On the Gruenberg–Kegel graphs of finite groups

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Abstract

Let G be a finite group. The spectrum of G is the set $\omega(G)$ of orders of all its elements. The subset of prime elements of $\omega(G)$ is called the prime spectrum of G and is denoted by $\pi(G)$. The spectrum $\omega(G)$ of a group G defines its Grunberg–Kegel graph (or prime graph) $\Gamma(G)$ with vertex set $\pi(G)$, in which any two different vertices r and s are adjacent if and only if the number rs belongs to the set $\omega(G)$. We discuss some problems concerning coincidence of Gruenberg–Kegel graphs of non-isomorphic finite groups and of realizability of a graph as the Gruenberg–Kegel graph of a finite group.

1 Terminology and Notation

During this paper by "group" we mean "a finite group" and by "graph" we mean "a finite undirected graph without loops and multiple edges". Our notation and terminology are mostly standard and can be found in [1, 3, 4, 7, 16].

Let G be a group. Denote by $\pi(G)$ the set of all prime divisors of the order of G and by $\omega(G)$ the *spectrum* of G, i.e. the set of all its element orders. The set $\omega(G)$ defines the *Gruenberg–Kegel graph* (or the *prime graph*) $\Gamma(G)$ of G; in this graph the vertex set is $\pi(G)$ and different vertices p and q are adjacent if and only if $pq \in \omega(G)$. Denote the number of connected components of $\Gamma(G)$ by $s(G)$ and the set of connected components of $\Gamma(G)$ by $\{\pi_i(G) \mid 1 \leq i \leq s(G)\}$; for a group G of even order, we assume that $2 \in \pi_1(G)$. Denote by $t(G)$ the greatest cardinality of a coclique in the Gruenberg–Kegel graph of the group G and by $t(r, G)$ the greatest cardinality of a coclique in the Gruenberg–Kegel graph of the group G containing the prime r.

Let π be a set of primes. Denote by π' the set of the primes not in π. Given a natural n, denote by $\pi(n)$ the set of its prime divisors. Then $\pi(|G|)$ is exactly $\pi(G)$ for any group G. If $|\pi(G)| = n$ then G is called *n-primary.* A subgroup H of a group G is called a *π-Hall subgroup* if $\pi(H) \subseteq \pi$ and $\pi(|G : H|) \subseteq \pi'$.

We will denote by $S(G)$ the *solvable radical* of a group G (i. e. the largest solvable normal subgroup of G), by $F(G)$ the *Fitting subgroup* of G (i. e. the largest nilpotent normal subgroup of G), by $Soc(G)$ the *socle* of G (i. e. the subgroup of G generated by the set of all non-trivial minimal normal subgroups of G) and by $O_p(G)$ the largest normal p-subgroup of G. A group G is *almost simple* if $Soc(G)$ is a simple group. It's well known, G is almost simple if and only if there exists a simple group S such that $S \cong Inn(S) \trianglelefteq G \leq Aut(S)$. In this case, $S \cong Soc(G)$.

If G and H are groups, then we use notation $G : H$ $(G \leftthreetimes H)$ for a split extension (semidirect product) of G by (with, on) H.

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Recall, a group G is a *Frobenius group* if there is a non-trivial subgroup C of G such that $C \cap gCg^{-1} = \{1\}$ whenever $g \notin C$. A subgroup C is a *Frobenius complement* of G. Let

$$K = \{1\} \cup (G \setminus \bigcup_{g \in G} gCg^{-1}).$$

Then K is a normal subgroup of Frobenius group G with a Frobenius complement C which is called *Frobenius core* of G. Note, C and K are Hall subgroups of G and $G = K \rtimes C$.

Recall, that a *n-clique* (resp. a *n-coclique*) is a graph with n vertices in which all the vertices are pairwise adjacent (resp. non-adjacent). A graph Γ is called *bipartite* with parts of sizes m and n if its vertices can be divided into two non-empty disjoint subsets V_m and V_n such that $|V_m| = m$, $|V_n| = n$ and the vertices from the same subset are non-adjacent. We will denote by $K_{m,n}$ a *complete bipartite graph* whose vertices are adjacent if and only if they belong to different subsets.

