $u^d \in L^2(\Gamma_r)$) describes the field measured in some subdomain $Q \subset \Omega^c$ or on the boundary Γ_r of the disc B_r of the radius r containing Ω inside. In the case when $u^d = u^{inc}$ the functional $I_1(U)$ (or $I_2(U)$) has the sense of squared mean-square integral norm of the scattered field u^s over Q (or over Γ_r). As controls we choose index of refraction n and surface conductivity η. We assume that n and η are elements of Sobolev spaces $H^r(\Omega)$ and $H^s(\Gamma_I)$ and define the following regularized functional:

$$J_j(U, n, \eta) = \frac{\alpha_0}{2} I_j(U) + \frac{\alpha_1}{2} \|n\|^2_{H^r(\Omega)} + \frac{\alpha_2}{2} \|\eta\|^2_{H^s(\Gamma_I)}. \tag{5}$$

Here $j = 1$ or 2, α_0, α_1 and α_2 are nonnegative parameters specifying the relative importance of each term in (5). We want to find controls n, η and the associated state – electromagnetic field $U = (w, u)$, such that the functional $J_j(U, n, \eta)$ defined in (5) is minimized subject to state equations (1)–(3).

The rest of the paper is organised as follows. In Section 2 we reduce unbounded problem (1)–(3) to an equivalent problem considered in bounded domain and prove the correct solvability of the latter problem. In Section 3 we prove the existence of a solution of control problem and derive an optimality system. Based on analysis of the optimality system we prove further in Section 4 uniqueness and stability of optimal solutions. Then in Section 5 we propose and discuss two numerical algorithms for solving our control problem.

2 Functional Spaces. Solvability of Direct Problem

Let us introduce the function spaces to be used in the subsequent analysis. Let B_R be the disc of radius R containing Ω, $\Omega_e := \Omega^c \cap B_R$. We will use the spaces $H^s(\Omega)$,

$H^1(\Omega_e)$, $H^1(B_R)$, $L^2(Q)$, $L^2(\Gamma_I)$, $H^{1/2}(\Gamma_R)$, $H^{-1/2}(\Gamma_R)$, $L^\infty(\Gamma_I)$, $H^s(\Gamma_I)$ with norms $\|\cdot\|_{s,\Omega}$, $\|\cdot\|_{1,\Omega_e}$, $\|\cdot\|_{1,B_R}$, $\|\cdot\|_Q$, $\|\cdot\|_{\Gamma_I}$, $\|\cdot\|_{1/2,\Gamma_R}$, $\|\cdot\|_{-1/2,\Gamma_R}$, $\|\cdot\|_{L^\infty(\Gamma_I)}$ and $\|\cdot\|_{s,\Gamma_I}$, respectively. The scalar products and norms in $H^r(\Omega)$, $L^2(\Omega)$, $H^s(\Gamma_I)$ and $L^2(\Gamma_I)$ will be denoted by $(\cdot,\cdot)_{r,\Omega}$, $\|\cdot\|_{r,\Omega}$, $(\cdot,\cdot)_\Omega$, $\|\cdot\|_\Omega$, $(\cdot,\cdot)_{s,\Gamma_I}$, $\|\cdot\|_{s,\Gamma_I}$ and $(\cdot,\cdot)_{\Gamma_I}$, $\|\cdot\|_{\Gamma_I}$, respectively.

Along with the space $H^1(\Omega)$ we will consider it's subspace $H^1(\Delta,\Omega) := \{w : w \in H^1(\Omega),\ \Delta w \in L^2(\Omega)\}$ equipped with the norm $\|w\|^2_{H^1(\Delta,\Omega)} = \|w\|^2_{1,\Omega} + \|\Delta w\|^2_\Omega$. It is well known (see [9, p. 28]) that any function $w \in H^1(\Delta,\Omega)$ has the trace $\gamma_1 w \equiv \partial w/\partial\nu|_\Gamma \in H^{-1/2}(\Gamma)$ and the following Green formula holds:

$$(\Delta w, w)_\Omega = -(\nabla w, \nabla w)_\Omega + \int_\Gamma \frac{\partial w}{\partial \nu} w d\sigma \quad \forall w \in H^1(\Omega). \tag{6}$$

Here and below integral over Γ (or over Γ_R) denotes the duality pairing $\langle\cdot,\cdot\rangle_\Gamma$ between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ (or between $H^{1/2}(\Gamma_R)$ and $H^{-1/2}(\Gamma_R)$). Similar formula holds and for the domain Ω_e.

We also need the space $X = H^1(B_R)$ with the norm $\|\cdot\|_X := \|\cdot\|_{1,B_R}$ and the space $H^{inc} \equiv H^{inc}(\Omega_e) = \{u \in H^1(\Omega_e) : \Delta u + k^2 u = 0 \text{ in } \Omega_e\}$ with the norm $\|u\|_{1,\Omega_e}$. These spaces will be used for describing properties of the weak solutions of problem (1)–(3) and for describing incident waves u^{inc}, respectively. By X^* we denote dual of space X. Let $L^\infty_{n_0}(\Omega) = \{n \in L^\infty(\Omega) : n(x) \geq n_0\}$, $H^r_{n_0}(\Omega) = \{n \in H^r(\Omega) : n(x) \geq n_0\}$, $L^\infty_{\eta_0}(\Gamma_I) = \{\eta \in L^\infty(\Gamma_I) : \eta(x) \geq \eta_0\}$ and $H^s_{\eta_0}(\Gamma_I) = \{\eta \in H^s(\Gamma_I) : \eta(x) \geq \eta_0\}$ where $n_0 = \text{const} > 0$, $\eta_0 = \text{const} > 0, r > 0, s > 0$. These sets will serve for describing properties of index of refraction n and conductivity η. We note that continuous compact embeddings $H^r(\Omega) \subset L^\infty(\Omega)$ at $r > 1$ and $H^s(\Gamma_I) \subset L^\infty(\Gamma_I)$ at $s > 1/2$ take place (if $\Gamma_I \in C^{1,1}$) and the following estimates hold:

