Solution of the Contact Elasticity Problem Based on an Iterative Proximal Regularization Method for the Modified Lagrangian Functional

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Abstract. The method of successive approximation is considered for solving the contact elasticity problem which corresponds to the quasivariational Signorini problem. Auxiliary problems with given frictions arising from each external step of this method are solved by the Uzawa method with iterative proximal regularization of the modified Lagrangian functional. Stabilization of the sequence of auxiliary finite-element solutions of external steps of successive approximation is investigated. Numerical results are considered.

Keywords: proximal regularization, Lagrangian functional, saddle point, Uzawa method, Delaunay triangulation, finite element method

1 Introduction

In this paper, we consider the contact problem of elasticity with friction between an elastic body Ω and absolutely rigid foundation (see [1,2]). Such problem corresponds to quasivariational Signorini inequality. The method of successive approximation is used for solving such inequality, on each external step of which the auxiliary contact problem with given friction arises. For solving semicoercive auxiliary problem we consider Uzawa method, based on modified Lagrangian functional and investigated in [3].

To overcome the problem of singularity (semicoercivity) we use the iterative proximal regularization method of modified Lagrangian functional, considered in [4].

In this paper, we consider the finite element solution of quasivariational inequality. Domain Ω is decomposed by Delaunay triangulation with the help of mesh concentration near the contact zone between elastic body and absolutely rigid foundation.

2 Formulation of the Problem

Let $\Omega \subset R^2$ be a domain with a sufficiently regular boundary Γ, $\Gamma = \Gamma_0 \bigcup \Gamma_1 \bigcup \Gamma_2$, where Γ_0, Γ_1, Γ_2 are open, mutually disjoint subsets of Γ; moreover, Γ_0 and Γ_2 are

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nonempty (that is, $mes\Gamma_0 > 0$ and $mes\Gamma_2 > 0$) (see fig. 1). We consider the contact problem of elasticity with friction between elastic body Ω and absolutely rigid foundation.

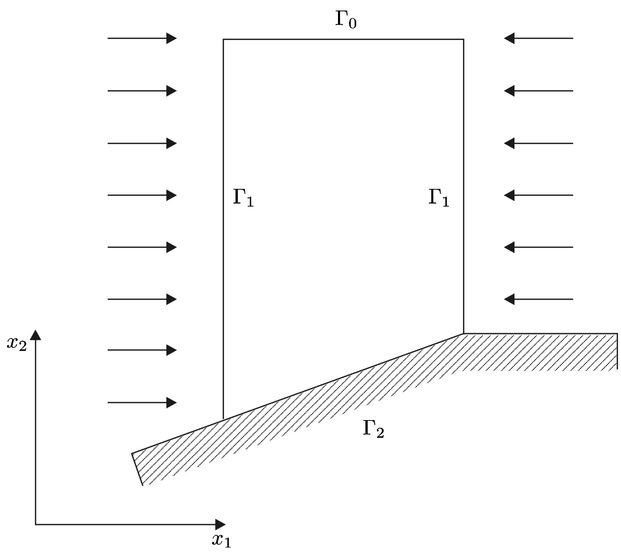


Fig. 1: Contact between elastic body and absolutely rigid foundation

For displacement vector $v = (v_1, v_2)$, we define the strain tensor

$$\varepsilon_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \qquad i, j = 1, 2,$$

and the stress tensor

$$\sigma_{ij}(v) = c_{ijkm}\varepsilon_{km}(v).$$

Here, $i, j, k, m = 1, 2$; $c_{ijkm} = c_{jimk} = c_{kmij}$, and repeated indices indicate summation.

For given functions $f = (f_1, f_2)$, $p = (p_1, p_2)$ and F consider the following boundary value problem (see [1]):

$$\begin{gathered} -\frac{\partial \sigma_{ij}}{\partial x_j} = f_i \ \text{ in } \ \Omega, \ i = 1, 2 \\ u_n = 0, \ \sigma_\tau = 0 \ \text{ on } \ \Gamma_0 \\ \sigma_{ij} n_j = p_i \ \text{ on } \ \Gamma_1, \ i = 1, 2 \end{gathered} \tag{1}$$

