Discrete Optimization of Unsteady Fluid Flows

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Abstract. The paper is devoted to discrete optimization of the viscous heat-conducting fluid flows. In the problem under consideration we want to create a flow with desired properties using optimal heating or cooling on some boundary sections. Optimal control approach reduces this problem to the constrained minimization. In this case the cost functional describes the objectives, mathematical model is the constraint and temperature boundary value is the control. If the flow is unsteady and the objectives data is discrete we need to use discrete optimization methods for numerical solution of this problem. We propose new numerical algorithm that does not use the first order necessary optimality conditions and based on the finite dimensional minimization.

Keywords: Partial differential equations, unsteady problems, discrete optimization, hydrodynamics

1 Introduction

Optimization problems in hydrodynamics have a large number of applications in science and engineering. Usually these problems are formulated as minimization problems for suitable cost functionals and can be analyzed and solved by applying a unified approach based on the constrained optimization theory (see, for example, [1–6]). The same approach can be applied for inverse and parameter estimation problems [7].

Numerical optimization for complicated mathematical models is connected with a number of difficulties. When we solve the minimization problem the solution must satisfy the nonlinear system of partial differential equations. Usually a constrained minimization problem is rewritten as an unconstrained minimization problem using the Lagrange multipliers. Then one can formally calculate the derivatives of the Lagrangian and derive the necessary optimality conditions. However, numerical solution of the optimality system for nonstationary mathematical models is very difficult therefore we propose a new numerical algorithm to solve minimization problem under consideration. In this algorithm we calculate the state and the control simultaneously at each time step. This approach does not require iterations and it is simpler to implement.

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2 Problem Statement

In the bounded two-dimensional domain Ω on the time interval $(0, t_{max})$ we consider the following dimensionless system of partial differential equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{\mathrm{Re}} \Delta \mathbf{v} + \nabla p = \frac{\mathrm{Gr}}{\mathrm{Re}^2} T \mathbf{j} \text{ in } \Omega, \tag{1}$$

$$\operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \quad \mathbf{v}|_{t=0} = \mathbf{v}_0 \text{ in } \Omega, \tag{2}$$

$$\mathbf{v} = \mathbf{g} \text{ on } \Gamma_{\mathrm{in}}, \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0 \cup \Gamma_c, \quad p\mathbf{n} - \frac{1}{\mathrm{Re}} \frac{\partial \mathbf{v}}{\partial \mathbf{n}} = \mathbf{0} \text{ on } \Gamma_{\mathrm{out}}, \tag{3}$$

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T - \frac{1}{\mathrm{RePr}} \Delta T = 0 \text{ in } \Omega, \quad T|_{t=0} = T_0 \text{ in } \Omega, \tag{4}$$

$$T = 0 \text{ on } \Gamma_{\mathrm{in}}, \quad T = \phi \text{ on } \Gamma_c, \quad \frac{\partial T}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_0 \cup \Gamma_{\mathrm{out}} \tag{5}$$

describing the time evolution of the viscous heat-conducting fluid flow. Here $\mathbf{v}$, p and T are the dimensionless velocity, pressure and temperature, $\mathbf{j} = \{0, 1\}$ is the unit vector directed upwards. Reynolds number Re, Grashof number Gr and Prandtl number Pr are dimensionless parameters of this problem.

In this paper we shall consider the unsteady flow in the open cavity (see Fig. 1). The boundary Γ consists of four parts: inlet section Γ_{in}, outlet section Γ_{out}, solid walls Γ_0 and control section Γ_c. For the velocity vector $\mathbf{v}$ we prescribe the inflow parabolic profile $\mathbf{g}$ on Γ_{in}, no-slip boundary condition on $\Gamma_0 \cup \Gamma_c$ and "do nothing" boundary condition on the outlet Γ_{out}. The temperature boundary value ϕ on the control section Γ_c will play the role of control.