$$\|n\|_{L^\infty(\Omega)} \leq C'_r \|n\|_{r,\Omega}\ \forall n \in H^r(\Omega),\ \|\eta\|_{L^\infty(\Gamma_I)} \leq C_s \|\eta\|_{s,\Gamma_I} \quad \forall \eta \in H^s(\Gamma_I). \tag{7}$$

Here C'_r (or C_s) is the constant dependent on r and Ω (or on s and Γ_I).

Now we are in position to study problem (1)–(3). We begin with reducing problem (1)–(3) to an equivalent problem considered in the disc B_R. For this purpose we define the Dirichlet-to-Neumann (DtN) operator $T : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$ that maps every function $g \in H^{1/2}(\Gamma_R)$ to a function $\partial\tilde{u}/\partial\nu \in H^{-1/2}(\Gamma_R)$ where $\tilde{u}$ is a solution of the exterior Dirichlet problem for the Helmholtz equation $\Delta\tilde{u} + k^2\tilde{u} = 0$ in $\Omega^c \backslash \overline{B}_R$ with condition $\tilde{u}|_{\Gamma_R} = g$. It is well known that problem (1)–(3) is equivalent to problem (1), (2) considered in the disc B_R under the following boundary condition for scattered field u^s on Γ_R:

$$\partial u^s/\partial\nu = T u^s \quad \text{on} \quad \Gamma_R. \tag{8}$$

We will refer to (1) in $\Omega \cup \Omega_e$, (2) and (8) as problem 1.

Let us multiply the first equation in (1) by $\overline{\Phi}|_\Omega$ where $\Phi \in X$ is a test function, integrate over Ω and apply the Green formula (6). We obtain

$$\int_\Omega \left(\nabla\overline{\Phi} \cdot \nabla w - k^2 n \overline{\Phi} w\right) dx = \int_\Gamma \overline{\Phi} \frac{\partial w}{\partial \nu} d\sigma. \tag{9}$$

Here and below $\overline{\Phi}$ denotes the complex conjugate of Φ. In a similar manner, we multiply the second equation in (1) by $\overline{\Phi}|_{\Omega_e}$, integrate over Ω_e, apply the Green formula for the domain Ω_e and add with (9). Using the boundary conditions in (2) and condition (8) for u^s we arrive at the following identity with respect to function $U := (w, u) \in X$:

$$a^\lambda(U, \Phi) := a_0(U, \Phi) - a_n(U, \Phi) - a_\eta(U, \Phi) = \langle f, \Phi \rangle \quad \forall \Phi \in X. \tag{10}$$

Here and below index λ denotes the pair (n, η), a_0, a_n, a_η and f are sesquilinear and linear forms defined by

$$a_0(U, \Phi) := \int_\Omega \nabla\overline{\Phi} \cdot \nabla U dx + \int_{\Omega_e} (\nabla\overline{\Phi} \cdot \nabla U - k^2 \overline{\Phi} U) dx - \int_{\Gamma_R} \overline{\Phi} T U d\sigma, \tag{11}$$

$$a_n(U, \Phi) = k^2 (nU, \Phi)_\Omega := k^2 \int_\Omega n \overline{\Phi} U dx,\ a_\eta(U, \Phi) := i(\eta U, \Phi)_{\Gamma_I} := i \int_{\Gamma_I} \eta \overline{\Phi} U d\sigma, \tag{12}$$

$$\langle f, \Phi \rangle := -\int_{\Gamma_R} \overline{\Phi} T u^{inc} d\sigma + \int_{\Gamma_R} \overline{\Phi} \frac{\partial u^{inc}}{\partial \nu} d\sigma,\ \lambda = (n, \eta). \tag{13}$$

The solution $U \in X$ of problem (10) is called a weak solution of problem 1.

