A Minimization Algorithm with Approximation of an Epigraph of the Objective Function and a Constraint Set

Igor Zabotin, Oksana Shulgina, and Rashid Yarullin

Kazan (Volga Region) Federal University, Institute of Computational Mathematics and Information Technologies, Kremlyovskaya str. 35, 420008 Kazan, Russian
{iyazabotin,onshul}@mail.ru, yarullinrs@gmail.com
http://www.kpfu.ru

Abstract. An algorithm is suggested for solving a convex programming problem which belongs to a class of cutting methods. In the algorithm an epigraph of the objective function and a feasible solutions set of the problem are embedded into some auxiliary sets to construct iteration points. Since these embedded sets are constructed as polyhedral sets in the algorithm, then each iteration point is found by solving a linear programming problem independently of the type of functions which define the initial problem. The suggested algorithm is characterized by the following fact. Sets which approximate the epigraph of the objective function can be updated periodically on the base of discarding cutting planes.

Keywords: cutting-plane methods, minimization methods, approximation sets, an epigraph, a constraint set

1 Introduction

Cutting methods (e. g. [1–8]) are quite often applied to solve both application tasks and auxiliary problems of constructing iteration points in the famous methods of constrained optimization. It can be explained, in particular, by the fact that there are usually some possibilities to estimate proximity of the current value of the objective function to its optimal value.

Among this class of methods there are ones which use approximation both an epigraph of the objective function and a feasible set of the initial problem to construct iteration points (e. g. [6,9]). These methods are convenient from the practical viewpoint, because in these ones iteration points can be obtained on the base of solving auxiliary linear programming problems.

The cutting algorithm proposed in this paper for solving a convex programming problem also uses embedding procedures of both mentioned sets. Note that the constraint set can be embedded partially, and, moreover, in the algorithm there are some opportunities of periodically dropping cutting planes which form sets for approximating the epigraph.

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2 Problem Setting

Let $f(x)$, $F(x)$ be convex functions defined in an n-dimensional Euclidian space $\mathrm{R_n}$, and the function $f(x)$ reaches its minimal value on the set $D = \{x \in \mathrm{R_n} : F(x) \leq 0\}$.

We solve the problem

$$\min\{f(x) : x \in D\}. \tag{1}$$

Let $f^* = \min\{f(x) : x \in D\}$, $X^* = \{x \in D : f(x) = f^*\}$, $X^*(\varepsilon) = \{x \in D : f(x) \leq f^* + \varepsilon\}$, where $\varepsilon \geq 0$, $\mathrm{epi}(f, \mathrm{R_n}) = \{(x, \gamma) \in \mathrm{R_{n+1}} : x \in \mathrm{R_n}, \gamma \geq f(x)\}$, $\partial f(x)$, $\partial F(x)$ be subdifferentails of functions $f(x)$ and $F(x)$ at point $x \in \mathrm{R_n}$ respectively, $K = \{0, 1, \ldots\}$, $x^* = X^*$.

3 The Cutting Algorithm and Discussion

The proposed algorithm for solving problem (1) constructs an auxiliary sequence of approximations $\{y_i\}$, $i \in K$, and a basic sequence $\{x_k\}$, $k \in K$, by the following rule. A convex bounded closed set $M_0 \subset \mathrm{R_n}$ and a convex closed set $G_0 \subset \mathrm{R_{n+1}}$ are formed such that

$$x^* \in M_0, \quad \mathrm{epi}(f, \mathrm{R_n}) \subset \mathrm{G_0}. \tag{2}$$

Generate numbers $\bar{\gamma}$ and $\varepsilon_k > 0$, $k \in K$, according to conditions $\bar{\gamma} \leq f(x)$ for all $x \in M_0$, $\varepsilon_k \to 0$, $k \to \infty$. Assign $i = 0$, $k = 0$.

1. Find a point (y_i, γ_i), where $y_i \in \mathrm{R_n}$, $\gamma_i \in \mathrm{R_1}$, as a solution of the problem

$$\min\{\gamma : (x, \gamma) \in G_i,\ x \in M_k,\ \gamma \geq \bar{\gamma}\}. \tag{3}$$

If $y_i \in D$ and $f(y_i) = \gamma_i$, then $y_i \in X^*$, and minimization process is finished. If $y_i \in D$ and at the same time the inequality

$$f(y_i) - \gamma_i \leq \varepsilon_k \tag{4}$$

is fulfilled, then $y_i \in X^*(\varepsilon_k)$, and the ε_k-solution of problem (1) is found.

2. Select an element $b_i \in \partial f(y_i)$. If

$$f(y_i) - \gamma_i > \varepsilon_k, \tag{5}$$

then assign

$$G_{i+1} = S_i \bigcap \{(x, \gamma) \in \mathrm{R_{n+1}} : f(y_i) + \langle b_i, x - y_i \rangle \leq \gamma\}, \tag{6}$$

where $S_i = G_i$, and go to Step 1, increase the value of i by one. Otherwise go to Step 3.