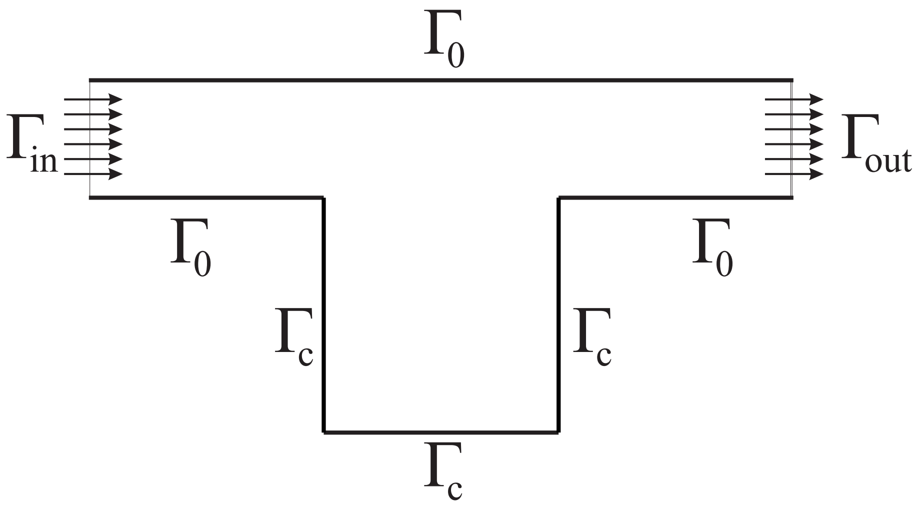

Fig. 1. Flow domain.

In our optimization problem we want to obtain the velocity field $\mathbf{v}$ close to a given "optimal" vector field $\mathbf{v}_d$ using temperature boundary control on Γ_c. We assume that values of the vector $\mathbf{v}_d$ are given only at fixed times $t_1, t_2, \ldots, t_N = t_{max}$. So we have a discrete set of data and need to solve the discrete optimization problem.

2.1 Time Discretization

At the beginning we split the time interval $(0, t_{max})$ into N parts (t_{n-1}, t_n) of length $\tau_n = t_n - t_{n-1}$, $n = 1, 2, \ldots, N$ and write the following semidiscrete approximation for the problem (1)–(5):

$$\frac{\mathbf{v}^n - \mathbf{v}^{n-1}}{\tau_n} + (\mathbf{v}^{n-1} \cdot \nabla)\mathbf{v}^n - \frac{1}{\mathrm{Re}} \Delta \mathbf{v}^n + \nabla p^n = \frac{\mathrm{Gr}}{\mathrm{Re}^2} T^n \mathbf{j} \ \text{ in } \ \Omega, \tag{6}$$

$$\operatorname{div} \mathbf{v}^n = 0 \ \text{ in } \ \Omega, \quad \mathbf{v}^0 = \mathbf{v}_0 \ \text{ in } \ \Omega, \tag{7}$$

$$\mathbf{v}^n = \mathbf{g}(t_n) \text{ on } \Gamma_{\text{in}}, \quad \mathbf{v}^n = \mathbf{0} \text{ on } \Gamma_0 \cup \Gamma_c, \quad p^n \mathbf{n} - \frac{1}{\mathrm{Re}} \frac{\partial \mathbf{v}^n}{\partial \mathbf{n}} = \mathbf{0} \text{ on } \Gamma_{\text{out}}, \tag{8}$$

$$\frac{T^n - T^{n-1}}{\tau_n} + \mathbf{v}^{n-1} \cdot \nabla T^n - \frac{1}{\mathrm{RePr}} \Delta T^n = 0 \ \text{ in } \ \Omega, \ T^0 = T_0 \ \text{ in } \ \Omega, \tag{9}$$

$$T^n = 0 \text{ on } \Gamma_{\text{in}}, \quad T^n = \phi(t_n) \text{ on } \Gamma_c, \quad \frac{\partial T^n}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_0 \cup \Gamma_{\text{out}}. \tag{10}$$

Here $\mathbf{v}^n$, p^n, T^n $(n = 1, 2, \ldots, N)$ are the new unknown functions that depend only on the space variables. Let us note that we use an implicit difference scheme for time discretization. It ensures the stability of our numerical solutions even for large time intervals.