Using the embedding theorems, trace theorem and the properties of DtN operator T it is easy to derive the following estimates for forms a_0, a_n, a_η, f:

$$\|a_0\| \le C_1, \|a_n\| \le C_1 \|n\|_{L^\infty(\Omega)},\ \|a_\eta\| \le C_1 \|\eta\|_{L^\infty(\Gamma_I)},\ \|f\|_{X^*} \le C_1 \|u^{inc}\|_{1,\Omega_e}. \tag{14}$$

Here C_1 is a constant dependent on Ω, k and R. We note that the sesquilinear form a^λ introduced in (10) defines operator $A_\lambda : X \to X^*$ by $\langle A_\lambda U, \Phi \rangle = a^\lambda(U, \Phi)$ for all $U \in X$, $\Phi \in X$ and problem (10) for $U \in X$ is equivalent to equation

$$A_\lambda U = f. \tag{15}$$

Using properties of forms a_0, a_n, a_η and operator T one can prove arguing as in [2] that the operator A_λ is an isomorphism. Denote by $A_\lambda^{-1} : X^* \to X$ the inverse of the operator A_λ. Let $\tilde{C}_\lambda = \|A_\lambda^{-1}\|$. It follows from the results above that for any element $f \in X^*$ equation (15) has a unique solution $U_\lambda \in X$ which satisfies the estimate $\|U_\lambda\|_X \le \tilde{C}_\lambda \|f\|_{X^*}$. Besides, in the case when index of refraction n and conductivity η belong to nonempty bounded sets $K_1 \subset H^r_{n_0}(\Omega)$, $r > 1$ and $K_2 \subset H^s_{\eta_0}(\Gamma_I)$, $s > 1/2$ one can show proceeding as in [2] that the solution U_λ of (15) satisfies the estimate $\|U_\lambda\|_X \le \tilde{C}_0 \|f\|_{X^*}$ where constant $\tilde{C}_0$ is independent of λ. Using estimate $\|f\|_{X^*} \le C_1 \|u^{inc}\|_{1,\Omega_e}$ from (14) and setting $C_0 = \tilde{C}_0 C_1$ we rewrite this estimate as

$$\|U_\lambda\|_X \le C_0 \|u^{inc}\|_{1,\Omega_e} \quad \forall \lambda = (n, \eta) \in K_1 \times K_2. \tag{16}$$

Let us formulate the result obtained as the next theorem.

Theorem 1. *Let $\Gamma \in C^{0,1}$, $\Gamma_I \in C^{1,1}$ and let $K_1 \subset H^r_{n_0}(\Omega)$ and $K_2 \subset H^s_{\eta_0}(\Gamma_I)$ be nonempty bounded sets where $r > 1$, $s > 1/2$. Let $u^{inc} \in H^{inc}$ be an incident field. Then for any pair $(n, \eta) \in K_1 \times K_2$ problem (10) has a unique solution $U_\lambda \in X$ which satisfies estimate (16) with constant C_0 independent of λ.*

3 Solvability of Control Problem. Optimality System

In this Section we formulate and study our control problem. We assume that controls n and η can change in certain sets K_1 and K_2. More precisely, the following conditions are assumed to hold:

(j) $\Gamma \in C^{0,1}$ $\Gamma_I \in C^{1,1}$; $\alpha_0 > 0$; $K_1 \subset H^r_{n_0}(\Omega)$ and $K_2 \subset H^s_{\eta_0}(\Gamma_I)$ are nonempty convex closed sets where n_0=const > 0, η_0=const > 0, $r > 1$, $s > 1/2$.

Let $K = K_1 \times K_2$, $\lambda = (n, \eta)$. Defining the operator $G : X \times K \times H^{inc} \to X^*$ by $\langle G(U, \lambda, u^{\rm inc}), \Phi\rangle = a_0(U, \Phi) - k^2(nU, \Phi)_\Omega - i(\eta U, \Phi)_{\Gamma_I} - \langle f, \Phi\rangle$ for all $\Phi \in X$ consider the constrained minimization problem

$$J(U, \lambda) := \frac{\alpha_0}{2} I(U) + \frac{\alpha_1}{2}\|n\|^2_{r,\Omega} + \frac{\alpha_2}{2}\|\eta\|^2_{s,\Gamma_I} \to \inf,$$

$$G(U, \lambda, u^{inc}) = 0,\ (U, \lambda) \in X \times K,\ \lambda = (n, \eta). \tag{17}$$

Here $I : X \to \mathbb{R}$ is a weakly lower semicontinuous cost functional. Denote by $Z_{ad} = Z_{ad}(u^{inc}) := \{(U, \lambda) \in X \times K : G(U, \lambda, u^{\rm inc}) = 0, J(U, \lambda) < \infty\}$ the set of admissible pairs for problem (17).

Theorem 2. *Let under conditions (j) $I : X \to \mathbb{R}$ be a weakly lower semicontinuous functional, $u^{inc} \in H^{inc}$ and Z_{ad} be nonempty set. Suppose that $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and K is bounded set or $\alpha_1 > 0$, $\alpha_2 > 0$ and functional I is bounded below. Then problem (17) has at least one solution $(U, \lambda) \in X \times K$.*

Proof. Let $(U_m, \lambda_m) \in Z_{ad}$ where $\lambda_m := (\eta_m, n_m)$ be minimizing sequence for which

$$\lim_{m\to\infty} J(U_m, \lambda_m) = \inf_{(U,\lambda)\in Z_{ad}} J(U, \lambda) := J^*.$$

From conditions (j) and Theorem 1 the following estimates follow:

$$\|n_m\|_{r,\Omega} \leq c_1,\ \ \|\eta_m\|_{s,\Gamma_I} \leq c_2,\ \ \|U_m\|_X \leq c_3.$$

Here c_1, c_2, c_3 are some constants which are independent of $m \in \mathbb{N} = \{1, 2, ...\}$. By definition of Z_{ad} the pair (U_m, λ_m) satisfies the identity

$$a_0(U_m, \Phi) - k^2(n_m U_m, \Phi)_\Omega - i(\eta_m U_m, \Phi)_{\Gamma_I} = \langle f, \Phi\rangle\ \forall\Phi \in X,\ m \in \mathbb{N}. \tag{18}$$