2 Finite groups and their Gruenberg-Kegel graphs

In the finite group theory many researchers are interested in various problems of the study of groups by their arithmetical properties. One of such problems is the problem of the study of a group by some properties of its Gruenberg–Kegel graph.

The Gruenberg–Kegel graph of a group is its fundamental arithmetical invariant, having numerous applications.

General problem. Describe all groups whose Gruenberg–Kegel graphs have a given property.

One of the first results concerning prime graphs of groups was the following unpublished result of K. W. Gruenberg and O. Kegel, which was published later in the paper of J. S. Williams [15, Theorem A].

Gruenberg–Kegel Theorem. If G is a group with disconnected Gruenberg–Kegel graph, then one of the following statements holds:

(1) G is a Frobenius group;

(2) G is a 2-Frobenius group, i. e., $G = ABC$, where A and AB are normal subgroups of G, AB and BC are Frobenius groups with cores A and B and complements B and C, respectively;

(3) G is an extension of a nilpotent $\pi_1(G)$-group by a group A, where $S \trianglelefteq A \leq \mathrm{Aut}(S)$, S is a simple non-abelian group with $s(G) \leq s(S)$, and A/S is a $\pi_1(G)$-group.

Gruenberg–Kegel Theorem implies the complete description of solvable group with disconnected prime graphs: they are exactly groups from items (1) and (2). Moreover, if G is a Frobenius group or 2-Frobenius group, then $\Gamma(G)$ has exactly two connected components (see [17]); if G is solvable, then each connected component of $\Gamma(G)$ is a clique. Williams and Kondrat'ev gave a description of connected components of the Gruenberg–Kegel graphs of all simple non-abelian groups [15, 9]. In particular, they proved, if S is a simple non-abelian group then $s(S) \leq 6$. An adjacency criterion of the Gruenberg–Kegel graphs of simple groups and cocliques of maximal sizes in the Gruenberg–Kegel graphs of simple groups are known (see [13, 14]).

In 1999 M. S. Lucido [10, Theorem 1] proved, if G is a group whose Gruenberg–Kegel graph contains a 3-coclique, then G is non-solvable. But this result follows directly from earlier results by G. Higman and P. Hall. Indeed, let G be a solvable group whose Gruenberg–Kegel graph $\Gamma(G)$ contains a 3-coclique $\{p_1, p_2, p_3\}$. Using Hall Theorem [3, Theorem 6.4.1] we conclude, that there exists a solvable $\{p_1, p_2, p_3\}$-Hall subgroup H of G, such that its Gruenberg–Kegel graph $\Gamma(H)$ is a 3-coclique. In view of Higman Theorem [8, Theorem 1] a solvable group in which every element has prime power order is at most 2-primary. A contradiction.

A. V. Vasil'ev [12, Propositions 2, 3] proved a theorem which widely generalize Gruenberg–Kegel Theorem.

Theorem (A. V. Vasil'ev, 2005). Let us assume, that $t(G) \geq 3$ and $t(2, G) \geq 2$ for some group G. Then the following conditions hold:

(1) there exists a simple non-abelian group S such that $S \trianglelefteq G/S(G) \leq Aut(S)$;

(2) if $\rho \subseteq \pi(G)$ is a coclique in $\Gamma(G)$ with $|\rho| \geq 3$, then at most one of the primes from ρ divides the product $|S(G)| \cdot |G/S(G) : S|$. In particular, $t(S) \geq t(G) - 1$;

(3) one of the following conditions holds:

(a) every $p \in \pi(G)$ which is non-adjacent to 2 in $\Gamma(G)$ doesn't divide the product $|S(G)| \cdot |G/S(G) : S|$. In particular, $t(2, S) \geq t(2, G)$;

(b) there exists $r \in \pi(S(G))$ which is non-adjacent to 2 in $\Gamma(G)$; in this case $t(G) = 3$, $t(2, G) = 2$ and $S \cong A_7$ or $PSL_2(q)$ for any odd q.