On the contact surface Γ_2 of the elastic body and the absolutely rigid support, we impose the conditions:

$$\begin{gathered} u_n \le 0, \ \sigma_n \le 0, \ u_n\sigma_n = 0 \ \text{ on } \ \Gamma_2 \\ |\sigma_\tau| \le F|\sigma_n|, \ (F|\sigma_n| - |\sigma_\tau|)u_\tau = 0, \ u_\tau \cdot \sigma_\tau \le 0 \ \text{ on } \ \Gamma_2 \end{gathered} \tag{2}$$

Here, $n = (n_1, n_2)$ is the unit outward-pointing normal to Γ, $u_n = u \cdot n$, $u_\tau = u - u_n n$; $\sigma_i = \sigma_{ij} n_j$, $i = 1, 2$; $\sigma = (\sigma_1, \sigma_2)$, $\sigma_n = \sigma_{ij} n_i n_j$, $\sigma_\tau = \sigma - \sigma_n n$; and frictional coefficient $F \geq 0$ on Γ_2.

The main difficulty in the study and construction of numerical methods for solving this nonlinear boundary value problem is that the frictional force $F|\sigma_n(u)|$ is a function of desired solution u.

Define the sets

$$V = \left\{ v \in [W_2^1(\Omega)]^2 : \ v_n \equiv v_2 = 0 \text{ on } \Gamma_0 \right\},$$
$$K = \{ v \in V : \ v_n \leq 0 \text{ on } \Gamma_2 \}.$$

Assume that functions $c_{ijkm} \in L_\infty(\Omega)$ $(i, j, k, m = 1, 2)$, $f \in [L_2(\Omega)]^2$, $p \in [L_2(\Gamma_1)]^2$ and $F \in L_2(\Gamma_2)$. Suppose that the solution u to the boundary value problem (1), (2) exists and belongs to the space $[W_2^2(\Omega)]^2$. Then u satisfies the quasivariational Signorini inequality for $\forall\ v \in K$ (see [1,2])

$$a(u, v - u) + \int_{\Gamma_2} F|\sigma_n(u)|(|v_\tau| - |u_\tau|)\, d\Gamma \geq \int_{\Omega} f \cdot (v - u)\, d\Omega + \int_{\Gamma_1} p \cdot (v - u)\, d\Gamma \ , \quad (3)$$

where

$$a(u, v) = \int_{\Omega} \sigma_{ij}(u) \varepsilon_{ij}(v)\, d\Omega = \int_{\Omega} c_{ijkm} \varepsilon_{km}(u) \varepsilon_{ij}(v)\, d\Omega \ . \quad (4)$$

We use the successive approximation method for solving the quasivariational inequality (see [1,2]):

1. Set starting friction force $g^0 \in W_2^{1/2}(\Gamma_2)$, $g^0 \geq 0$.
2. Find u^k as a solution of auxiliary variational inequality

$$a(u^k, v - u^k) + \int_{\Gamma_2} g^k(|v_\tau| - |u_\tau^k|)\, d\Gamma \geq \int_{\Omega} f \cdot (v - u^k)\, d\Omega + \int_{\Gamma_1} p \cdot (v - u^k)\, d\Gamma \ . \quad (5)$$

3. Correct friction force $g^{k+1} = F|\sigma_n(u^k)|$.

The convergence of the successive approximation method to the solution to the quasivariational Signorini inequality (3) is still an open question. The existence of the solution is proved in the coercive case for a sufficiently small coefficient F only (see [1]).

Variational inequality (5) is called the problem with given friction g^k. It is equivalent to the following constrained non-differentiable minimization problem:

$$\begin{cases} J(v) = \dfrac{1}{2} a(v, v) + \int_{\Gamma_2} g^k |v_\tau|\, d\Gamma - \int_{\Omega} f \cdot v\, d\Omega - \int_{\Gamma_1} p \cdot v\, d\Gamma \to min \ , \\ v \in K \ . \end{cases} \quad (6)$$

Under geometric form of domain Ω shown in fig. 1, functional $J(v)$ is not strongly convex on all space V (see [1,2]). The kernel of bilinear form $a(u, v)$ is not trivial and consists of vector-functions $\rho = (\overline{a}, 0)$, where $\overline{a}$ is an arbitrary number.

However, if

$$\int_{\Omega} f_1 \, d\Omega + \int_{\Gamma_1} p_1 \, d\Gamma > 0$$

then

$$J(v) \to +\infty \quad \text{under} \quad \|v\|_{[W_2^1(\Omega)]^2} \to \infty, \quad v \in K,$$

that is, J is a coercive functional on the set K and therefore problem (6) is solvable.

