Generalized Metrics and Their Relevance for FCA and Closure Operators

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Abstract. We provide an approach to generalized metrics that covers various concepts of distance. In particular, we consider functorial maps which are weakly positive. Here, we focus on the supermodular case which generalizes dimension functions. We give a lattice-theoretically based construction for supermodular functorial maps, which generalize those arising from Dempster-Shafer-Theory. Within this framework, generalized metrics relevant for FCA and closure operators are discussed.

Keywords: Generalized metric, supermodular, formal concept analysis, Dempster-Shafer-Theory, closure operators

1 Introduction

Generalized metrics recently have become of increased interest for modelling a concept of directed distances with values in a qualitative measurement space. In particular, they allow to distinguish between *deletion* and *error* within the context of transferred information. We propose a general modeling including lattices and ordered monoids. Here, our goal is to construct *generalized metrics* relevant for FCA and closure operators [3, 8, 7]. For our approach it turns out that *supermodularity* plays an important role, which goes beyond ideas of measurement accociated with Dempster-Shafer-Theory [8].

Our modeling of *generalized metrics* can be very helpful to improve and better understand the mapping of ratings, i. e. compare the rating methodologies of different rating agencies with different result scales.

2 Motivation

Before we present *generalized metrics* in an abstract setting, we want to discuss a motivating special situation where we collect properties relevant for our general approach.

We start with a lattice $\mathbb{L} = (L, \leq_{\mathbb{L}})$. Then, we consider the ordered monoid $\mathcal{M} = (M, \cup, \emptyset, \subseteq)$ with $M = 2^L$. Furthermore, we set

$$\downarrow x \coloneqq \{t \in L \mid t \leq_{\mathbb{L}} x\}$$

and define the maps

$$\lambda \colon L \longrightarrow M : x \mapsto \downarrow x$$

and

$$D_\lambda \colon \ \leq_{\mathbb{L}} \longrightarrow M : (x, y) \mapsto \downarrow y - \downarrow x. \tag{1}$$

We can easily see that $D_\lambda(x, y)$ is equal to the set $\{t \in L \mid t \leq_{\mathbb{L}} y \text{ and } t \not\leq_{\mathbb{L}} x\}$, where $x, y \in L$ and $x \leq_{\mathbb{L}} y$.

Observation Obviously, D_λ fulfils the following properties:

- $D_\lambda(x, x) = \emptyset$ holds for all $x \in L$, since $D_\lambda(x, x) = \downarrow x - \downarrow x = \emptyset$.
- $D_\lambda(x, y) \cup D_\lambda(y, z) = D_\lambda(x, z)$ holds for all $x, y, z \in L$ with $x \leq_{\mathbb{L}} y \leq_{\mathbb{L}} z$, since

$$\begin{aligned} D_\lambda(x, y) \cup D_\lambda(y, z) &= (\downarrow y - \downarrow x) \ \cup \ (\downarrow y - \downarrow z) \\ &= \ \downarrow z - \downarrow x \\ &= D_\lambda(x, z). \end{aligned}$$

Satisfying these conditions, D_λ will be called *functorial* (see definition 2).

Now we want to put the previous considerations in a slightly more general setting. As above, the starting point is a lattice $\mathbb{L} = (L, \leq_{\mathbb{L}})$. For a given subset A of L we consider the ordered monoid $\mathcal{M} = (M, \cup, \emptyset, \subseteq)$ with $M = 2^A$. Then, for all $x \in L$ we define

$$Ax \coloneqq \{a \in A \mid a \leq_{\mathbb{L}} x\}.$$

Based on this setup, we consider the following maps:

$$\begin{aligned} \lambda \colon L &\longrightarrow M : x \mapsto Ax, \\ D_\lambda \colon \ \leq_{\mathbb{L}} &\longrightarrow M : (x, y) \mapsto Ay - Ax. \end{aligned} \tag{2}$$

Claim 1 *D_λ is functorial w. r. t. $(\mathbb{L}, \mathcal{M})$.*

Proof.
- Firstly, $D_\lambda(x, x) = \emptyset$ obviously holds for all $x \in L$.
- Secondly, we have to show that $D_\lambda(x, y) \cup D_\lambda(y, z) = D_\lambda(x, z)$ holds for all $x, y, z \in L$ with $x \leq_{\mathbb{L}} y \leq_{\mathbb{L}} z$:

