Convergence of Solutions of an Optimal Control Problem for SP1 and Rosseland Approximations

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Abstract. The optimal control problems for SP_1 and Rosseland approximations of evolution radiative heat transfer are considered. The problems are solved by weak form technique and Lagrange method. Numerical experiments for borosilicate glass are done. Numerical convergence of optimal control problem for SP_1 approximation to Rosseland is studied.

1 Introduction

The interest in studying problems for complex heat transfer models [1], where the radiative, convective, and conductive contributions are simultaneously taken into account, is motivated by their importance for many engineering applications. The common feature of such processes is the radiative heat transfer dominating at high temperatures. The radiative heat transfer equation is a first order integro-differential equation governing the radiation intensity. The radiation traveling along a path is attenuated as a result of absorption and scattering. Solutions to the radiative transfer equation can be represented in the form of the Neumann series whose terms are powers of an integral operator applied to a certain start function. The terms can be calculated using a Monte Carlo method (see e.g. [2], [3]), which may be interpreted as tracking the history of energy bundles from emission to absorption at a surface or within a participating medium.

Numerical and theoretical analysis of one-dimensional heat transfer models coupled with the radiative transfer equation can be found in [4]–[8]. In particular, efficient numerical algorithms are proposed in [6], [7]. In [5]–[8] the unconditional unique solvability of one-dimensional steady-state complex heat transfer problems is proved. Papers [9]–[12] state conditional unique solvability of three dimensions problems for complex heat transfer models. In [13] unconditional unique solvability of boundary-value problem for P1 approximation of 3D complex heat transfer model is proved.

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A considerable number of works of optimal control problems for complex heat transfer models is devoted to the problems of controlling evolutionary systems (see e.g. [14]–[17]). In [18], [19] problems of optimal boundary control for a steady-state complex heat transfer model were considered. On the basis of new a priori estimates of solutions of the control systems, the solvability of the optimal control problems was proved.

In paper [20] optimal control problems in radiative transfer are solved by means of the space mapping technique. Authors constructed fast numerical algorithm for solving problems using a hierarchy of approximate models.

The numerical analysis of optimal control problem of complex heat transfer for SP_1 approximation using weak form technique and Lagrange method was considered in [21]. Using a similar approach we study the optimal control problem for a simplified model given by the Rosseland approximation. Moreover, we further consider the numerical convergence of optimal control problem in SP_1 approximation to Rosseland.

2 Rosseland Approximation

Let, $x \in \Omega, t \in [0, \mathrm{T}], Q = \Omega \times [0, \mathrm{T}], \Sigma = \partial\Omega \times [0, \mathrm{T}]$. The normalized evolution diffusion model describing radiative, conductive, and convective heat transfer in a bounden region Ω has the following form:

$$\partial_t \theta = D\Delta\theta + b\Delta\theta^4, \tag{1}$$

here, θ denotes the normalized temperature, D and b denote the coefficient of diffusion, and radiation, respectively. The constant D and b are defined as follows:

$$D = \frac{k}{\rho c_v}, b = \frac{4\sigma n^2 T_{max}^3}{\rho c_v} \kappa_\alpha,$$

here, k denotes the heat conductivity, ρ the density, c_v the heat capacity, σ the Stephan-Boltzmann constant, n the refractive index, T_{max} the maximum temperature in the unnormalized model, and κ_a the absorption coefficient.

Assume that the function θ satisfy to the following condition on the boundary:

$$\boldsymbol{n} \cdot \nabla\theta = (u - \theta)h \quad \text{on } \partial\Omega, \tag{2}$$

and the initial condition:

$$\theta|_{t=0} = \theta_0. \tag{3}$$

Here, u denotes temperature of sources on the boundary $\partial\Omega$ and $h \in \mathbb{R}$ describes the reflective properties of the boundary.

For problem (1)-(3) we consider the cost functional of tracking type

$$J(\theta, u) = \frac{1}{2}\int_Q |\theta - \theta_d|^2 \, dxdt + \frac{\alpha}{2}\int_\Sigma |u|^2 \, dxdt, \tag{4}$$

where θ solves (1)–(3). Here, θ_d is a specified desired temperature profile. Furthermore, the positive constant α allows one to adjust the weight of the penalty term.

The main subject on the analysis in this paper is the following initial boundary control problem:

$$\begin{gathered} \min \quad J(\theta,u) \text{ w.r.t. } (\theta,u), \\ \text{subject to system (1)–(3)} \end{gathered} \tag{5}$$

This optimal control problem is considered as a constrained minimization problem and the adjoint variables are used for the construction of numerical algorithms.