3 Application of Dual Schemes for Solving the Auxiliary Problem with Given Friction

For problem (6), on set $V \times L_2(\Gamma_2)$ we define the classical Lagrangian functional

$$L(v,l) = J(v) + \int_{\Gamma_2} l v_n \, d\Gamma \; .$$

Denote by $(L_2(\Gamma_2))^+$ the set of nonnegative square integrable functions on Γ_2.

Definition 1. *A pair $(v^*, l^*) \in V \times (L_2(\Gamma_2))^+$ is called a saddle point of the Lagrangian functional $L(v,l)$ if it satisfies the two sided inequalities*

$$L(v^*, l) \le L(v^*, l^*) \le L(v, l^*) \quad \forall \; (v,l) \in V \times (L_2(\Gamma_2))^+ \; .$$

It was shown in [3] that, if a solution $\overline{u}$ of auxiliary problem (6) belongs to space $[W_2^2(\Omega)]^2$ and $mes\{x \in \Gamma_2 : \sigma_n(\overline{u}) < 0\} > 0$, then $\overline{u}$ is a unique solution to the problem (6) and the pair $(\overline{u}, -\sigma_n(\overline{u}))$ is a unique saddle point of $L(v,l)$.

Because of the second part of saddle point is equal to $-\sigma_n(u)$, we can find accurately the frictional force on the next step of successive approximation method.

However, application of the dual Uzawa method with the classical Lagrangian functional $L(v,l)$ does not guarantee convergence to a saddle point in the semicoercive case (see [6,7]).

To overcome this difficulty the modified Lagrangian functional $M(v,l)$ was considered on space $V \times L_2(\Gamma_K)$ (see [8],[9]):

$$M(v,l) = J(v) + \frac{1}{2r} \int_{\Gamma_2} \left\{ \left[(l + r v_n)^+ \right]^2 - l^2 \right\} d\Gamma \; ,$$

where $(l + r v_n)^+ \equiv \max\{0, l + r v_n\}$, $r > 0$ – const.

Definition 2. *A pair $(\overline{v}, \bar{l}) \in V \times (L_2(\Gamma_2))$ is called a saddle point of the modified Lagrangian functional $M(v,l)$ if it satisfies the two sided inequalities*

$$M(\overline{v}, l) \le M(\overline{v}, \bar{l}) \le M(v, \bar{l}) \quad \forall \; (v,l) \in V \times (L_2(\Gamma_2)) \; .$$

Definition 2 is different from Definition 1. Here we use all space $L_2(\Gamma_2)$ instead of $L_2(\Gamma_2)^+$ in the first definition.

However, it is known that the functionals $L(v,l)$ and $M(v,l)$ have the same set of saddle points (see [3], [10]).

For finding a saddle point of modified Lagrangian functional $M(v,l)$ we use the method based on the combination of Uzawa method with the proximal regularization (see [4]). We note that similar method in finite-dimensional case was investigated in [5].

According to this method, the sequence $\{(u^m, l^m)\}$ is generated as follows.

1. Assign the initial approximation $(u^0, l^0) \in V \times W_2^{\frac{1}{2}}(\Gamma_2)$.
2. Find u^{m+1} such that
$$\|u^{m+1} - \overline{u}^{m+1}\|_{[W_2^1(\Omega)]^2} \leq \delta_m, \tag{7}$$
where
$$\overline{u}^{m+1} = \arg\min_{v\in V} \left\{ M(v,l^m) + \frac{1}{2}\|v-u^m\|^2_{[L_2(\Omega)]^2} \right\}, \tag{8}$$
$$\delta_m > 0,\ \sum_{m=1}^{\infty} \delta_m < \infty, \tag{9}$$
3. Calculate the next value of the dual variable by the formula
$$l^{m+1} = (l^m + r u_n^m)^+. \tag{10}$$

Criterion (7) implies that the exact solution $\overline{u}^{m+1}$ is replaced by its approximation u^{m+1}, obtained by solving problem (8) numerically using the finite element method. In this case parameter δ_m can be interpreted as the error of the numerical solution.

The regularizing term $\frac{1}{2}\|v-u^m\|^2_{[L_2(\Omega)]^2}$ ensures that the functional to be minimized in (8) is strongly convex on V. This guarantees that the auxiliary problem (5) is uniquely solvable.