It follows from the estimates above that there exist weak limits $n_* \in K_1 \subset H^s_{n_0}(\Omega)$, $\eta_* \in K_2 \subset H^s_{\eta_0}(\Gamma_I)$ and $U_* \in X$ of some subsequences of the sequences $\{n_m\}$, $\{\eta_m\}$ and $\{U_m\}$. Using this fact and compactness of embeddings $H^r(\Omega) \subset L^\infty(\Omega)$ at $r > 1$, $H^s(\Gamma_I) \subset L^\infty(\Gamma_I)$ at s>1/2 we conclude (passing if necessary to subsequencies) that $U_m \to U_* \in X$ weakly in X and $U_m|_\Omega \to U_*|_\Omega$ weakly in $L^2(\Omega)$, $U_m|_{\Gamma_I} \to U_*|_{\Gamma_I}$ weakly in $L^2(\Gamma_I)$, $n_m \to n_* \in K_1$ strongly in $L^\infty(\Omega)$, $\eta_m \to \eta_* \in K_2$ strongly in $L^\infty(\Gamma_I)$.

Let us pass to the limit in (18) when $m \to 0$. Using (14) we obtain that the pair (U_*, λ_*) where $\lambda_* := (n_*, \eta_*)$ satisfies

$$a_0(U_*, \Phi) - k^2(nU_*, \Phi)_\Omega - i(\eta_* U_*, \Phi)_{\Gamma_I} = \langle f, \Phi\rangle\ \forall\Phi \in X. \tag{19}$$

This means that $G(U_*, \lambda_*, u^{\rm inc}) = 0$. Since J is weakly lower semicontinuous on $X \times K$, we have $J(U_*, \lambda_*) = J^*$ which proves the theorem.

We note that the assertion of Theorem 2 is valid for both functionals $I_1(U)$ and $I_2(U)$ since they are nonnegative and are weakly lower semicontinuous.

The next stage in the study of control problem (17) is to establish sufficient conditions on the input data under which its solution is unique and stable for particular cost functionals. For this purpose we make use the approach developed in [2, 3]. It is based on the derivation and analysis of an optimality system describing the first-order necessary conditions for an extremum in problem (17). Arguing as in [3] one can prove the following result.

Theorem 3. *Let under conditions (j) the triple* $(\hat{U}, \hat{n}, \hat{\eta}) \in X \times K$ *be a solution of problem (17) where the functional* $I(U)$ *is continuously differentiable with respect to U in the point* $\hat{U}$. *Then there exists a unique nonzero Lagrange multiplier* $P \in X$ *which satisfies the Euler-Lagrange equation*

$$a_0(\Psi, P) - k^2(\hat{n}\Psi, P)_\Omega - i(\hat{\eta}\Psi, P)_{\Gamma_I} = -(\alpha_0/2)\overline{\langle I'_U(\hat{U}), \Psi\rangle} \ \forall \Psi \in X \qquad (20)$$

and the minimum principle holds which is equivalent to inequalities

$$\alpha_1(\hat{n}, n - \hat{n})_{r,\Omega} - k^2 \mathrm{Re}((n - \hat{n})\hat{U}, P)_\Omega \geq 0 \ \ \forall n \in K_1, \qquad (21)$$

$$\alpha_2(\hat{\eta}, \eta - \hat{\eta})_{s,\Gamma_I} - \mathrm{Re}[i((\eta - \hat{\eta})\hat{U}, P)_{\Gamma_I}] \geq 0 \ \ \forall \eta \in K_2. \qquad (22)$$

Direct problem (10), the Euler-Lagrange equation (20) which has the sense of adjoint problem for the adjoint state $P \in X$ and variational inequalities (21), (22) constitute the optimality system for control problem (17). The optimality system plays an important role in studying the properties of solutions of the control problem. On its basis, efficient numerical algorithms of solving problem (17) can be developed. In addition, using analysis of the optimality system one can establish the sufficient conditions on the initial data providing the uniqueness and stability of solutions of particular extremal problems.

4 Uniqueness and Stability of Optimal Solutions

We assume that the incident field u^{inc} can change in a bounded set $K^{inc} \subset H^{inc}$. Denote by $(U_1, n_1, \eta_1) \in X \times K$ a solution of (17) corresponding to given field $u^{inc} = u_1^{inc} \in K^{inc}$. By $(U_2, n_2, \eta_2) \in X \times K$ we denote a solution of problem

$$\tilde{J}(U, \lambda) = \frac{\alpha_0}{2}\tilde{I}(U) + \frac{\alpha_1}{2}\|n\|^2_{r,\Omega} + \frac{\alpha_2}{2}\|\eta\|^2_{s,\Gamma_I} \to \inf,$$

$$G(U, \lambda, \tilde{u}^{inc}) = 0, \ (U, \lambda) \in X \times K, \ \ \lambda = (n, \eta). \qquad (23)$$