Multiplying equations in (6), (7), (9) by the corresponding test functions, integrating over Ω and using Green's formula for certain terms we obtain a variational formulation of the problem (6)–(10). It consists in finding functions $\mathbf{v}^n, p^n, T^n$ $(n = 1, 2, \ldots, N)$ satisfying equations

$$\frac{(\mathbf{v}^n - \mathbf{v}^{n-1}, \mathbf{w})}{\tau_n} + ((\mathbf{v}^{n-1} \cdot \nabla)\mathbf{v}^n, \mathbf{w}) + \frac{1}{\mathrm{Re}} (\nabla \mathbf{v}^n, \nabla \mathbf{w}) - (p^n, \operatorname{div} \mathbf{w}) \tag{11}$$

$$+ \frac{\mathrm{Gr}}{\mathrm{Re}^2} (T^n \mathbf{j}, \mathbf{w}) = 0 \ \ \forall \mathbf{w} \in \mathbf{W}, \quad (\chi, \operatorname{div} \mathbf{v}^n) = 0 \ \ \forall \chi \in X, \tag{12}$$

$$\mathbf{v}^n = \mathbf{g}^n \text{ on } \Gamma_{\text{in}}, \quad \mathbf{v}^n = \mathbf{0} \text{ on } \Gamma_0 \cup \Gamma_c, \tag{13}$$

$$\frac{(T^n - T^{n-1}, s)}{\tau_n} + (\mathbf{v}^{n-1} \cdot \nabla T^n, \, s) + \frac{1}{\mathrm{RePr}} (\nabla T^n, \nabla s) = 0 \ \ \forall s \in S, \tag{14}$$

$$T^n = 0 \text{ on } \Gamma_{\text{in}}, \quad T^n = \phi^n \text{ on } \Gamma_c. \tag{15}$$

Here the function $\mathbf{v}^0 = \mathbf{v}_0$, $\theta^0 = \theta_0$ are determined from the initial conditions of the problem (1)–(5); $\mathbf{g}^n = \mathbf{g}(t_n)$, $\phi^n = \phi(t_n)$; $(\cdot, \cdot)$ is the scalar product in the space $L^2(\Omega)$. If we already know all values at time $t = t_{n-1}$ then this problem is linear for unknowns $(\mathbf{v}^n, p^n, T^n)$. We will use this linearity in the construction of our numerical algorithm.

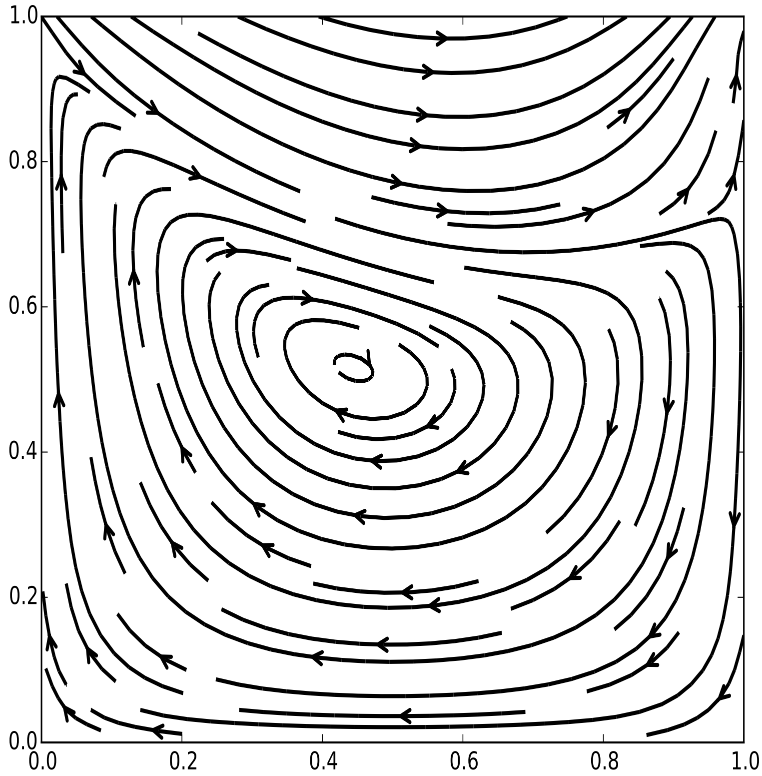

Fig. 2. Uncontrolled flow in the cavity.