Application of the modified Lagrangian fucntional allows as to find saddle point efficiently in comparison with duality methods based on classical Lagrangian functional.

It should be noted that in article [11] it was built and investigated a wide class of iterative solution methods of the semicoercive variational inequalities including the iterative proximal regularization method.

Discuss discretization of the variational problem (8).

4 Finite Element Discretization

Domain Ω is taken in the form of a trapezoid (fig. 1) with sides 2, 1, 1.5, $\sqrt{5}/2$. Under the proposed algorithm (7)–(10) on each step of the iteration process, we regard the minimization problem of the strongly convex functional

$$M_m(v) = M(v,l^m) + \frac{1}{2}\|v-u^m\|^2_{[L_2(\Omega)]^2} \to min, \quad v \in V \tag{11}$$

For the solving of the problem (11) finite element method have been applied. Using the Delaunay triangulation, we triangulate Ω into a set of triangles T_k (fig. 2), so that $\Omega = \bigcup_1^{N_t} T_k$, where N_t is the number of triangles. Thus we have a finite element consisting of triangles, which have one point in common at the triangulation nodes. Condensation of the mesh occurs near the contact zone Γ_2.

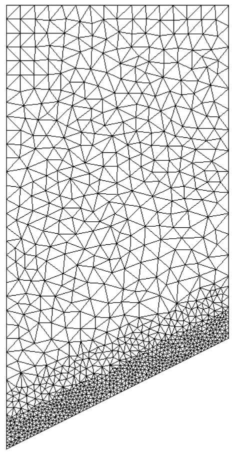


Fig. 2: Delaunay triangulation of Ω

Enumerate triangulation nodes from the top down, from 1 to N. For each node i it is defined basis function $\varphi_i(x, y)$, for which $\varphi_i(x^i, y^i) = 1$ and for all neighboring nodes j: $\varphi_i(x^j, y^j) = 0$. For basis functions φ_i we take piecewise linear functions (see [12]).

Let us introduce the following notation: h - maximum edge length of the triangulation T_k, $P_h = \{D_1, \cdots, D_N\}$ - triangulation node set , $I_h = \{M_1, \cdots, M_R\}$ - boundary node set on Γ_2, V_h - linear span of the basis functions $\varphi_i(x, y)$, $u_h = (u_1^h, u_2^h)$ - piecewise interpolation of the exact solution u:

$$u_i^h(x, y) = \sum_{j=1}^{N} t_j^{(i)} \varphi_j(x, y), \quad \text{for } i = 1, 2 \text{ and } t_j^{(i)} \in R\,.$$

Note that since Ω is a polygon, then embedding $V_h \subset V$ is provided. Thus we can substitute the problem (8) by finite-element problem:

$$u^{m+1} = \arg\min_{v \in V_h} \{M_m(v)\}\,. \tag{12}$$

There is shown in [13] that error estimate is true for the exact solution sequence $\{\overline{u}^m\}$:

$$\|u^m - \overline{u}^m\|_{[W_2^1(\Omega)]^2} \le Ch^{1/2}, \quad C > 0 - const, \quad m = 1, 2, \ldots, \tag{13}$$

and finite-element solution sequence $\{u^m\}$ converges on V to solution of the auxiliary problem (6) under $h \to 0$.

Let us introduce the vector $t = (t_1, t_2, \dots, t_{2N})$, where the first N its components correspond to $u_1^h(D_i(x^i, y^i))$ and last N components correspond to $u_2^h(D_i(x^i, y^i))$. Then the minimization problem (11) reduces to finding optimal values t_i. For this purpose, we use coordinate descent method.

Let stiffness matrix A and vector F be defined as follows

$$A = \left\{ a(\psi_i, \psi_j) + \int\limits_{\Omega} \psi_i \psi_j \, d\Omega \right\}_{i,j=\overline{1,2N}} ,$$

$$F = \left\{ \int\limits_{\Omega} (f + u^m)\psi_i \, d\Omega + \int\limits_{\Gamma_1} p\psi_i \, d\Gamma \right\}_{i=\overline{1,2N}} ,$$

where ψ_i are vector-functions such that

$$\psi_i = \begin{cases} (\varphi_i, 0) \ , & i \le N \ , \\ (0, \varphi_{(i-N)}) \ , & i > N \ . \end{cases}$$