It is obtained from (17) by replacing functional $I(U)$ by another functional $\tilde{I}(U)$ and replacing incident field u^{inc} by another incident field $\tilde{u}^{inc} = u_2^{inc} \in K^{inc}$. We assume that the set K is bounded and derive one important inequality for the difference of solutions of problems (17) and (23). We note firstly that by Theorem 1 the following estimates hold for U_l, $l=1,2$:

$$\|U_l\|_X \leq M_U = C_0 \sup \|u^{inc}\|_{1,\Omega_e}, \quad u^{inc} \in K^{inc}. \tag{24}$$

Denote by $P_l \in X$, $l = 1,2$ Lagrange multipliers corresponding to solutions (U_l, n_l, η_l). By Theorem 3 $P_l, l=1,2$ satisfy identity

$$a_0(\Psi, P_l) - k^2(n_l\Psi, P_l)_\Omega - i(\eta_l\Psi, P_l)_{\Gamma_I} = -(\alpha_0/2)\overline{\langle I'_U(U_l), \Psi\rangle}\ \forall\Psi \in X. \tag{25}$$

Set $u^{inc} = u_1^{inc} - u_2^{inc}$,

$$n = n_1 - n_2,\ \eta = \eta_1 - \eta_2,\ U = U_1 - U_2, P = P_1 - P_2,\ f = f_1 - f_2. \tag{26}$$

We subtract the identity (10) written for U_2, n_2, η_2, u_2^{inc} from (10) for U_1, n_1, η_1, u_1^{inc} to obtain the following equation for the difference $U = U_1 - U_2$:

$$\begin{aligned} &a_0(U,\Phi) - k^2(n_2U,\Phi)_\Omega - i(\eta_2U,\Phi)_{\Gamma_I} = \\ &= k^2(nU_1,\Phi)_\Omega + i(\eta U_1,\Phi)_{\Gamma_I} + \langle f,\Phi\rangle\ \ \forall\Phi \in X. \end{aligned} \tag{27}$$

Using estimates (7), (14) and (24) we deduce that

$$|k^2(nU_1,\Phi)_\Omega| \leq C_1C'_r\|n\|_{r,\Omega}M_U\|\Phi\|_X,\ |(\eta U_1,\Phi)_{\Gamma_I}| \leq C_1C_s\|\eta\|_{s,\Gamma_I}M_U\|\Phi\|_X. \tag{28}$$

It follows from (28) and Theorem 1 applied to problem (27) for U that

$$\|U\|_X \leq C_0(C'_rM_U\|n\|_{r,\Omega} + C_sM_U\|\eta\|_{s,\Gamma_I} + \|u^{inc}\|_{1,\Omega_e}),\ C_0 := C_1\tilde{C}_0. \tag{29}$$

We set $n = n_1$ in (21) written at $\hat{n} = n_2$, $\hat{U} = U_2$, $P = P_2$ and then we set $n = n_2$ in (21) written at $\hat{n} = n_1$, $\hat{U} = U_1$, $P = P_1$. Using (26) we obtain

$$\alpha_1(n_2,n)_{r,\Omega} - k^2\mathrm{Re}[(n\,U_2,P_2)_\Omega] \geq 0,\ -\alpha_1(n_1,n)_{r,\Omega} + k^2\mathrm{Re}[(n\,U_1,P_1)_\Omega] \geq 0.$$

Adding these inequalities we arrive at the following inequality for n, U and P:

$$-k^2\mathrm{Re}[(n\,U,P_1)_\Omega + (n\,U_2,P)_\Omega] \leq -\alpha_1\|n\|^2_{r,\Omega}. \tag{30}$$

In a similar manner we obtain the following inequality for η, U and P:

$$-\mathrm{Re}[i(\eta\,U,P_1)_{\Gamma_I} + i(\eta\,U_2,P)_{\Gamma_I}] \leq -\alpha_1\|\eta\|^2_{s,\Gamma_I}. \tag{31}$$

Let us subtract (25) at $l=2$ from (25) at $l=1$. Setting $\Psi = U$ we obtain

$$\begin{aligned} &a_0(U,P) - k^2(n_2U,P)_\Omega - k^2(nU,P_1)_\Omega - i(\eta_2U,P)_{\Gamma_I} - \\ &-i(\eta U,P_1)_{\Gamma_I} = -(\alpha_0/2)\overline{\langle I'_U(U_1) - \tilde{I}'_U(U_2),U\rangle}. \end{aligned} \tag{32}$$

We set $\Phi = P$ in (27), subtract from (32) and add the real part of obtained result with (30) and (31). Using relation $(nU,P_1)_\Omega + (nU_2,P)_\Omega - (nU_1,P)_\Omega = (nU,P_1)_\Omega - (nU,P)_\Omega = (nU,P_2)_\Omega$ and analogous one for terms in (32) containing η we arrive at the inequality

$$\begin{aligned} &(\alpha_0/2)\mathrm{Re}[\langle I'_U(U_1) - \tilde{I}'_U(U_2),U\rangle] \leq -\alpha_1\|n\|^2_{r,\Omega} - \alpha_2\|\eta\|^2_{s,\Gamma_I} + \\ &+\mathrm{Re}[k^2(nU,P_1+P_2)_\Omega + i(\eta U,P_1+P_2)_{\Gamma_I} - \langle f,P\rangle]. \end{aligned} \tag{33}$$

Let us formulate obtained results as the Lemma.