For solving (5) and finding the optimal pair (θ, u) authors use the method of Lagrange multipliers. First of all, denote the week form for (1)–(3) as follows

$$\begin{aligned} < e,p >:= -\int_\Omega \theta_0 \hat{p} dx - \int_Q \theta \partial_t p\, dxdt - D\int_\Sigma (u-\theta)hp\, dxdt + \\ +D\int_Q \nabla\theta\cdot\nabla p\, dxdt - 4b\int_\Sigma \theta^3(u-\theta)hp\, dxdt + 4b\int_Q \theta^3\nabla\theta\cdot\nabla p\, dxdt, \end{aligned} \tag{6}$$

where p denotes a test function in the Sobolev space $H^1(Q)$, $\hat{p} = p|_{t=0}$.

Denote the Lagrange function

$$L(\theta,p,u) := \lambda J(\theta,u) + < e,p >, \tag{7}$$

with $\lambda > 0$. Here we only consider the case $\lambda = 1$. It is worth to note that p is typically called a Lagrange multiplier or adjoint variable.

For the minimization of (7), we solve the first-order optimality system

$$\nabla L = 0. \tag{8}$$

The state equation for (8) is defined as follows

$$L_p[v_p] := < e, v_p > = 0. \tag{9}$$

Here and in the following, $v_p, v_\theta \in H^1(Q)$ denote several test functions.

The adjoint equation is defined as follows

$$\begin{aligned} L_\theta[v_\theta] := \int_Q v_\theta(\theta-\theta_d)\, dxdt - \int_Q v_\theta \partial_t p\, dxdt + D\int_\Sigma v_\theta hp\, dxdt + \\ +D\int_Q \nabla v_\theta\cdot\nabla p\, dxdt - 12b\int_\Sigma \theta^2 v_\theta puh\, dxdt + 16b\int_\Sigma \theta^3 v_\theta ph\, dxdt \\ + 12b\int_Q \theta^2 v_\theta \nabla\theta\cdot\nabla p\, dxdt + 4b\int_Q \theta^3 v_\theta\cdot\nabla p\, dxdt = 0, \end{aligned} \tag{10}$$

By denoting $\hat{J}(u) := J(\theta(u),u)$, the gradient of $\hat{J}$ may be expressed as

$$\nabla \hat{J} \coloneqq \alpha u + Dhp - 4b\theta^3 hp \tag{11}$$

For solving problem (5) we use an iterative algorithm. On the first stage, the state equation (9) is solved by a time-discrete approximation and semi-implicit scheme. On the second stage, the adjoint equation (10) is solved by time-discrete approximation, starting from the terminal time ($t = T$). On the third stage, a new control variable u is computed by using (11) and the Armijo rule. Afterwards, this algorithm is repeated until the relative gradient of $\hat{J}$ is less than a small parameter δ. The construction of the algorithm is presented below.

1. Choose $u_0 = 0$
2. $k = 1$
3. REPEAT
   (a) Choose $\theta^0 = \theta_0$
   (b) FOR $i = 1, 2, \dots, N$ DO
   Find $\theta^i(u_k^i)$ from (9)
   (c) $p^N = 0$
   (d) FOR $i = N, N-1, \dots, 1$ DO
   Find p^i from (10)
   (e) $s = 1$
   (f) REPEAT
   (i) FOR $i = 1, 2, \dots, N$ DO
   $u_{k+1}^i = u_k^i - s \cdot \nabla \hat{J}(u_k^i)$ using (11)
   (ii) FOR $i = 1, 2, \dots, N$ DO
   Find $\theta^i(u_{k+1}^i)$ from (9)
   (iii) $s = s/2$
   (g) UNTIL $\sum_1^N J(u_k^i) - \sum_1^N J(u_{k+1}^i) > \gamma \left\| \sum_1^N \nabla \hat{J}(u_k^i) \right\|^2$
   (h) $k = k + 1$
4. UNTIL $\dfrac{\left\| \sum_1^N \nabla \hat{J}(u_k^i) \right\|}{\left\| \sum_1^N \nabla \hat{J}(u_0^i) \right\|} > \delta$

In the algorithm, we used the following notation for variables: $\theta^i \coloneqq \theta(x, t_i)$, $p^i := p(x, t_i)$, $u_k^i = u_k(x, t_i)$, where u_k denotes the control function on step $\#k$. The set $\{t_i\}_{i=0}^N$ denotes a uniform grid on the interval $[0, \mathrm{T}]$. It is worth noting that step 3(f-g) describes the Armijo rule.

Further, we consider a glass cooling process which was already considered for P_n in [22]. A numerical experiment for borosilicate glass is investigated in the domain $\Omega \coloneqq \{(x, y) \in \Omega: 0 \le x \le 5, 0 \le y \le 5\}$ and $t \in [0{,}300]$.