Thus we have the computational formula for all nodes $D_i \in P_h \setminus I_h$

$$t_i^{s+1} = -\frac{1}{A_{ii}} \left(\sum_{j<i} A_{ij} t_j^{s+1} + \sum_{j>i} A_{ij} t_j^s - F_i \right) . \tag{14}$$

Consider next a problem for nodes $D_i \in I_h$. For this purpose, we pass from variables u_1, u_2 to u_n, u_τ, where α is an acute angle of the domain Ω:

$$\begin{cases} u_1 = u_n \cos\alpha + u_\tau \sin\alpha \ , \\ u_2 = -u_n \sin\alpha + u_\tau \cos\alpha \ . \end{cases}$$

For evaluation the boundary integral on Γ_2

$$\frac{1}{2r} \int\limits_{\Gamma_2} \left\{ \left[(l^m + r v_n)^+ \right]^2 - (l^m)^2 \right\} d\Gamma$$

we use the trapezoidal rule for $v_n > \frac{l^m}{r}$

$$\frac{1}{2r} \int\limits_{\Gamma_2} \left\{ \left[(l^m + r v_n)^+ \right]^2 - (l^m)^2 \right\} d\Gamma = \int\limits_{\Gamma_2} \left(l^m v_n + \frac{r}{2} v_n^2 \right) d\Gamma \approx$$

$$\sum_{j=1}^{R-1} \left(l^m(M_j) v_n(M_j) + l^m(M_{j+1}) v_n(M_{j+1}) + \frac{r}{2} (v_n^2(M_j) + v_n^2(M_{j+1})) \right) \frac{|M_j, M_{j+1}|}{2} .$$

Let $|M_j, M_{j+1}| = H$ for $j = \overline{1, R-1}$, then introduce some notations

$$A_n = A_{ii} \cos^2\alpha + A_{i+Ni+N} \sin^2\alpha - 2A_{ii+N} \cos\alpha \sin\alpha \ ,$$

$$A_\tau = A_{ii}\cos\alpha\sin\alpha + A_{ii+N}\cos^2\alpha - A_{i+Ni}\sin^2\alpha - A_{i+Ni+N}\cos\alpha\sin\alpha\ ,$$

$$L(D_i) = \frac{\partial}{\partial t_n}\left(\int_{\Gamma_2} l^m v_n\, d\Gamma\right) \approx \begin{cases} l^m(D_i)H\ , & D_i \in \{M_2,\cdots,M_{R-1}\}\ ,\\ l^m(D_i)\dfrac{H}{2}\ , & D_i \in \{M_1, M_R\}\ ,\end{cases}$$

$$B(D_i) = \frac{\partial^2}{\partial t_n^2}\left(\int_{\Gamma_2} \frac{r}{2} v_n^2\, d\Gamma\right) \approx \begin{cases} rH\ , & D_i \in \{M_2,\cdots,M_{R-1}\}\ ,\\ r\dfrac{H}{2}\ , & D_i \in \{M_1, M_R\}\ ,\end{cases}$$

where $t_n = v_n(D_i)$.

To find the optimal t_n compute

$$\phi_i = -\frac{1}{A_n}\Bigg(t_\tau^s A_\tau + \sum_{j<i} A_{ij}\cos\alpha\ t_j^{s+1} + \sum_{\substack{j>i\\ j\neq i+N}} A_{ij}\cos\alpha\ t_j^s -$$

$$-\sum_{j<i} A_{i+Nj}\sin\alpha\ t_j^{s+1} - \sum_{\substack{j>i\\ j\neq i+N}} A_{i+Nj}\sin\alpha\ t_j^s - F_i\cos\alpha + F_{i+N}\sin\alpha\Bigg)\ .$$

Let us denote an expression within the parentheses by ω_i. Then

$$t_n^{s+1} = \begin{cases} \phi_i\ , & \phi_i \le -\frac{l^m(D_i)}{r}\ ,\\ -\dfrac{\omega_i + L(D_i)}{A_n + B(D_i)}\ , & \phi_i > -\frac{l^m(D_i)}{r}\ ,\end{cases} \tag{15}$$

A similar approach can be used for t_τ on $(i+N)th$ step of the Gauss-Seidel method. Approximate

$$\int_{\Gamma_2} g^k|v_t|\, d\Gamma \approx \sum_{j=1}^{R-1}\left(g^k(M_j)|v_\tau(M_j)| + g^k(M_{j+1})|v_\tau(M_{j+1})|\right)\frac{|M_j, M_{j+1}|}{2}\ .$$