Appling the finite element method for space disctretization we can solve the problem (11)–(15) numerically. Uncontrolled flow in the cavity for Reynolds number $\mathrm{Re} = 10$ at time $t = 10$ is shown in Fig. 2.

We can clearly see a large vortex occupying most of the cavity. Most of the fluid will never leave the cavity. In this case we have a stagnant zone in which the suspended particles are accumulated. We want to suppress this vortex. Therefore we choose the potential flow with zero vorticity as the desirable vector field $\mathbf{v}_d$. Usually the moving bottom wall is used for flow control (see [8, 9]). In this study we consider the heat-conducting fluid an can use the temperature boundary control. This control can be more effective then tangential velocity because the temperature can affect the entire fluid volume by means the buoyancy effect.

3 Discrete Optimization

Let $\Omega_d \subseteq \Omega$ be the subset of Ω (subdomain, surface or curve in the two-dimensional case). It will play the role of the observation set in our optimization problem for the system (11)–(15). At time step $t = t_n$ the problem of finding the boundary function ϕ^n for the vector field $\mathbf{v}_d^n = \mathbf{v}_d(t_n)$ is reduced to the minimization of the quality functional

$$J(\mathbf{v}^n, \phi^n) = \frac{1}{2} \int\limits_{\Omega_d} |\mathbf{v}^n - \mathbf{v}_d^n|^2 d\Omega + \frac{\mu}{2} \int\limits_{\Gamma_c} (\phi^n)^2 d\Gamma, \tag{16}$$

depending on the boundary function ϕ^n and corresponding velocity vector $\mathbf{v}^n$. Here $\mu = \text{const} \geq 0$ is the small regularization parameter.

The following method is based on the main idea of [10] where an optimal boundary control problem for the stationary Navier-Stokes equations was solved using the finite dimensional minimization approach.

We will find the unknown function ϕ^n in the following form:

$$\phi^n = \sum_{i=1}^{M} k_i \phi_i. \tag{17}$$

Here $\{\phi_i\}_{i=1}^M$ are the given basic functions on Γ_c while k_i, $i = 1, 2, \ldots, M$ are the unknown coefficients that must be found at each time step. The corresponding velocity vector then can be written as

$$\mathbf{v}^n = \mathbf{w}^n + \sum_{i=1}^{M} k_i \mathbf{v}_i. \tag{18}$$

Here $\mathbf{w}^n$ is the solution for homogeneous boundary conditions in (15) while $\mathbf{v}_i$ ($i = 1, \ldots, M$) are the solutions corresponding $\phi^n = \phi_i$ in (15) and homogeneous boundary conditions in (13). Then the functional (16) can be written as

$$J(\mathbf{v}^n, \phi^n) = \frac{1}{2} \int\limits_{\Omega_d} |\mathbf{v}^n - \mathbf{v}_d^n|^2 d\Omega + \frac{\mu}{2} \int\limits_{\Gamma_c} (\phi^n)^2 d\Gamma = \frac{1}{2} \int\limits_{\Omega_d} \left(\mathbf{w}^n - \mathbf{v}_d^n + \sum_{i=1}^{M} k_i \mathbf{v}_i \right)^2 d\Omega$$

$$+ \frac{\mu}{2} \int\limits_{\Gamma_c} \left(\sum_{i=1}^{M} k_i \phi_i \right)^2 d\Gamma = \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} a_{ij} k_i k_j - \sum_{j=1}^{M} b_j k_j + \frac{1}{2} c.$$

Here the coefficients

$$a_{ij} = (\mathbf{v}_i, \mathbf{v}_j)_{\Omega_d} + \mu(\phi_i, \phi_j)_{\Gamma_c}, \quad b_j = (\mathbf{v}_d^n - \mathbf{w}^n, \mathbf{v}_j)_{\Omega_d}, \quad c = \|\mathbf{v}_d^n - \mathbf{w}^n\|_{\Omega_d}^2 \tag{19}$$

can be easily calculated by known functions ϕ_i, $\mathbf{v}_i$, $\mathbf{v}_d^n$ and $\mathbf{w}^n$.