  Let x, y, z be elements in L such that $x \leq_{\mathbb{L}} y \leq_{\mathbb{L}} z$. For all $a \in A$, $a \in D_\lambda(x, z)$ is equivalent to

  $$a \not\leq_{\mathbb{L}} x \quad \text{and} \quad a \leq_{\mathbb{L}} z.$$

  Let $a \in D_\lambda(x, z)$. We distinguish two situations:

Case 1: $\boxed{a \leq_{\mathbb{L}} y}$ Then, $a \not\leq_{\mathbb{L}} x$ and $a \leq_{\mathbb{L}} y$, that is $a \in D_\lambda(x, y)$.

Case 2: $\boxed{a \not\leq_{\mathbb{L}} y}$ Then, $a \not\leq_{\mathbb{L}} y$ and $a \leq_{\mathbb{L}} z$, that is $a \in D_\lambda(y, z)$.

Hence, $a \in D_\lambda(x, y) \cup D_\lambda(y, z)$.

On the other hand, assume $a \in D_\lambda(x, y) \cup D_\lambda(y, z)$. Hence, $a \not\leq_{\mathbb{L}} x$ and $a \leq_{\mathbb{L}} y$, or $a \not\leq_{\mathbb{L}} y$ and $a \leq_{\mathbb{L}} z$. Then, $a \not\leq_{\mathbb{L}} x$ (since $x \leq_{\mathbb{L}} y$) and $a \leq_{\mathbb{L}} z$ (since $y \leq_{\mathbb{L}} z$) which yields $a \in D_\lambda(x, z)$. □

In (2), we introduced D_λ as a function with domain $\leq_{\mathbb{L}}$. Next, we want to look for an extension of D_λ onto $L \times L$. We achieve this by the following map:

$$d_\lambda \colon L \times L \longrightarrow M : (x, y) \mapsto D_\lambda(x \wedge y, y) \tag{3}$$

Claim 2 *The map d_λ is a* generalized quasi metric (GQM) *w. r. t. $(L, \mathcal{M})$, that is, the subsequent conditions are satisfied:*

(A0) *for all $x, y \in L$:* $\emptyset \subseteq d_\lambda(x, y)$,

(A1) *for all $x \in L$:* $d_\lambda(x, x) = \emptyset$,

(A2) *for all $x, y, z \in L$:* $d_\lambda(x, z) \subseteq d_\lambda(x, y) \cup d_\lambda(y, z)$.

We remind the reader that A is called *join-dense* in $\mathbb{L}$ if for all $x, y \in L$ with $x \not\leq_{\mathbb{L}} y$ there exists $a \in A$ such that $a \not\leq_{\mathbb{L}} y$ and $a \leq_{\mathbb{L}} x$.

Claim 3 *Let A be join-dense in $\mathbb{L}$. Then d_λ is a* generalized metric (GM) *w. r. t. $(L, \mathcal{M})$, that is, d_λ is a GQM which additionally satisfies:*

(A3) *For all $x, y \in L$:* $d_\lambda(x, y) = \emptyset = d_\lambda(y, x) \implies x = y$.

A more general definition for the underlying concepts will be given in definition 3.

Proof. (A0) Obviously, for all $x, y \in L$, the condition $\emptyset \subseteq d_\lambda(x, y)$ is satisfied.

(A1) Clear, since for all $x \in L : d_\lambda(x, x) = \emptyset$.

(A2) We have to show that $d_\lambda(x, z) \subseteq d_\lambda(x, y) \cup d_\lambda(y, z)$ holds for all $x, y, z \in L$. This is equivalent to

$$D_\lambda(x \wedge z, z) \subseteq D_\lambda(x \wedge y, y) \cup D_\lambda(y \wedge z, z).$$

To do so, let $a \in D_\lambda(x \wedge z, z)$. Hence, $a \not\leq_{\mathbb{L}} x \wedge z$ and $a \leq_{\mathbb{L}} z$, which implies

$$a \not\leq_{\mathbb{L}} x \quad \text{and} \quad a \leq_{\mathbb{L}} z.$$

We have to examine two cases:

Case 1: $\boxed{a \leq_{\mathbb{L}} y}$ Hence, $a \not\leq_{\mathbb{L}} x \wedge y$ and $a \leq_{\mathbb{L}} y$. It follows $a \in D_\lambda(x \wedge y, y)$.