Let

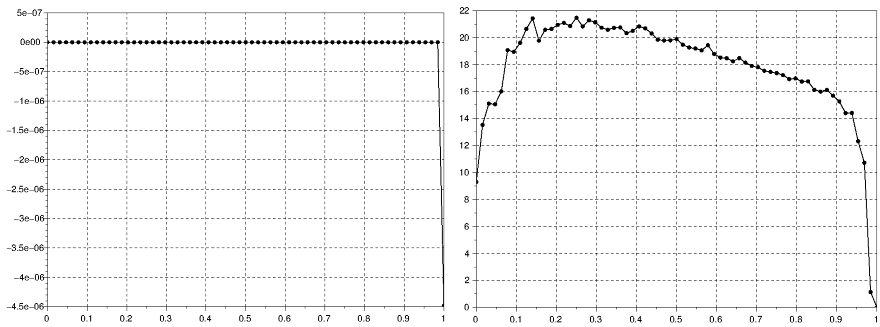
$$A_n^* = A_{ii}\cos\alpha\sin\alpha - A_{ii+N}\sin^2\alpha + A_{i+Ni}\cos^2\alpha - A_{i+Ni+N}\cos\alpha\sin\alpha\ ,$$

$$A_\tau^* = A_{ii}\sin^2\alpha + A_{i+Ni+N}\cos^2\alpha + 2A_{ii+N}\cos\alpha\sin\alpha\ ,$$

$$G(D_i) = \begin{cases} g(D_i)H\ , & D_i \in \{M_2,\cdots,M_{R-1}\}\ ,\\ g(D_i)\dfrac{H}{2}\ , & D_i \in \{M_1, M_R\}\ ,\end{cases}$$

$$\xi_i = t_n^s A_n^* + \sum_{\substack{j<i+N\\ j\neq i}} A_{ij}\sin\alpha\ t_j^{s+1} + \sum_{j>i+N} A_{ij}\sin\alpha\ t_j^s + \sum_{\substack{j<i+N\\ j\neq i}} A_{i+Nj}\cos\alpha\ t_j^{s+1} +$$

$$+\sum_{j>i+N} A_{i+Nj}\cos\alpha\ t_j^s - F_i\sin\alpha + F_{i+N}\cos\alpha\ .$$



(a) u_n on Γ_2 (b) $|\sigma_n|$ on Γ_2

Fig. 3: Computational result #1

Then the computational formula for t_τ can be written as follows

$$t_\tau^{s+1} = \begin{cases} -\dfrac{\xi_i + G(D_i)}{A_\tau^*} \ , & \xi_i < -G(D_i) \ , \\ -\dfrac{\xi_i - G(D_i)}{A_\tau^*} \ , & \xi_i > G(D_i) \ , \\ 0 \ , & -G(D_i) \leq \xi_i \leq G(D_i) \ . \end{cases} \tag{16}$$

Iterations over (14) – (16) stop when

$$\min_i |t_i^{s+1} - t_i^s| < 10^{-10} \ , \ i = \overline{1, 2N} \ .$$

Cessation condition of the iteration process for the Uzawa method and successive approximation method can be defined as follows

$$\min_{i=\overline{1,R}} |l_i^{m+1} - l_i^m| < 10^{-8} \ , \ \min_{i=\overline{1,R}} |g_i^{k+1} - g_i^k| < 10^{-8} \ .$$

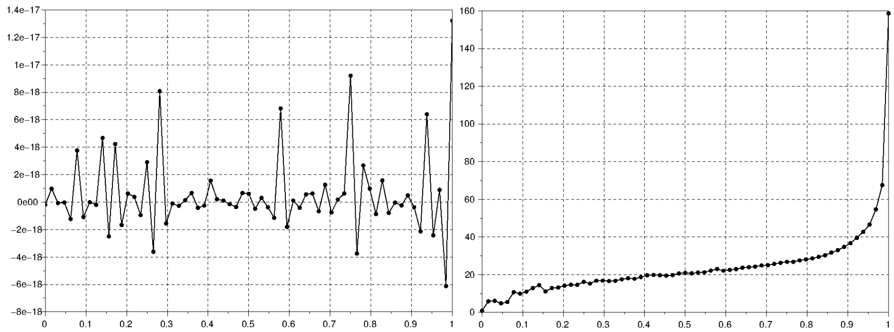
4.1 Numerical Solution of the Quasivariational Inequality