Area parameters are represented in Table 1.

Table 1. Area Parameters

$k, \dfrac{W}{cm \cdot ℃}$	$\rho, \dfrac{g}{cm^3}$	$c_v, \dfrac{J}{g \cdot ℃}$	$\sigma, \dfrac{W}{cm^2 \cdot ℃^4}$	n	$T_{max}, ℃$
$1.14 \cdot 10^{-2}$	2.23	0.75	$5.67 \cdot 10^{-12}$	1.474	1200

Consider experiment results (Fig. 1-2). It is worth noting that the proposed algorithm for this example converges after 4 iterations. Convergence of the cost functional J to the minimum value of 89.59 is given in graphical form in Fig. 1. Temperature profile of glass, limited to goal temperature θ_d, is shown in Fig. 2.

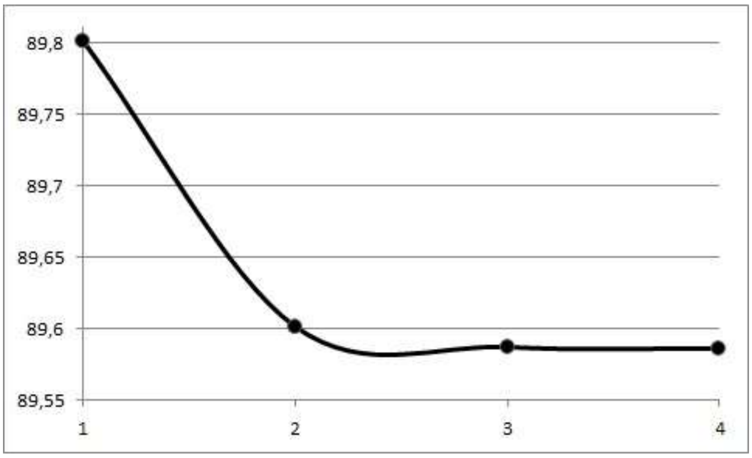

Fig. 1. $f(k) = J(u_k)$.

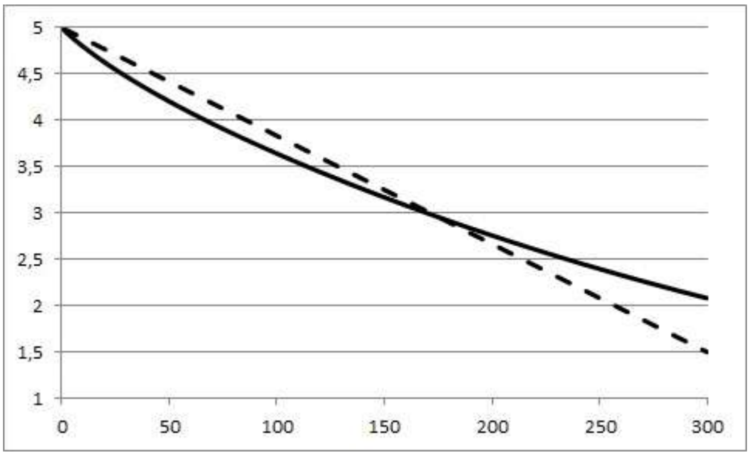

Fig. 2. $f_{b,d} = \|\theta\|_{\Omega}(t), \|\theta_d\|_{\Omega}(t)$

3 SP_1 Approximation

Furthermore, we consider one more diffusion model of complex heat transfer in SP_1 approximation. The process of propagation of heat transfer is investigated in the same medium Q with same boundary and initial conditions. It is known that the Rosseland approximation is a simplification of the SP_1 approximation [1], [23]. Rosseland approximation is valid when the medium is optically thick.

The normalized evolution diffusion model describing radiative, conductive, and convective heat transfer in a bounded region Ω has the following form [24]

$$\partial_t \theta = D\Delta\theta + b\Delta\varphi, \tag{12}$$

$$-\epsilon^2\Delta\varphi + \varphi = \theta^4, \tag{13}$$

$$\boldsymbol{n}\cdot\nabla\theta = (u-\theta)h \text{ on } \partial\Omega, \tag{14}$$

$$\boldsymbol{n}\cdot\nabla\varphi = u^4 - \varphi \text{ on } \partial\Omega, \tag{15}$$

$$\theta|_{t=0} = \theta_0, \tag{16}$$

here, $\epsilon = {}^{1}/_{3\kappa\kappa_a}$. Note the week form for (12), (14), (16) as follows

$$< e_1, p_1 >:= -\int_\Omega \theta_0\hat{p}_1 dx - \int_Q \theta\partial_t p_1\, dxdt - D\int_\Sigma (u-\theta)hp_1\, dxdt$$
$$+ D\int_Q \nabla\theta\cdot\nabla p_1\, dxdt + \frac{b}{\epsilon^2}\int_Q (\theta^4 - \varphi)p_1\, dxdt,$$

here p_1 denotes several test function in Sobolev space $H^1(Q)$, $\hat{p}_1 = p_1|_{t=0}$.