Case 2: $\boxed{a \not\leq_{\mathbb{L}} y}$ Hence, $a \not\leq_{\mathbb{L}} y \wedge z$ and $a \leq_{\mathbb{L}} z$. It follows $a \in D_\lambda(y \wedge z, z)$.

All in all, also (A2) is satisfied. Consequently, d_λ is a GQM.

(A3) Let $x, y \in L$. We suppose $d_\lambda(x, y) = \emptyset$. This is equivalent to

$$\begin{aligned} & Ay - A(x \wedge y) = \emptyset \\ \iff\quad & Ay = A(x \wedge y). \end{aligned}$$

Taking advantage of the precondition $d_\lambda(x, y) = \emptyset = d_\lambda(y, x)$, we follow that

$$Ay = A(x \wedge y) = A(y \wedge x) = Ax.$$

Hence, $y = x$, as A is join-dense.

All in all, d_λ is a GM w. r. t. $(L, \mathcal{M})$. □

Claim 4 *The map D_λ is* supermodular *w. r. t.* $(\mathbb{L}, \mathcal{M})$ [10]*, that is, for all $x, y \in L$, the following condition holds:*

(A4) $D_\lambda(x \wedge y, y) \subseteq D_\lambda(x, x \vee y)$

Proof. Let $a \in D_\lambda(x \wedge y, y)$. Since $D_\lambda(x \wedge y, y)$ equals $Ay - A(x \wedge y)$, we know that

$$a \in Ay \quad \text{and} \quad a \notin A(x \wedge y). \tag{4}$$

According to the definition of A, we obtain $a \leq y$ and $a \not\leq x \wedge y$. Hence, $a \leq x \vee y$.

Suppose $a \leq x$. As $a \leq y$, it follows $a \leq x \wedge y$ which is a contradiction to (4). Therefore, $a \in A(x \vee y) - Ax = D_\lambda(x, x \vee y)$. □

3 Abstract Approach

We want to put our recent examinations from the special case into a more general setting. For that, we start with some necessary definitions [1, 3, 10].

Definition 1 $\mathcal{M} = (M, *, \varepsilon, \leq)$ *is an* **ordered monoid** *if* $\mathbb{M} := (M, *, \varepsilon)$ *is a monoid and $(M, \leq)$ is a poset such that $a \leq b$ implies $c * a \leq c * b$ and $a * c \leq b * c$, for all $a, b, c \in M$.*

Definition 2 *Let $\mathbb{P} = (P, \leq_{\mathbb{P}})$ be a poset and $\mathcal{M} = (M, *, \varepsilon, \leq)$ be an ordered monoid. A function*

$$\Delta\colon\ \leq_{\mathbb{P}}\ \longrightarrow M$$

is called **functorial w. r. t.** $(\mathbb{P}, \mathcal{M})$*, if*

– *for all* $p \in P: \quad \Delta(p,p) = \varepsilon$,

– *for all* $p,t,q \in P$ *with* $p \leq_{\mathbb{P}} t \leq_{\mathbb{P}} q: \quad \Delta(p,t) * \Delta(t,q) = \Delta(p,q)$.

Furthermore, Δ *is called* **weakly positive**, *if* $\varepsilon \leq \Delta(p,q)$ *for all* $(p,q) \in\ \leq_{\mathbb{P}}$.

Δ *is called* **supermodular** *w. r. t.* $(\mathbb{P}, \mathcal{M})$, *if* $\Delta(p \wedge q, q) \leq \Delta(p, p \vee q)$ *holds for all* $(p,q) \in\ \leq_{\mathbb{P}}$.

Furthermore, Δ *is called* **submodular** *w. r. t.* $(\mathbb{P}, \mathcal{M})$, *if* $\Delta(p \wedge q, q) \geq \Delta(p, p \vee q)$ *holds for all* $(p,q) \in\ \leq_{\mathbb{P}}$.