As a result, we have the finite dimensional minimization problem for variables k_i, $i = 1, 2, \ldots, M$. Solution of this problem can be found by solving the following system of linear algebraic equations:

$$\sum_{i=1}^{M} a_{ij} k_i = b_j, \quad j = 1, 2, \ldots, M. \tag{20}$$

Substituting a_{ij} and b_j in this system and using (18) we obtain the following equalities

$$(\mathbf{v}^n - \mathbf{v}_d^n, \mathbf{v}_j)_{\Omega_d} + \mu(\phi^n, \phi_j)_{\Gamma_c} = 0, \;\; j = 1, 2, \ldots, M.$$

that have a clear meaning. If $\mu = 0$ then the difference $\mathbf{v}^n - \mathbf{v}_d^n$ must be orthogonal to all functions $\mathbf{v}_j$ in subdomain Ω_d. Solving the system (20) and substituting k_i in (17) we find the boundary temperature ϕ^n on the n-th time step.

3.1 Numerical Algorithm

Proposed numerical algorithm can be written as follows.

Step 0. Choose M basic functions ϕ_i on Γ_c. Set initial values $\mathbf{v}^0$ and T^0. Set $n = 1$.

Step 1. Assuming that $\mathbf{v}^{n-1}$ and T^{n-1} are already known, solve $M+1$ linearized boundary value problems to find $\mathbf{v}_i$, $i = 1, \ldots, M$ and $\mathbf{w}^n$.

Step 2. Calculate a_{ij} and b_j by (19). Solve linear system (20) to find k_i.

Step 3. Calculate ϕ^n by (17). Solve linear boundary value problem (11)–(15) to find $\mathbf{v}^n$, p^n, T^n.

Step 4. If $n < N$ then set $n := n + 1$ and go to step 1.

Let us note that this algorithm does not use the first order necessary optimality conditions (see [1, 3, 4, 6, 7]) and it is simpler to implement. It can be efficiently parallelized because $M+1$ boundary value problems to find $\mathbf{w}^n$ and $\mathbf{v}_i$ are solved independently. It must be noted that at each time step we find the control and the state simultaneously.

3.2 Computational Results

Now let us consider some computatiomal results for optimization of viscous heat-conducting fluid flow in the open cavity (see Fig. 1). In this problem we find optimal heating or cooling of the control boundary section Γ_c to create the velocity $\mathbf{v}$ closed to the given vector field $\mathbf{v}_d$ in the subdomain Ω_d. We want to suppress large vortex (see Fig. 2) in the cavity therefore we choose the potential flow with zero vorticity as the desirable vector field $\mathbf{v}_d$. The observation set Ω_d occupies the entire cavity. In these computations Reynolds number Re $= 10$, Grashof number Gr $= 10^4$ and Prandtl number Pr $= 7$. The dimensionless time interval $(0, t_{max})$ was chosen as $(0, 20)$ with $N = 100$.

The controlled flow at time $t = 10$ is shown in Fig. 3. It is clearly seen that the vortex is fully suppressed and the main flow covers the whole cavity.

Corresponding temperature values on the bottom boundary of the cavity are shown in Fig. 4. This is a dimensionless temperature deviation from the values given on the inlet boundary section.

Fig. 5 shows obtained temperature controls on the left (solid line) and right (dotted line) boundaries of the cavity. It is easy to notice that temperature on the right part is higher than on the left part.

These figures help us to understand the basic effect of the temperature boundary control. We see cooling on the left boundary and heating on the right boundary. Therefore the cold fluid with a higher density moves downwards along the left wall. The heated fluid with lower density goes up along the right wall. Heating and cooling