3. Assign $i_k = i$, and choose a point x_k such that

$$x_k \in M_k, \quad f(x_k) \leq f(y_{i_k}). \tag{7}$$

4. Choose a convex closed set $S_i \subset \mathrm{R_{n+1}}$ in accordance with

$$\mathrm{epi}(f, \mathrm{R_n}) \subset S_i \tag{8}$$

and construct a set G_{i+1} in the form of (6).

5. Construct a set M_{k+1} by the following rule. If $x_k \in D$, then assign $M_{k+1} = M_k$, else choose $a_k \in \partial F(x_k)$ and assign $M_{k+1} = M_k \bigcap \{x \in R_n : F(x_k) + \langle a_k, x - x_k \rangle \leq 0\}$.

6. Increment values of i and k by one, and go to Step 1.

Lets represent some remarks for the algorithm.

It is advisable to choose the initial approximating sets M_0, G_0 as polyhedral sets, because in this case on each iteration $i \in K$ problem (3) of constructing the auxiliary point y_i is a linear programming problem.

If the set D is bounded and polyhedral, then it is not necessary to approximate D by the sets M_k, $k \in K$. In accordance with (2) it is clear to assign $M_0 = D$ in the algorithm, and in view of Steps 3, 5 equalities $M_{k+1} = M_k = D$ will be fulfilled for all $k \in K$.

Condition (2) allows to assign $G_0 = \mathrm{epi}(f, \mathrm{R_n})$, and this way is suitable in case of determining the function $f(x)$ as maximum of linear functions. At the same time for each $i \in K$ inequality (4) is defined, and using $S_i = \mathrm{epi}(f, \mathrm{R_n})$ in (8) we have equalities $G_i = \mathrm{epi}(f, \mathrm{R_n})$, $i \in K$. Also note that it is possible to use $G_0 = \mathrm{R_{n+1}}$. In this case any point (y_0, γ_0), where $y_0 \in M_0$ and $\gamma_0 = \bar{\gamma}$, can be a solution of problem (3) for $i = 0$.

Note that one can assign $S_i = G_i$ for all $i \in K$ independently of conditions (4), (5). Then cutting planes formed approximating sets G_{i+1} will be accumulated infinitely. Notice that condition (8) of constructing sets S_i allows to update sets that approximate the epigraph in the process of forming G_{i_k+1} on iterations with numbers $i = i_k$. Namely, there is rejection of a finite number of cutting planes $f(y_i) + \langle b_i, x - y_i \rangle = \gamma$ by construction S_{i_k}, for example, on the base of any sets $G_0, \ldots, G_{i_k - 1}$ according to (8). In particular, if we assign $S_{i_k} = G_0$, then we discard all planes which are constructed to the step $i = i_k$.

Note that condition (7) of selection of the point x_k allows to assign, in particular, $x_k = y_{i_k}$ for all $k \in K$.

Lets observe some properties of the proposed algorithm.

Taking into account (2) it is easy to prove by induction that for all $k \in K$ and $i \in K$ inclusions $x^* \in M_k$, $(x^*, f^*) \in G_i$ are fulfilled.

On the base of these inclusions, the construction condition of the number $\bar{\gamma}$ and the type of constraints in problem (3) it is not difficult to observe that the inequality

$$\gamma_i \leq f^* \tag{9}$$

is determined for the solution (y_i, γ_i) of problem (3) under any $k \in K$, $i \in K$.

Obviously the stopping criterion represented in Step 1 of the algorithm is proved according to condition (9). If the point (y_i, γ_i) is constructed such that $y_i \in D$ and condition (4) is determined, then from (9) it follows that $f(y_i) \leq f^* + \varepsilon_k$, and, consequently, the approximation y_i is an ε_k-solution of the initial problem.

If the equation $S_i = G_i$ is defined for all $i \in K$ in the algorithm, then taking into account (9) it is not difficult to prove the limit expression $\lim\limits_{i \in K}(f(y_i) - \gamma_i) = 0$. On the base of this equality and in view of the methology [10] it is clear to observe the following

Lemma 1. *If the sequence $\{(y_i, \gamma_i)\}$ is constructed by the suggested algorithm, then there exist a number $i = i_k \in K$ for all $k \in K$ such that equality (4) is fulfilled.*

According to Lemma 1 the sequence $\{x_k\}$, $k \in K$, will be formed with the sequence $\{y_i\}$, $i \in K$, by the algorithm.

Theorem 1. *The inclusion $\bar{x} \in X^*$ is valid for any limit point $\bar{x}$ of the sequence $\{x_k\}$ constructed by the algorithm.*

Proof. This theorem is proved by the following schema. Firstly, taking into account the approach of constructing sets M_k the inclusion $\bar{x} \in D$ is proved, consequently, the inequality

$$f(\bar{x}) \geq f^* \tag{10}$$

is observed too. Further, from (4), (7), (9) and according to the technique of establishing $\{\varepsilon_k\}$ it is easy to get contradiction with inequality (10).

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