Definition 3 *Let* P *be a set, and* $\mathcal{M} = (M, *, \varepsilon, \leq)$ *be an ordered monoid. A function* $d\colon P \times P \longrightarrow M$ *is called* **generalized quasi-metric (GQM) w. r. t.** $(P, \mathcal{M})$, *if*

(A0) *for all* $(p,q) \in\ \leq_{\mathbb{P}}\ : \quad \varepsilon \leq d(p,q)$

(A1) *for all* $p \in P: \quad d(p,p) = \varepsilon$

(A2) *for all* $p,t,q \in P: \quad d(p,t) * d(t,q) \leq d(p,q)$

If in addition, (A3) holds, d *is a* **generalized metric (GM) w. r. t.** $(P, \mathcal{M})$:

(A3) *for all* $(p,q) \in P \times P: \quad d(p,q) = \varepsilon = d(q,p) \quad \Longrightarrow \quad p = q$

For a given $\Delta\colon \leq_{\mathbb{P}} \longrightarrow M$, does there exist a generalized quasi-metric $d\colon P \times P \longrightarrow M$ w. r. t. $(P, \mathcal{M})$ which extends Δ such that $d|_{\leq_{\mathbb{P}}} = \Delta$?

Theorem 1 *Let* $\mathbb{P} = (P, \leq_{\mathbb{P}})$ *be a lattice. If a map* $\Delta\colon \leq_{\mathbb{P}}\ \longrightarrow M$ *is weakly positive, supermodular and functorial w. r. t.* $(\mathbb{P}, \mathcal{M})$, *then*

$$d\colon P \times P \longrightarrow M, (p,q) \mapsto \Delta(p \wedge q, q)$$

is a GQM w. r. t. $(P, \mathcal{M})$.

Proof. Obviously, conditions (A0) and (A1) from definition 3 hold for d.

For (A2), we have to show that $d(p,q) \leq_{\mathbb{P}} d(p,t) * d(t,q)$ holds for all $p,t,q \in P$. According to the definition of d this means

$$\Delta(p \wedge q, q) \leq_{\mathbb{P}} \Delta(p \wedge t, t) * \Delta(t \wedge q, q).$$

We will prove this inequality immediately in Claim 3 below. However, first of all, we need to show two properties in preparation for that.

Claim 1 (Interval Property)

$$t \leq_{\mathbb{P}} x \leq_{\mathbb{P}} y \leq_{\mathbb{P}} z \quad \Longrightarrow \quad \Delta(x,y) \leq \Delta(t,z).$$

Proof. Since Δ is functorial, we obtain

$$\Delta(t,z) = \Delta(t,x) * \Delta(x,y) * \Delta(y,z).$$

As Δ is weakly positive, we get

$$\Delta(x,y) = \varepsilon * \Delta(x,y) * \varepsilon \leq \Delta(t,z).$$

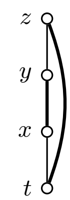

Fig. 1

$\diamond$

Claim 2 (Meet Property)

$$x \leq_{\mathbb{P}} y \quad \Longrightarrow \quad \Delta(x \wedge z, y \wedge z) \leq \Delta(x,y).$$

Proof. To show this implication, we rewrite the right hand side:

$$\Delta(x \wedge z, y \wedge z) = \Delta(x \wedge (y \wedge z), y \wedge z)$$

We continue denoting $y \wedge z$ by y' and derive

$$\begin{aligned} \Delta(x \wedge (y \wedge z), y \wedge z) &= \Delta(x \wedge y', y') \\ &\leq \Delta(x, x \vee y'), \end{aligned}$$

due to supermodularity of Δ. We know that $x \vee y' \leq_{\mathbb{P}} y$, since $x \leq_{\mathbb{P}} y$ and $y' \leq_{\mathbb{P}} y$. Hence, with Claim 1, we get