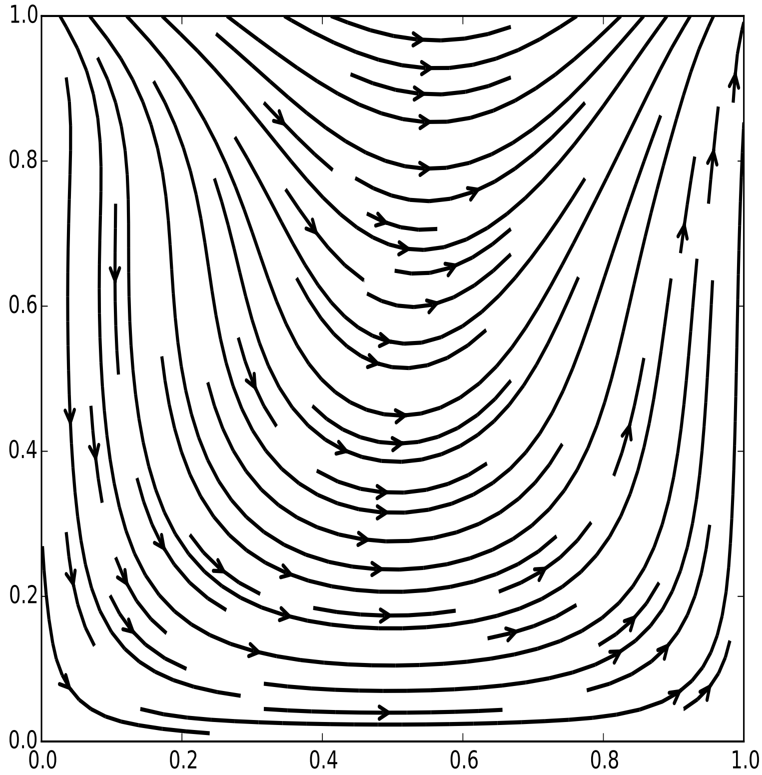

Fig. 3. Controlled flow in the cavity.

change the fluid density and thus affect the fluid flow. Let us note that in order to take into account of such effects we need to consider more sophisticated mathematical model in comparison with the works of other authors devoted to the boundary control problems for Navier-Stokes equations.

The open source software freeFEM++ [11] was used for the discretization and numerical solving boundary value problems by the finite element method. The main goal of the computational experiments was to determine the dependence of the solution accuracy on the choice of the problem parameters. We choose different ways to specify the given velocity data $\mathbf{v}_d$. For example, the observation domain Ω_d can coincide with the whole flow domain or can be some subdomain. The location of the small observation domain with fixed size can be also very important. Numerical experiments show that the solution is more accurate if the observation domain Ω_d is located closer to the control boundary Γ_c. The number of basis functions M also affects the numerical