Lemma 1. *Let in addition to conditions (j) K and $K^{inc} \subset H^{inc}$ be bounded sets and let the triples (U_1, n_1, η_1) and (U_2, n_2, η_2) be solutions of problems (17) at $u^{inc} = u_1^{inc} \in K^{inc}$ and (23) at $\tilde{u}^{inc} = u_2^{inc} \in K^{inc}$, respectively. Let functionals $I(U)$ and $\tilde{I}(U)$ be continuously differentiable and let P_l be Lagrange multipliers corresponding to (U_l, n_l, η_l), $l = 1, 2$. Then the estimate (29) for the difference $U := U_1 - U_2$ and inequality (33) for differences defined in (26) hold.*

Based on Lemma 1 we are able now to study uniqueness and stability of solutions of control problem

$$J_j(U, n, \eta) \to \inf,\ G(U, \lambda, u^{inc}) = 0,\ (U, \lambda) \in X \times K,\ \ \lambda = (n, \eta),\ j = 1, 2 \tag{34}$$

for particular cost functionals defined in (4), (14). We begin with the case $j = 1$ corresponding to functional $I_1(U) := \|U - u^d\|_Q^2$. Denote by (U_1, n_1, η_1) the solution of (34) at $j = 1$ corresponding to given functions $u^d = u_1^d \in L^2(Q)$ and $u^{inc} = u_1^{inc} \in K^{inc} \subset H^{inc}$. By (U_2, n_2, η_2) we denote the solution of (34) at $j = 1$ corresponding to perturbed functions $\tilde{u}^d = u_2^d \in L^2(Q)$ and $\tilde{u}^{inc} = u_2^{inc} \in K^{inc}$. Setting $u^d = u_1^d - u_2^d$ in addition to relation (26) we have

$$\langle I_1'(U_l), \Psi \rangle = 2(U_l - u_l^d, \Psi)_Q,\ \ \langle I_1'(U_1) - \tilde{I}_1'(U_2), U) \rangle = 2(\|U\|_Q^2 - (U, u^d)_Q).$$

Then the identity (25) for Lagrange multiplier $P_l \in X$ and inequality (33) for differences (26) take the form

$$a_0(\Psi, P_l) - k^2(n_l\Psi, P_l)_\Omega - i(\eta_l\Psi,\ P_l)_{\Gamma_I} = -\alpha_0(\Psi, U_l - u_l^d)_Q\ \forall \Psi \in X, \tag{35}$$

$$\alpha_0(\|U\|_Q^2 - \mathrm{Re}(U, u^d)_Q) \le \mathrm{Re}[k^2(nU, P_1 + P_2)_\Omega +$$
$$+ i(\eta U, P_1 + P_2)_{\Gamma_I} - \langle f, P \rangle] - \alpha_1\|n\|_{r,\Omega}^2 - \alpha_2\|\eta\|_{s,\Gamma_I}^2. \tag{36}$$

Firstly we estimate multipliers P_1, P_2 and the term $\langle f_i, P \rangle$ entering into the right-hand side of (36). To this end we consider the problem (35) for Lagrange multiplier P_l which is equivalent to the following equation: $A_{\lambda_l}^* P_l = \alpha_0 f_l \in X^*$, $\langle f_l, \Psi \rangle = -(\Psi, U_l - u_l^d)_Q$, $l = 1, 2$. Here $A_{\lambda_l}^*$ is an adjoint operator of A_{λ_l}. It is defined by $\langle A_{\lambda_l}^* P, \Phi \rangle = \overline{a_{\lambda_l}(\Phi, P)} = \overline{\langle A_{\lambda_l}\Phi, P \rangle}$ for all $P \in X$, $\Phi \in X$. Since $|(\Psi, U_l - u_l^d)_Q| \le [\|U_l\|_X + \max(\|u_1^d\|_Q, \|u_2^d\|_Q)]\|\Psi\|_X$ then using the properties of adjoint operators and Theorem 1 we derive the estimate

$$\|P_l\|_X \le \tilde{C}_0\alpha_0 M_U^0,\ l = 1, 2,\ M_U^0 = M_U + \max(\|u_1^d\|_Q, \|u_2^d\|_Q). \tag{37}$$

Taking into account (14), (26) and (37) we deduce that

$$|\langle f, P \rangle| \le C_1\|u^{\mathrm{inc}}\|_X\|P_1 + P_2\|_X \le \alpha_0 a\|u^{inc}\|_X,\ a := 2C_0 M_U^0. \tag{38}$$

Using estimates (7), (14), (29), (37) and Young's inequality $2cd \le \epsilon c^2 + (1/\epsilon)d^2$ for all $c \ge 0$, $d \ge 0$, $\epsilon > 0$ at $\epsilon = 1$ we have

$$|k^2(nU,\ P_1 + P_2)_\Omega| \le C_1\|n\|_{L^\infty(\Omega)}\|U\|_X\|P_1 + P_2\|_X \le$$
$$\le 2C_1C_0\tilde{C}_0\alpha_0 M_U^0 C_r'\|n\|_{r,\Omega}(C_r' M_U\|n\|_{r,\Omega} + C_s M_U\|\eta\|_{s,\Gamma_I}$$

$$+\|u^{inc}\|_{1,\Omega_e}) \le \alpha_0 b(4C_r'^2 M_U^2 \|n\|^2_{r,\Omega} + C_s^2 M_U^2 \|\eta\|^2_{s,\Gamma_I} + \|u^{inc}\|^2_{1,\Omega_e}), \quad (39)$$

$$|i(\eta U,\, P_1+P_2)_{\Gamma_I}| \le C_1 \|\eta\|_{L^\infty(\Gamma_I)} \|U\|_X \|P_1+P_2\|_X \le$$

$$\le \alpha_0 b(C_r'^2 M_U^2 \|n\|^2_{r,\Omega} + 4C_s^2 M_U^2 \|\eta\|^2_{s,\Gamma_I} + \|u^{inc}\|^2_{1,\Omega_e}),\ b := C_0^2 M_U^0 M_U^{-1}. \quad (40)$$