$$\Delta(x \wedge z, y \wedge z) \leq \Delta(x,y).$$

$\diamond$

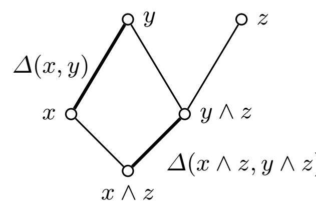

Fig. 2

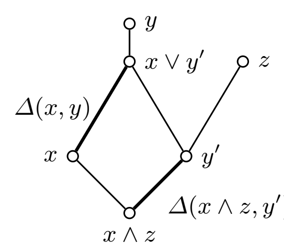

Fig. 3

Claim 3

$$\Delta(p \wedge q, q) \leq \Delta(p \wedge t, t) * \Delta(t \wedge q, q).$$

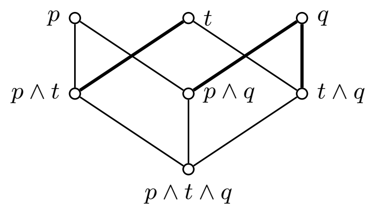

Fig. 4

Proof. Taking advantage of Claim 1, we know that

$$\Delta(p \wedge q, q) \leq \Delta(p \wedge t \wedge q, q).$$

Since Δ is functorial, it follows

$$\Delta(p \wedge t \wedge q, q) = \Delta(p \wedge t \wedge q, t \wedge q) * \Delta(t \wedge q, q).$$

With Claim 2, we finally receive

$$\Delta(p \wedge t \wedge q, t \wedge q) * \Delta(t \wedge q, q) \leq \Delta(p \wedge t, t) * \Delta(t \wedge q, q).$$

Therefore, $\Delta(p \wedge q, q) \leq \Delta(p \wedge t, t) * \Delta(t \wedge q, q)$. □

The latter theorem can be applied to various concepts of distance between objects of a given lattice. In the following, we will study some interesting applications in different context, starting with formal concept analysis.

4 Application to FCA

Let $\mathbb{K} = (G, M, I)$ be a finite formal context. Then the set of formal concepts of $\mathbb{K}$ is given by

$$B\mathbb{K} := \{(X, Y) \in 2^G \times 2^M \mid X' = Y \text{ and } , Y' = X\}$$

and the formal concept lattice of $\mathbb{K}$ is defined as

$$\mathfrak{B}\mathbb{K} := (B\mathbb{K}, \leq_{\mathfrak{B}\mathbb{K}})$$

with $c_1 \leq_{\mathfrak{B}\mathbb{K}} c_2$ iff $A_1 \subseteq A_2$ holds for all $c_1 = (A_1, B_1), c_2 = (A_2, B_2) \in B\mathbb{K}$.

Remarkably, the map

$$d_{ext} \colon B\mathbb{K} \times B\mathbb{K} \longrightarrow \mathbb{N} \quad \text{such that} \quad (c_1, c_2) \mapsto \#(A_2 - A_1)$$

is a GM w. r. t. $(B\mathbb{K}, \mathcal{M})$ with $\mathcal{M} := (\mathbb{N}, +, 0, \leq)$. The reason for this is based in Theorem 1, as we will outline below.

For all $c_1 = (A_1, B_1), c_2 = (A_2, B_2) \in B\mathbb{K}$ it follows

$$d_{ext}(c_1, c_2) = \#A_2 - \#(A_1 \cap A_2),$$

since $A_2 - A_1 = A_2 - (A_1 \cap A_2)$ and $\#$ is the counting measure.

To verify that d_{ext} is a GM, we define

$$D_{ext}\colon \leq_{\mathfrak{B}\mathbb{K}} \longrightarrow \mathbb{N} \quad \text{such that} (c_1, c_2) \mapsto \#A_2 - \#A_1.$$

Claim 4 *D_{ext} is functorial w. r. t. $(\mathfrak{B}\mathbb{K}, \mathcal{M})$, weakly positive and supermodular.*

Proof. The properties of being weakly positive and functorial are clear due to the definition of D_{ext} via the counting measure. Let us have a closer look at the supermodularity:

Let $c_1, c_2 \in B\mathbb{K}$. We have to show that

$$D_{ext}(c_1 \wedge c_2, c_2) \overset{!}{\leq} D_{ext}(c_1, c_1 \vee c_2).$$

Transforming the left hand side, we obtain

$$\begin{aligned} D_{ext}(c_1 \wedge c_2, c_2) &= D_{ext}\Big(\big(A_1 \cap A_2, (A_1 \cap A_2)'\big), \big(A_2, B_2\big)\Big) \\ &= \#A_2 - \#(A_1 \cap A_2) \\ &= d_{ext}(c_1, c_2). \end{aligned}$$