We present the results of numerical computations for solving problem (3). We assume a volume load $f = (f_1, f_2) = (0, 0)$, boundary loading from the left $p_1|_{\Gamma_1} = 27$ mPa and from the right $p_1|_{\Gamma_1} = -27$ mPa sides, $p_2|_{\Gamma_1} = 0$, frictional coefficient $F = 0.3$, Young's modulus $E = 73000$ mPa, Poisson's ratio $\mu = 0.34$, constant $r = 10^8$.

Numerical solution of the considering problem is shown in fig. 3. The graphs of u_n and $|\sigma_n|$ show that body Ω is detached from absolutely rigid foundation at the vertex of the obtuse angle. It follows from the fact that $u_n < 0$ and $|\sigma_n| = 0$ at this vertex.

Let us introduce the example, when body Ω sticks together with absolutely rigid foundation. We change boundary loading from the right side $p_1|_{\Gamma_1} = -21.6$ mPa.

Figure 4 shows that u_n takes on a value close to zero on Γ_2 (less than calculation accuracy $\varepsilon = 10^{-10}$), $|\sigma_n|$ is everywhere positive, whence it follows that elastic body comes in contact with a rigid foundation at all points. The graph of $|\sigma_n|$ states that body is compressed and the maximum stress is achieved at the vertex of the obtuse angle.



(a) u_n on Γ_2 (b) $|\sigma_n|$ on Γ_2

Fig. 4: Computational result #2

Numerical results confirmed that modified Lagrangian functionals effectively remove conditions like $u_n \leq 0$ on Γ_2 when passing to the unconstrained minimization problem. Besides, it was revealed that successive approximation method is more effective at larger r ($r = 10^6, 10^8$).

References

1. Hlaváček, I., Haslinger, J., Nečas, J., Lovíšek, J.: Solution of Variational Inequalities in Mechanics. Springer, New York (1988)
2. Kikuchi, N., Oden, T.: Contact problem in elasticity: a study of variational inequalities and finite element methods. SIAM, Philadelphia (1988)
3. Vikhtenko, E.M., Namm, R.V.: Duality scheme for solving the semicoercive signorini problem with friction. Comput. Math. Math. Phys. 47(12), 2023–2036 (2007)
4. Vikhtenko, E.M., Namm, R.V.: Iterative proximal regularization of the modified Lagrangian functional for solving the quasi-variational Signorini inequality. Comput. Math. Math. Phys. 48(9), 1–9 (2008)
5. Rockafellar, R.T.: Augmented lagrangians and applications of the proximal point algorithm in convex programming, Math. Oper. Res. 1, 97–116 (1976)
6. Glowinski, R., Lions, J.L., Tremolieres, R.: Numerical analysis of variational inequalities. North-Holland, Amsterdam (1981)
7. Glowinski, R.: Numerical methods for nonlinear variational problems. Springer-Verlag, New York (1984)
8. Woo, G., Namm, R.V., Sachkoff, S.A.: An iterative method based on a modified Lagrangian functional for finding a saddle point in the semicoercive Signorini problem. Comput. Math. Math. Phys. 46(1), 26–36 (2006)
9. Woo, G., Kim, S., Namm, R.V., Sachkoff, S.A.: Iterative proximal regularization method for finding a saddle point in the semicoercive Signorini problem. Comput. Math. Math. Phys. 46(11), 2024–2031 (2006)
10. Vikhtenko, E.M., Woo, G., Namm, R.V.: Solution methods for semicoerive variational inequalities of mechanics based on modified Lagrangian functionals. Dal'nevost. Mat. Zh. 14(1), 6–17 (2014)
11. Konnov, I., Gwinner, J.: A strongly convergent combined relaxation in Hilbert Spaces. Numer. Func. Anal. Opt. 35, 1066–1077 (2014)
12. Marchuk, G.I., Agoshkov, V.I.: An Introduction to Projective Grid Methods [in Russian]. Fizmatlit, Moscow (1981)
13. Namm, R.V., Sachkoff, S.A.: Solving the quasi-variational Signorini inequality by the method of successive approximations. Comput. Math. Math. Phys. 49(5), 805–814 (2009)