We assume that the following conditions take place:

$$\alpha_1(1-\varepsilon) > 5\alpha_0 b C_r'^2 M_U^2,\ \alpha_2(1-\varepsilon) > 5\alpha_0 b C_s^2 M_U^2 \quad (41)$$

where $\varepsilon \in (0,1)$ is an arbitrary constant. Using (41) we derive from (39), (40)

$$\mathrm{Re}|k^2(nU, P_1+P_2)_\Omega + i(\eta U, P_1+P_2)_{\Gamma_I}| \le$$

$$\le \alpha_1(1-\varepsilon)\|n\|^2_{r,\Omega} + \alpha_2(1-\varepsilon)\|\eta\|^2_{s,\Gamma_I} + 2\alpha_0 b \|u^{inc}\|^2_{1,\Omega_e}. \quad (42)$$

Taking into account (38) and (42) from (36) we obtain

$$\alpha_0\|U\|^2_Q \le \alpha_0 \mathrm{Re}(U, u^d)_Q - \varepsilon\alpha_1\|n\|^2_{r,\Omega} - \varepsilon\alpha_2\|\eta\|^2_{s,\Gamma_I} + \alpha_0\varphi(\|u^{inc}\|_{1,\Omega_e}). \quad (43)$$

Here function $\varphi(\cdot)$ is defined by

$$\varphi(\|u^{inc}\|_{1,\Omega_e}) = \big(a\|u^{inc}\|_{1,\Omega_e} + 2b\|u^{inc}\|^2_{1,\Omega_e}\big)^{1/2} \quad (44)$$

where constants a and b are defined in (38) and (39). Omitting nonpositive term $-\varepsilon\alpha_1\|n\|^2_{r,\Omega} - \varepsilon\alpha_2\|\eta\|^2_{s,\Gamma_I}$ in the right-hand side of (43) we have $\|U\|^2_Q \le \|U\|_Q\|u^d\|_Q + a\|u^{inc}\|_{1,\Omega_e} + 2b\|u^{inc}\|^2_{1,\Omega_e}$. This is a quadratic inequality for $\|U\|_Q$. Solving it for $\|U\|_Q = \|U_1 - U_2\|_Q$ we obtain the estimate

$$\|U_1 - U_2\|_Q \le \|u_1^d - u_2^d\|_Q + \varphi(\|u_1^{inc} - u_2^{inc}\|_{1,\Omega_e}). \quad (45)$$

If $u_1^{inc} = u_2^{inc}$ (45) transforms to the estimate $\|U_1 - U_2\|_Q \le \|u_1^d - u_2^d\|_Q$.

Using estimate (45) and inequality $\|U\|_Q\|u^d\|_Q \le \|U\|^2_Q + (1/4)\|u^d\|^2_Q$ which follows from Young's inequality we obtain from (43) that

$$\varepsilon\alpha_1\|n\|^2_{r,\Omega} + \varepsilon\alpha_2\|\eta\|^2_{s,\Gamma_I} \le \alpha_0[(1/2)\|u^d\|_Q + \varphi(\|u^{inc}\|_{1,\Omega_e})]^2. \quad (46)$$

From (46) and (29) we deduce the estimates:

$$\|n_1 - n_2\|_{r,\Omega} \le \sqrt{\alpha_0/\varepsilon\alpha_1}\Delta,\ \|\eta_1 - \eta_2\|_{s,\Gamma_I} \le \sqrt{\alpha_0/\varepsilon\alpha_2}\Delta,$$

$$\|U_1 - U_2\|_X \le C_0(C_r' M_U\sqrt{\alpha_0/\varepsilon\alpha_1}\Delta + C_s M_U\sqrt{\alpha_0/\varepsilon\alpha_2}\Delta + \|u_1^{inc} - u_2^{inc}\|_{1,\Omega_e}) \quad (47)$$

where

$$\Delta = (1/2)\|u_1^d - u_2^d\|_Q + \varphi(\|u_1^{inc} - u_2^{inc}\|_{1,\Omega_e}). \quad (48)$$

The estimates (47) have the sense of stability estimates of the solution $(\hat{U}, \hat{n}, \hat{\eta})$ of problem (34) at $j = 1$ with respect to small perturbations of functions $u^d \in L^2(Q)$ and $u^{inc} \in H^{inc}$. We formulate the obtained result as

Theorem 4. *Let in addition to conditions (j)* $K := K_1 \times K_2$ *and* $K^{inc} \subset H^{inc}$ *be bounded sets and let the triple* $(U_l, n_l, \eta_l) \in X \times K$ *be a solution of problem (34) at* $j = 1$ *corresponding to given functions* $u_l^d \in L^2(Q)$ *and* $u_l^{inc} \in K^{inc}$, $l = 1, 2$, *where* $Q \subset \Omega_e$ *is a nonempty open subset. Suppose that conditions (41) take place. Then the stability estimates (45) and (47) hold where* Δ *is given by (48).*

Similar result holds and for problem (34) at $j = 2$ corresponding to $I_2(U)$.