On the right hand side, we get

$$\begin{aligned} D_{ext}(c_1, c_1 \vee c_2) &= D_{ext}\Big(\big(A_1, A_2\big), \big(\underbrace{(B_1 \cap B_2)'}_{=(A_1 \cup A_2)''}, B_1 \cap B_2\big)\Big) \\ &= \#\big((A_1 \cup A_2)''\big) - \#A_1 \\ &\geq \#(A_1 \cup A_2) - \#A_1 \\ &= \#A_2 - \#(A_1 \cap A_2) \\ &= d_{ext}(c_1, c_2). \end{aligned}$$

Hence,

$$D_{ext}(c_1 \wedge c_2, c_2) \leq D_{ext}(c_1, c_1 \vee c_2)$$

and the supermodularity is shown. □

Obviously, by theorem 1 together with claim 4 it immediately follows that d_{ext} is a GM w. r. t. $(B\mathbb{K}, \mathcal{M})$.

Remark. In analogy to the above, the map

$$d_{int}\colon B\mathbb{K} \times B\mathbb{K} \longrightarrow \mathbb{N} \quad \text{such that} \quad (c_1, c_2) \mapsto \#(B_1 - B_2)$$

is a GM w. r. t. $(B\mathbb{K}, \mathcal{M})$.

5 Application to Dempster-Shafer-Theory

Choosing $L \coloneqq 2^U$ and $A \coloneqq L$, where U is a finite set, allows us a link to *Dempster-Shafer-Theory*.

Let m be a *mass function on* L, that is

$$m \colon L \longrightarrow \mathbb{R}_{\geq 0} : X \mapsto mX \quad \text{is a map such that } m(\emptyset) = 0 \text{ and } \sum_{X \in L} mX = 1.$$

We define

$$\mathrm{Bel}_m \colon L \longrightarrow \mathbb{R}_{\geq 0} : X \mapsto \sum_{T \subseteq X} mT$$

as the so-called *belief map* w. r. t. m and

$$\mathrm{Pl}_m \colon L \longrightarrow \mathbb{R}_{\geq 0} : X \mapsto \sum_{T \in L : T \cap X \neq \emptyset} mT$$

as the so-called *plausibility map* w. r. t. m.

Obviously, for all $X \in \mathrm{Bel}_m$, the equation $\mathrm{Bel}_m X + \mathrm{Pl}_m(U - X) = 1$ holds. That is:

$$\begin{aligned} 1 - \mathrm{Pl}_m(U - X) &= \mathrm{Bel}_m X \\ 1 - \mathrm{Bel}_m X &= \mathrm{Pl}_m(U - X). \end{aligned} \tag{5}$$

Claim 5 *Let Δ be the function which maps every pair (X, Y) with $X \subseteq Y \subseteq U$ to*

$$\Delta(X, Y) \coloneqq \mathrm{Bel}_m Y - \mathrm{Bel}_m X. \tag{6}$$

Δ is functorial, weakly positive, and supermodular w. r. t. the power set lattice of U into the naturally ordered additive monoid of non-negative real numbers.

Applying this claim to theorem 1, we receive that

$$d \colon L \times L \longrightarrow \mathbb{R}_{\geq 0},\ (X, Y) \mapsto \Delta(X \cap Y, Y) = \mathrm{Bel}_m Y - \mathrm{Bel}_m(X \cap Y)$$

is a GM w. r. t. the power set of U into the naturally ordered additive monoid of non-negative real numbers.

Remark. With the *plausibility map* introduced above, a submodular pendant to the supermodular map Δ in (6) can be constructed via

$$\widetilde{\Delta}(X, Y) \coloneqq \mathrm{Pl}_m Y - \mathrm{Pl}_m X \qquad \text{where } X \subseteq Y \subseteq U.$$

$\widetilde{\Delta}$ is indeed submodular, as the following inequation holds:

$$\mathrm{Pl}_m(X \cup Y) + \mathrm{Pl}_m(X \cap Y) \leq \mathrm{Pl}_m X + \mathrm{Pl}_m Y.$$

This can be shown by using the equality

$$\operatorname{Pl}_m X = 1 - \operatorname{Bel}_m(U - X)$$

that we have already observed in (5).