Moreover, Vasil'ev's Theorem is valid for any non-solvable group G such that $t(2, G) \geq 2$.

3 On the realizability of a graph as the Gruenberg-Kegel graph of a group

The Gruenberg–Kegel graph of a group G can be considered as a graph with $|\pi(G)|$ vertices, where the vertices are labeled by different primes and two different vertices p and q are adjacent if and only if there is an element $g \in G$ such that the order of g is pq. The following problem arises.

Problem 1. Let Γ be a graph with vertices labeled by different primes. Is there a group such that Γ is its Gruenberg–Kegel graph?

Problem 1 has a negative solution in the general case. It can be shown by the following example.

Example 1 (see [2, § 3]). Let Γ be a 3-coclique with $V(\Gamma) = \{p, q, r\}$, where p, q and r are pairwise distinct odd primes. Assume, G is a group such that $\Gamma(G) = \Gamma$. Then by Feit-Thompson theorem G is solvable and by Gruenberg—Kegel theorem it is a Frobenius group or a 2-Frobenius group. Gruenberg—Kegel graphs of Frobenius groups and of 2-Frobenius groups have exactly two connected components, but $\Gamma(G)$ has three connected components. This contradiction proves that there is no a group G such that $\Gamma(G) = \Gamma$.

We say that a graph Γ with $|\pi(G)|$ vertices is *realizable as the Gruenberg–Kegel graph of a group G* if there exists an one-to-one correspondence ϕ between the vertex set of Γ and $\pi(G)$ such that the vertices x and y are adjacent in Γ if and only if $\phi(x)\phi(y) \in \omega(G)$. In other words, there exists a vertices marking of Γ by different primes from $\pi(G)$ such that the marked graph is equal to $\Gamma(G)$. A graph Γ is *realizable as the Gruenberg–Kegel graph of a group* if Γ is realizable as the Gruenberg–Kegel graph of an appropriate group G. In other words, Γ is isomorphic to the Gruenberg–Kegel graph of an appropriate group G.
As a generalization of Problem 1, we obtain the following

Problem 2. Let Γ be a graph. Is Γ realizable as the Gruenberg–Kegel graph of a group?

In view of Lucido's result, it's easy to see, 3-coclique is not realizable as the Gruenberg—Kegel graph of a solvable group, but it is realizable as the Gruenberg—Kegel graph of the group A_5. However, Problem 2 has a negative solution in the general case. It can be shown by the following example.

Example 2 (see [11, Lemma 5]). Let Γ be a n-coclique, where $n \geq 5$. Then Γ is not realizable as the Gruenberg–Kegel graph of a group. Suppose the converse. If Γ is realizable as the Gruenberg–Kegel graph of a group G, then $s(G) = n \geq 5$ and $t(G) = n \geq 5$. In view of Lucido's result, G is non-solvable. By Gruenberg–Kegel Theorem there exists a simple non-abelian group S such that $S \trianglelefteq G/F(G) \leq Aut(S)$ and $s(S) \geq s(G) = n$. According to [9, 15] we have $n \leq s(S) \leq 6$ and S is isomorphic either to J_4 or to $E_8(q)$ for $q \equiv 0, 1, 4 \pmod 5$, and consequently $|\pi(S)| > 6$. A contradiction.

There are just a few works devoted to problems of realizability of a graph as the Gruenberg–Kegel graph of a group.
In 2008 I. N. Zharkov, who was a student of V. D. Mazurov, proved in his unpublished graduate work [18] the following theorem.

Theorem (I. N. Zharkov, 2008). A chain is realizable as the Gruenberg–Kegel graph of a group if and only if the length of this chain is at most 4.