5 Numerical Algorithms

Optimality system (10), (20), (21), (22) derived above can be used to design efficient numerical algorithms for solving control problem (34). The simplest one (Algorithm 1) for $I_1(U)$ can be obtained by applying simple iteration method for solving the optimality system. The m-th iteration of this algorithm consists of finding unknown values U^m, P^m, n^{m+1} and η^{m+1} for given n^m and η^m by sequentially solving following problems:

$$a_0(U^m, \Phi) - k^2(n^m U^m, \Phi)_\Omega - i(\eta^m U^m, \Phi)_{\Gamma_I} = \langle f^{inc}, \Phi\rangle \;\forall \Phi \in X, \tag{49}$$

$$a_0(\Psi, P^m) - k^2(n^m \Psi, P^m)_\Omega - i(\eta^m \Psi, P^m)_{\Gamma_I} = -\alpha_0(\Psi, U^m - u^d)_Q \;\forall \Psi \in X, \tag{50}$$

$$\alpha_1(n^{m+1}, n - n^m)_{r,\Omega} - k^2 \mathrm{Re}((n - n^{m+1})U^m, P^m)_\Omega \geq 0 \;\forall n \in K_1, \tag{51}$$

$$\alpha_2(\eta^{m+1}, \eta - \eta^m)_{s,\Gamma_I} - \mathrm{Re}[i((\eta - \eta^{m+1})U^m, P^m)_{\Gamma_I}] \geq 0 \;\forall \eta \in K_2. \tag{52}$$

For discretization and solving problems (49), (50) one can use open source software free FEM++ (www.freefem.org) based on using finite element method. For discretization of (51), (52) it is convenient to look for solutions n and η as

$$n(x) = \sum_{j=1}^{N} n_j \varphi_j(x),\ x \in \Omega,\ \eta(x) = \sum_{k=1}^{M} \eta_k \psi_k(x),\ x \in \Gamma_1. \tag{53}$$

Here $\varphi_j \in H^r(\Omega)$ and $\psi_k \in H^s(\Gamma_I)$ are nonnegative basis functions in $H_+^r(\Omega)$ and $H_+^s(\Gamma_I)$ and $n_j \geq 0$ and $\eta_k \geq 0$ are unknown coefficients. Similar algorithm which is based on the strategy "optimize-then-discretize" can be used and for functional $I_2(U)$.

Now we discuss another algorithm (Algorithm 2) which is based on the opposite strategy: "discretize-then-optimize". The idea of this algorithm consists of seeking unknown controls – refraction index n and surface conductivity η in the form (53). Here n_j and η_k are unknown coefficients which one should define from the condition of minimum of the discrete analogue of functional $I_1(U)$ (or $I_2(U)$) in (4) which has the form

$$I_1(n_1, \ldots, n_N, \eta_1, \ldots, \eta_M) = \int_Q |U(n_1, \ldots, n_N, \eta_1, \ldots, \eta_M) - u^d|^2 dx. \tag{54}$$

Here $U(n_1, \ldots, n_N, \eta_1, \ldots, \eta_M)$ is a solution of the direct problem (1)–(3) for the case when parameters $n(x)$ and $\eta(x)$ have the form (53). In such a case the discrete analogue of problem (34) takes the form

$$\frac{\alpha_0}{2} I_1(n_1, \ldots, n_N, \eta_1, \ldots, \eta_M) + \frac{\alpha_1}{2}\sum_{j=1}^{N} n_j^2 + \frac{\alpha_2}{2}\sum_{k=1}^{M} \eta_k^2 \to \inf, \tag{55}$$

where $n_j \geq 0, \eta_k \geq 0,\ j = 1, \ldots, N,\ k = 1, \ldots, M$.

Problem (55) represents the finite-dimensional problem of conditional minimization which can be solved numerically using known methods of solution of discrete extremum problems. The formal comparison of both algorithms shows that Algorithm 1 is more complicated and more expensive in terms of CPU time and memory space. This is due to the fact that Algorithm 1 is based on solving the optimality system (10), (20), (21), (22) which involves a coupled system of state and adjoint equation together with two variational inequalities for sought for controls.

6 Concluding Remarks

We studied control problems for the 2D electromagnetic field model describing scattering TM-polarized electromagnetic waves by a penetrable inhomogeneous dielectric obstacle. These problems arise when optimization method is applied for solving cloaking problems for respective scattering model. The refraction index $n(x)$ of the inhomogeneous medium filling the obstacle and the boundary conductivity $\eta(x)$ of the coated part of the boundary play the role of controls. We studied some new properties of solutions of the direct problem, proved the solvability of control problems and derived the optimality systems describing the necessary conditions of extremum. Based on analysis of the optimality system we established the uniqueness and stability estimates of optimal solutions. Besides, we proposed two numerical algorithms for solving our cloaking problems. Separate paper by the authors will be devoted to comparative study of properties of these algorithms and to detailed analysis of results of numerical experiments.

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