Remark. Dually to theorem 1, $\widetilde{\Delta}$ induces a GM $\tilde{d}$ via

$$\tilde{d} \colon L \times L \longrightarrow \mathbb{R}_{\geq 0},\ (X,Y) \mapsto \widetilde{\Delta}(X, X \cup Y) = \operatorname{Pl}_m(X \cup Y) - \operatorname{Pl}_m X.$$

6 A Fundamental Construction of Generalized Quasi Metrics

Let $\mathbb{P} = (P, \leq_{\mathbb{P}})$ be a poset, $\mathbb{M} = (M, *, \varepsilon)$ be a monoid, and $* \colon P \times M \longrightarrow P$ be a map such that the following properties are satisfied:

① For all $p \in P$ and all $x, y \in M$: $\quad p * (x * y) = (p * x) * x$

② For all $p \in P$: $\quad p * \varepsilon = p$

③ For all $p, y \in P, x \in M$: $\quad p \leq_{\mathbb{P}} q \implies p * x \leq_{\mathbb{P}} q * x$

Then we call the triple $(\mathbb{P}, \mathbb{M}, *)$ a *poset right monoid action*.

In this setup, we consider the map $\nabla \colon P \times P \longrightarrow M$ defined by

$$\nabla(p, q) := \{x \in M \mid q \leq_{\mathbb{P}} p * x\} \text{ for all } p, q \in P.$$

Claim 6 *For all $p, q, r \in P$ the following reverse triangle inequality holds:*

$$\nabla(p,q) * \nabla(q,r) \subseteq \nabla(p,r)$$

Proof. We choose $z \in \nabla(p,q) * \nabla(q,r)$. That is, there exits $x \in \nabla(p,y)$ and $y \in \nabla(q,r)$ such that $z = x * y$.

Since $x \in \nabla(p,q)$, we know that $q \leq_{\mathbb{P}} p * x$. Analogously, $y \in \nabla(q,r)$ implies $r \leq_{\mathbb{P}} q * y$.

Using property ③, we get

$$\begin{aligned} r \leq_{\mathbb{P}} q * y &\overset{③}{\leq_{\mathbb{P}}} (p * x) * y \\ &\overset{①}{=} p * (x * y) \\ &= p * z. \end{aligned}$$

All in all, $r \leq_{\mathbb{P}} p * z$ which implies $z \in \nabla(p,r)$. □

Let $(L, *, \epsilon, \leq)$ be a *residual complete lattice*, that is an ordered monoid for which $*$ preserves arbitrary infima in each component. Furthermore, we consider a map $\nu \colon M \longrightarrow L$ which satisfies the condition

$$\nu(x * y) \leq \nu x * \nu y \quad \text{for all } x, y \in M. \tag{7}$$

On this basis, we construct the following map

$$d \colon P \times P \longrightarrow L \quad \text{via} \quad (p, q) \mapsto \inf \nu\big(\nabla(p, q)\big).$$

Claim 7 *The map d satisfies the triangle inequality, that is*

$$d(p, r) \leq d(p, q) * d(q, r) \quad \textit{holds for all } p, q, r \in P.$$

Proof. Let $p, q, r \in P$. First, we transform the inequality's right hand side:

$$\begin{aligned} d(p, q) * d(q, r) &= \inf \nu\big(\nabla(p, q)\big) * \inf \nu\big(\nabla(q, r)\big) \\ &= \inf\Big(\nu\big(\nabla(p, q)\big) * \nu\big(\nabla(q, r)\big)\Big). \end{aligned}$$

Hence, we have to show that $\inf \nu\big(\nabla(p, q)\big) \leq \inf\Big(\nu\big(\nabla(p, q)\big) * \nu\big(\nabla(q, r)\big)\Big)$.