In the paper [2] Problem 2 was solved for graphs with at most 5 vertices by A. L. Gavrilyuk, I. V. Khramtsov, A. S. Kondrat'ev and the author. This investigation was initiated by A. Gavrilyuk in 2012 on the 43th International Youth School-conference "Modern Problems in Mathematics and its Applications" and was terminated by I. Khramtsov, A. Kondrat'ev and the author. The following theorem was proved.

Theorem (A. L. Gavrilyuk, I. V. Khramtsov, A. S. Kondrat'ev, N. M., 2014). Let Γ be a graph with at most five vertices. Then
(1) If Γ is 5-coclique then Γ is not realizable as the Gruenberg-Kegel graph of a group.
(2) If Γ is not 5-coclique then Γ is realizable as the Gruenberg-Kegel graph of a group.

4 On the realizability of graphs with 3-cocliques as the Gruenberg-Kegel graphs of non-solvable groups

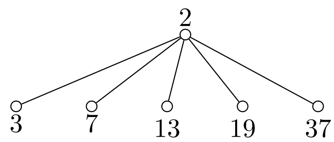
It's interesting to obtain solutions of Problem 2 for some classes of graphs contain 3-cocliques. In [11] Problem 2 was solved for complete bipartite graphs, moreover for any complete bipartite graph which is realizable as the Gruenberg–Kegel graph of a group it was obtained the "number" of realizations. The following theorem was proved.

Theorem 1 (N.M., D. Pagon, 2016). Let Γ be a complete bipartite graph $K_{m,n}$, where $m \leq n$. Then the following statements hold:

(1) Γ is realizable as the Gruenberg–Kegel graph of a group if and only if $m + n \leq 6$ and $(m, n) \neq (3, 3)$;

(2) if $m + n \leq 6$ and $(m, n) \neq (3, 3), (1, 5)$, then there exist infinitely many sets T of primes such that Γ is realizable as the Gruenberg–Kegel graph of a group G and $T = \pi(G)$;

(3) if $(m, n) = (1, 5)$ and Γ is realizable as the Gruenberg–Kegel graph of a group G, then $\pi(G) = \{2, 3, 7, 13, 19, 37\}$, $O_2(G) \neq 1$, and $G/O_2(G) \cong {}^2G_2(27)$.



The only marking of $K_{1,5}$

5 Realizability of a graph as the Gruenberg-Kegel graph of a solvable group

In 2015 A. Gruber, T. Keller, M. Lewis, K. Naughton and B. Strasser [6] obtained a complete description of graphs which are realizable as Gruenberg–Kegel graphs of solvable groups. They proved the following theorem.

Theorem (A. Gruber, T. Keller, M. Lewis, K. Naughton, B. Strasser, 2015). A graph Γ is realizable as the Gruenberg–Kegel graph of a solvable group if and only if its complement is 3-colorable and triangle free.

Recently I. B. Gorshkov and the author investigated the question of isomorphism of Gruenberg–Kegel graphs of solvable groups and almost simple groups. In [5, Theorem 1] the following theorem was proved.

Theorem 2 (I. B. Gorshkov, N.M., 2016). Let G be an almost simple group. Then the following conditions are equivalent:

(1) $\Gamma(G)$ does not contain a 3-coclique;

(2) $\Gamma(G)$ is isomorphic to the Gruenberg–Kegel graph of a solvable group;

(3) $\Gamma(G)$ is equal to the Gruenberg–Kegel graph of an appropriate solvable group.

The following question naturally arises.

Question. Is there a graph without 3-cocliques, whose complement is not 3-colorable, but which is isomorphic to the Gruenberg–Kegel graph of an appropriate non-solvable group? In the other words, is there a graph which is realizable as the Gruenberg–Kegel graph of an appropriate non-solvable group, but is not realizable as the Gruenberg–Kegel graph of any solvable group?