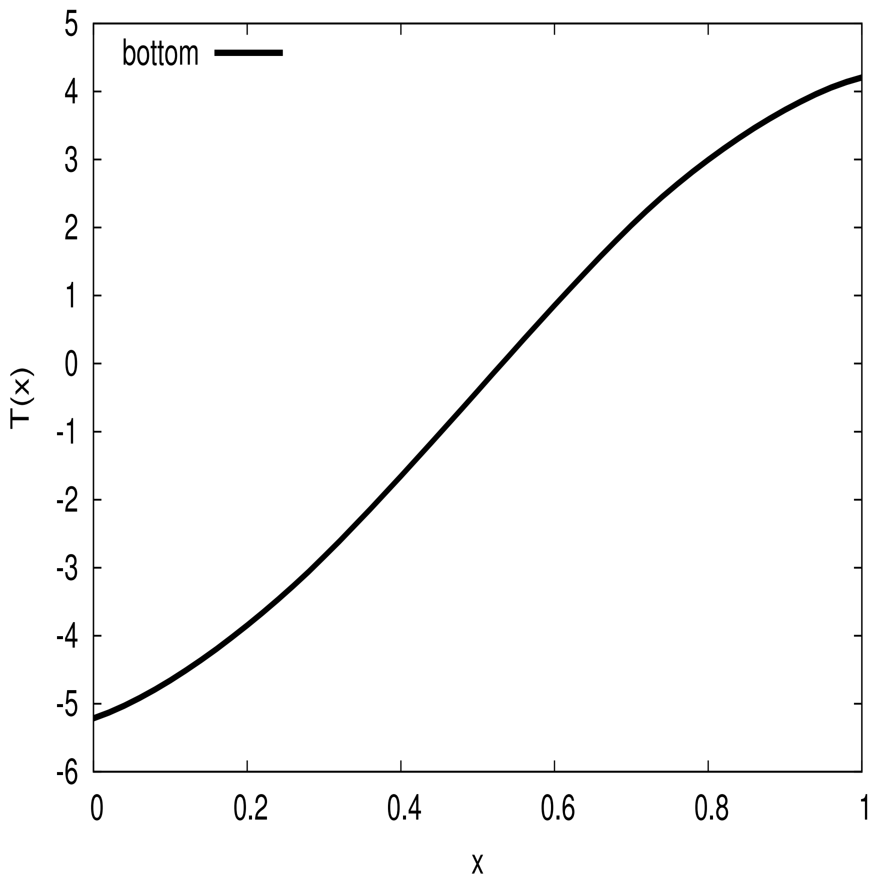

Fig. 4. Temperature $T(x)$ on the bottom boundary of the cavity.

solution. Based on the analysis of computational results we can choose optimal values and develop some recommendations for future applications.

4 Conclusion

In this paper we considered the discrete optimization problem for the unsteady viscous heat-conducting fluid flows. This problem consists in creating a flow with desired properties using optimal heating or cooling on some boundary sections. We have proposed new numerical algorithm that does not use the first order necessary optimality conditions and based on the finite dimensional minimization. Computational experiments have shown the ability to change the flow with small Reynolds numbers by means of the temperature boundary control.

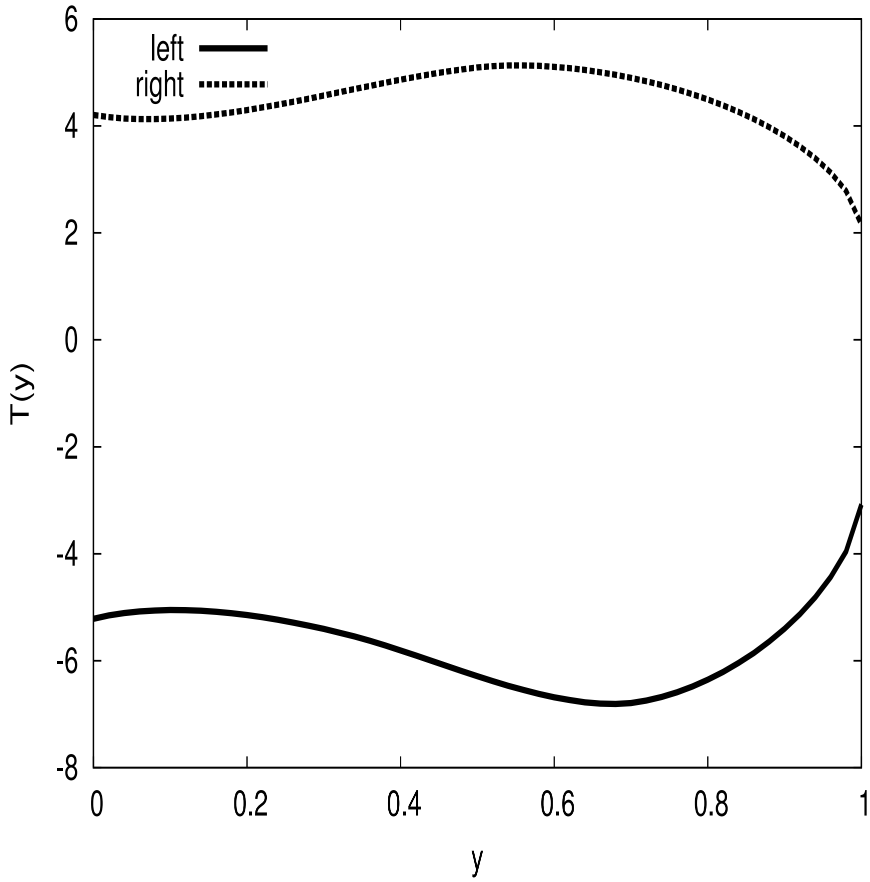

Fig. 5. Temperature $T(y)$ on the left and right boundaries of the cavity.

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