Note the week form for (13), (15) as follows

$$< e_2, p_2 >:= -\epsilon^2\int_\Sigma (u^4-\varphi)p_2\, dxdt + \epsilon^2\int_Q \nabla\varphi\cdot\nabla p_2\, dxdt$$
$$+ \int_Q (\varphi - \theta^4)p_2\, dxdt,$$

here p_1 denotes several test function in Sobolev space H^1 (Q), analogically.

For problem (12)–(16) authors consider the same cost functional (4) and solve analogical initial-boundary control problem

$$\begin{gathered}\min \quad J(\theta,u) \text{ w.r.t. } (\theta,\varphi,u),\\ \text{subject to system (12)–(16)}\end{gathered} \tag{17}$$

We then construct the Lagrange function

$$L(\theta,\varphi,p_1,p_2,u) \coloneqq \lambda J(\theta,u) + < e_1, p_1 > + < e_2, p_2 >, \tag{18}$$

in a similar fashion, where $< e_1, p_1 >$ and $< e_2, p_2 >$ denote the weak form of (12), (14), (16) and (13), (15) respectively.

For the minimization of (18), we solve the corresponding optimality system

$$\nabla L = 0. \tag{19}$$

We designed a similar algorithm for solving (19). The analysis of numerical experiments for SP_1 approximation for borosilicate glass is considered in [21].

4 Studying a convergence of solutions

In this section, we consider a numerical convergence of the optimal control problem in SP_1 approximation to solution in Rosseland when $\epsilon \to 0$

$$\operatorname{argmin} J_\epsilon \to \operatorname{argmin} J_0 \ \text{ as } \ \epsilon \to 0. \tag{22}$$

In the following, we denote $(\bar{\theta}_\epsilon, \bar{u}_\epsilon)$ as the optimal solution of (17) for SP_1 approximation with the cost functional J_ϵ and $(\bar{\theta}_0, \bar{u}_0)$ denotes the optimal solution of (5) for Rosseland approximation with cost functional J_0.

The numerical experiment for borosilicate glass is done. An exemplary case study is a thin bar (5x5 cm) that has been cooled during 300 sec. On the Fig. 3, we present the values of the cost functional J_ϵ which depends on ϵ for (17) and J_0 for (5). One clearly observes that the functional values of (17) converge to the one of (5).

Thus, we have shown, numerically, that solutions of the optimal control problem for the SP_1 approximation converges to the solution of optimal control problem for the Rosseland approximation for any initial temperature and medium when $\epsilon \to 0$.

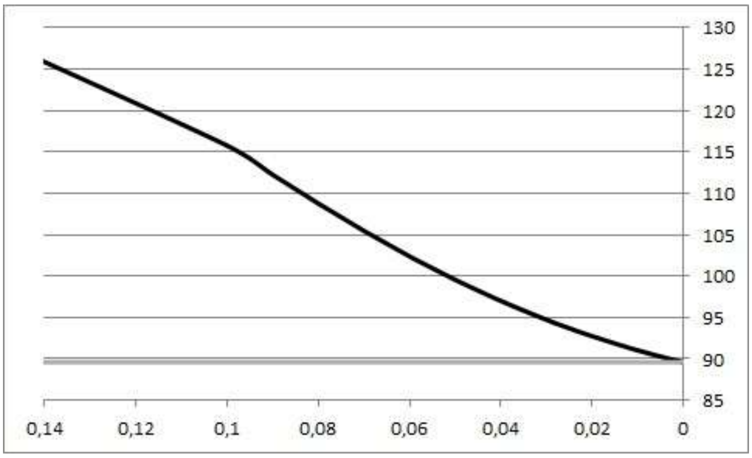

Fig. 3. Functions: $f_1(\epsilon) = J_\epsilon(\bar{\theta}_\epsilon, \bar{u}_\epsilon)$ – black curve; $f_2(\epsilon) = J_0(\bar{\theta}_0, \bar{u}_0)$ gray curve.

Conclusions

We studied the optimal control problems of complex heat transfer with diffusion approximations. For a special cost functional that allows one to find the optimal temperature of sources on the boundary and get target temperature in the medium, we designed iterative algorithms. For some examples we showed a convergence of the optimal costs in the SP_1 approximations to optimal cost in the Rosseland approximation in the $\epsilon \to 0$ limit. Since the simplified Rosseland model is more applicable for computing because it needs less computation for solving, such results could be used in glass production.

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