Let $t \in \nu\big(\nabla(p, q)\big) * \nu\big(\nabla(q, r)\big)$. It follows that there exists $x \in \nabla(p, q)$ and $y \in \nabla(q, r)$ such that $t = \nu x * \nu y$, which is greater than or equal to $\nu(x * y)$ due to property, i. e. we obtain

$$\nu(x * y) \leq t. \tag{8}$$

Consequently, with claim 6, we get

$$x * y \in \nabla(p, q) * \nabla(q, r) \subseteq \nabla(p, r).$$

Applying ν on both sides yields

$$\nu(x * y) \in \nu\big(\nabla(p, r)\big).$$

Hence,

$$\inf \nu\big(\nabla(p, r)\big) \leq \nu(x * y) \overset{(8)}{\leq} t$$

and the triangle inequality of d is shown. □

Definition 4 *Let $\mathbb{M} = (M, *, \varepsilon)$ be a monoid and let $\mathcal{L} = (L, *, \epsilon, \leq)$ be an ordered monoid. Then, a map $\nu \colon M \longrightarrow L$ is a* **monoid norm w. r. t.** $(\mathbb{M}, \mathcal{L})$ *if $\nu(\varepsilon) = \epsilon$ and $\nu(x * y) \leq \nu x * \nu y$ holds for all $x, y \in M$.*

Theorem 2 *Let $(\mathbb{P}, \mathbb{M}, *)$ be a poset right monoid action with $\mathbb{P} = (P, \leq_{\mathbb{P}})$ and $\mathbb{M} = (M, *, \varepsilon)$. Further, let $\mathcal{L} = (L, *, \epsilon, \leq)$ be a residual complete lattice and $\nu \colon M \longrightarrow L$ be a monoid norm w. r. t. $(\mathbb{M}, \mathcal{L})$.*

Then

$$d \colon P \times P \longrightarrow L, (p, q) \mapsto \inf \nu\big(\nabla(p, q)\big)$$

is a GQM w. r. t. $(P, \mathcal{L})$.

7 Application to join geometries

Let $\mathbb{P} = (P, \leq)$ be a complete lattice. Then an element $x \in P$ is called *compact in* $\mathbb{P}$ if for every subset T of P with $x \leq \sup T$ there exists a finite subset U of T such that $x \leq \sup U$.

Definition 5 *A* **join geometry** *is defined as a pair* $(\mathbb{P}, E)$ *consisting of a complete lattice* $\mathbb{P}$ *and a join-dense subset* E *consisting of compact elements in* $\mathbb{P}$*.*

For the following, let $(\mathbb{P}, E)$ be a join geometry such that for all $p, q \in P$ there exists a compact element $r \in P$ such that $q \leq p \vee r$.

Then the triple $(\mathbb{P}, \mathbb{M}, *)$ is a *poset right monoid action* for $\mathbb{M} = (M, \cup, \emptyset)$ with $M := 2^E_{fin}$ and

$$*\colon P \times M \longrightarrow P, \quad (x, D) \mapsto x \vee \sup D.$$

Moreover, the map

$$\nu\colon M \longrightarrow \mathbb{N} \cup \{\infty\}, \quad D \mapsto \#D$$

is a monoid norm w. r. t. $(\mathbb{M}, \mathcal{L})$ for

$$\mathcal{L} := \big(\mathbb{N} \cup \{\infty\}, +, 0, \leq\big)$$

(which forms a residual complete lattice). Obviously, by theorem 2 it follows that

$$\begin{aligned} d\colon P \times P \longrightarrow \mathbb{N} \cup \{\infty\}, \quad (p, q) \mapsto & \inf \nu\big(\nabla(p, q)\big) \\ & = \min\{\#D \mid D \in 2^E_{fin} : q \leq p \vee \sup D\} \end{aligned}$$

is a GM w. r. t. $(P, \mathcal{L})$.

This result has an important specialisation for closure operators on power sets of finite sets.

8 Application to Closure Operators

Let U be a finite set and γ be a closure operator on $\mathbb{P} := \big(P, \subseteq\big)$ with $P := 2^U$. Further, let $\mathcal{M} := \big(\mathbb{N}, +, 0, \leq\big)$. Then the map

$$d\colon P \times P \longrightarrow \mathbb{N}, (X, Y) \mapsto \min\big\{\#T \mid T \in P : Y \subseteq \gamma(X \cup T)\big\}$$

is a GQM w. r. t. $(P, \mathcal{M})$, which we want to call the *closure distance*.

In particular, the restriction of d onto $\gamma P \times \gamma P$ is a GM w. r. t. $(\gamma P, \mathcal{M})$.

In context of information pooling, for a group of received elements, we can construct the corresponding closure and with the *closure distance* d from above, the distance to a given closure can be evaluated. This works for arbitrary closure operators, which also includes closure systems of a *matroid*, for instance.

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