Let $q = p^m$ and G be an almost simple group such that $S = Soc(G)$ is isomorphic to one of the following simple groups: A_n, where $n \geq 5$; $PSL_n(q)$, where $n \geq 2$ and $(n, q) \neq (2, 2), (2, 3)$; $PSU_n(q)$, where $n \geq 3$ and $(n, q) \neq (3, 2)$; $PSp_n(q)$, where $n \geq 4$ is even; $P\Omega_n(q)$, where $n \geq 7$ is odd; $P\Omega_n^{\pm}(q)$, where $n \geq 8$ is even; an exceptional group of Lie type over the field of order q; a sporadic simple group. Let f be the standard field automorphism of S and g be the standard graph automorphism of S (see [4, 2.5.12, 2.5.13]).

In [19, Theorems 1, 3] M. R. Zinov'eva and V. D. Mazurov described simple groups whose Gruenberg–Kegel graphs coincide with Gruenberg–Kegel graphs of Frobenius groups or 2-Frobenius groups. Using [13, 14] it is not difficult to prove these simple groups are exactly all simple groups whose Gruenberg–Kegel graphs coincide with Gruenberg–Kegel graphs of solvable groups. In [5, Theorem 2] the following theorem was proved.

Theorem 3 (I. Gorshkov, N.M., 2016). Let G be an almost simple group and $S = Soc(G)$. Then $\Gamma(G)$ does not contain a 3-coclique if and only if $\pi(G) = \pi(S)$ and one of the following conditions holds:

(1) S is isomorphic to one of the following groups: A_9, A_{10}, A_{12}, $PSU_3(9)$, $PSU_4(2)$, $PSp_6(2)$, $P\Omega_8^+(2)$, ${}^3D_4(2)$;

(2) G is isomorphic to one of the following groups: S_5, S_6, $PGL_2(9)$, M_{10}, $Aut(A_6)$, S_8, $Aut(PSL_2(8))$, $Aut(PSL_3(2))$, $PGL_3(4)\langle f\rangle$, $PGL_3(4)\langle g\rangle$, $Aut(PSL_3(4))$, $PSL_4(4)\langle f\rangle$, $PSL_4(4)\langle g\rangle$, $Aut(PSL_4(4))$, $Aut(PSU_5(2))$;

(3) $G \cong PGL_2(p)$, where p is either a Fermat prime or a Mersenne prime;

(4) $S \cong PSL_2(2^m)$, where $m \geq 4$ is even and $\{2\} \subseteq \pi(G/S)$;

(5) $S \cong PSL_3(p)$, where p is a Mersenne prime and $(p-1)_3 \neq 3$;

(6) $S \cong PSL_3(p)$, where p is either a Fermat prime or a Mersenne prime, $(p-1)_3 = 3$ and $Inndiag(S) \leq G \leq Aut(S)$;

(7) $S \cong PSL_3(2^m)$, where $m \geq 3$, $(2^m-1)_3 = 3$ and $Inndiag(S)\langle g\rangle \leq G \leq Aut(PSL_3(2^m))$;

(8) $S \cong PSL_3(2^m)$, where $m \geq 3$, $(2^m-1)_3 \neq 3$ and $S\langle g\rangle \leq G \leq Aut(PSL_3(2^m))$;

(9) $S \cong PSL_4(2^m)$, where $m \geq 3$ and $S\langle g\rangle \leq G \leq Aut(PSL_4(2^m))$;

(10) $S \cong PSU_3(p)$, where p is a Fermat prime and $(p+1)_3 \neq 3$;

(11) $S \cong PSU_3(p)$, where p is a Fermat prime, $(p+1)_3 = 3$ and $Inndiag(S) \leq G \leq Aut(S)$;

(12) $S \cong PSU_3(2^m)$, where $m \geq 2$, $(2^m-1)_3 = 3$, $\{2\} \subseteq \pi(G/S)$ and $Inndiag(S) \leq G \leq Aut(S)$;

(13) $S \cong PSU_3(2^m)$, where $m \geq 2$, $(2^m-1)_3 \neq 3$ and $\{2\} \subseteq \pi(G/S)$;

(14) $S \cong PSU_4(2^m)$, where $m \geq 2$ and $\{2\} \subseteq \pi(G/S)$;

(15) $S \cong PSp_4(q)$.

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