Application of method of special series with functional arbitrariness in construction of solutions of nonlinear partial differential equations

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Abstract

In the paper an analytical method for constructing solutions of nonlinear partial differential equations based on the method of special series is considered. The essence of this approach is in representation of solutions of an original equation in the form of series with recurrently coefficients computed in powers of specially selected functions. Such functions may depend on several independent variables, and also contain arbitrary functions that may depend on a smaller number of independent variables. The paper gives an answer to A.F. Sidorov's question about using functional arbitrariness in study of convergence of the constructed series. There are examples how to prove global convergence by choosing an arbitrary function.

1 Introduction

The most well known and commonly used methods for representation of solutions of nonlinear partial differential equations are Taylor series and Fourier series (especially for linear equations of mathematical physics). However, generally, Taylor series have only local convergence which is often quite slow, and Fourier series when applied to nonlinear equations lead to finding the coefficients as solutions of infinite systems of nonlinear equations, which have to be truncated to find an approximate ones for the Fourier coefficients. A.F. Sidorov has formulated the following requirements for the series used for constructing solutions of nonlinear partial differential equations [1]:

- the series have to converge in sufficiently large domains;
- the series have to converge fast (several first terms of the series have to correctly represent a basic features of the solution);
- the coefficients of the series have to be computed with using non-expensive and effective algorithms;
- a way of construction of the series has to be applicable to a wide class of equations.

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To ensure the above requirements, it is desirable to get the coefficients of the series not by successive differentiating (as in the Taylor series) but by integrating some simple recurrent systems of ordinary differential equations. It is also beneficial that in the case of a nonlinear problem the initial part of the chain of ordinary differential equations is nonlinear, then it makes possible to pass this short segment catching some basic features of the nonlinear boundary value problem, and the other terms could be determined from a system of linear differential equations of simpler structure. The following series allow to satisfy these requirements to some extent.

Picard's method of iteration for ordinary differential equations leads to construction of solutions in the form of power series. Investigation of the structure of the resulting successive approximations allowed Cauchy to prove his theorem for analytic equations. Direct transfer of these approach to partial differential equation is impossible. A method of special series with recurrently computed coefficients, elaborated to represent solutions of nonlinear partial differential equations [2, 3, 4] is more close to the idea of representation of solutions of nonlinear ordinary differential by series in powers of functions which are solutions of other equations. An example of a series with recurrently computed coefficients used to represent solutions of nonlinear ordinary differential equations is the result of N.P. Erugin's [5], concerning reexpansion of solutions of equation

$$\dot{y} = -y^2 + \sum_{k=3}^{\infty} a_k y^k, \quad a_k = \text{const}, \quad y(0) = y_0 > 0$$

in power of z, which is a solution of "reduced" equation

$$\dot{z} = -z^2, \quad z(0) = z_0 > 0.$$

In this case, for sufficiently small y_0, z_0 a solution of the original equation can be represented as a convergent series

$$y = z\left[1 + z\left(c + \sum_{k=1}^{\infty} \alpha_k(c) z^k\right)\right], \quad c = \text{const}$$

with recurrently calculated coefficients $\alpha_k(c)$.

This approach is most ideologically close to the proposed method. In fact, for a class of nonlinear ordinary differential equations were constructed series convergent to solutions of these equations. Coefficients of these series are recurrently computed as solutions of other (perhaps simpler) ordinary differential equations. Solutions of these simpler equations we shall call *basic functions*, in accordance with the terminology of the method of special series. In this example, the basic function is a solution of the "reduced" equation z.

2 The method of special series

Let us consider one of constructions of special series for solving Cauchy problem for nonlinear partial differential equations of the form

$$u_t = F\left(t, u, \frac{\partial u}{\partial x}, \cdots, \frac{\partial^m u}{\partial x^m}\right), \quad u(0, x) = u^0(x), \tag{1}$$

where F is a polynomial of the unknown function $u(t, x)$ and its derivatives with respect to the space variable. The solution is represented by the series

$$u(t, x) = \sum_{n=0}^{\infty} u_n(t) P^n(t, x) \tag{2}$$

by the powers of basic function $P(t, x)$ satisfying the overdetermined system

$$P_x = A(t, P), \quad P_t = B(t, P) \tag{3}$$

with functions $A(t, P)$ and $B(t, P)$ being analytic respect to P and such that $A(t, 0) \equiv 0$ and $B(t, 0) \equiv 0$. It was shown that if the initial conditions are written in the form

$$u^0(x) = \sum_{n=0}^{\infty} u_n^0 P^n(0, x), \tag{4}$$

then substituting series (2) into equation (1), collecting similar terms, and taking into account relations (3), we obtain the sequence of first-order ordinary differential equations for the coefficients $u_n(t)$

$$u_n' = F_n(t, u_n, \ldots, u_0), \qquad u_n(0) = u_n^0, \quad n = 1, 2, \ldots, \tag{5}$$

where the right-hand sides F_n include only u_j with $j \leq n$ and the coefficients u_n may be linearly contained only. Convergence of some special series in representing solutions of Cauchy problems is proved for different nonlinear partial differential equations [2, 7, 8]. It is also possible to use this method to solve initial-boundary value problems with exact satisfaction of zero boundary conditions [9].

As an example, we mention the construction of the series (2) for Korteweg-de Vries equation with initial condition $u(0,x)=u^0(x)$ in the form of (4). Basic function

$$P(x,t)=\frac{1}{\exp x+f(t)}, \quad f(t) \in C^1[0,\infty) \tag{6}$$

is used, which satisfies system (3) for

$$A(t,P)=-P+f(t)P^2, \quad B(t,P)=-f(t)'P^2$$

with an arbitrary function $f(t)$. Convergence of series (2), (6) is proved [2] for all $x\geq 0$, $t\geq 0$, if $0\leq f(t)\leq(\sqrt[3]{2}-1)/(2-\sqrt[3]{2})$. At $x=0$ the arbitrary function $f(t)$ gives rise a new boundary condition $u(0,t)=h(t)$. We can try to find the function $f(t)$ from the function $h(t)$ [6].

However, such an arbitrary functions can also be used to prove global convergence of special series with recurrently computed coefficients. Note that condition (3) is not necessary to find the coefficients recurrently. This recurrency is due to specifics of the studied nonlinear equations [3, 10]. In this case, such special series are called *consistent special series*. Further, we consider examples of consistent series.

3 Construction of consistent special series

It is convenient to show construction and application of consistent series on the base of example of model nonlinear equation

$$u_t = G(t, u, xu_x, u_x/x, u_x^2), \tag{7}$$

where G is a polynomial by $u, xu_x, u_x/x, u_x^2$. Consider for this equation Cauchy problem

$$u(x,0) = u_0(x), \quad -\infty < x < \infty. \tag{8}$$

Because the right-hand side of equation (7) contains independent variable x, then its solution cannot be represented in the form of (2), (3). However, due to specific of the equation, we can find a basis function that allows the series coefficients to be computed recurently.

Note that equation (7) is reduced to the transport equation by some contact transformation, direct application of the proposed method to the equation (7) is possible.

Assume that the coefficients of the polynomial G are continuous and limited functions for all $t \geq 0$. Solution of equation (7) will construct in the form of series

$$u(x,t)=\sum_{n=0}^{\infty} u_n(t)R_1^n(x,t), \tag{9}$$

where

$$R_1(x,t)=\frac{1}{x^2+f(t)}, \qquad f(t)\in C^1[0,\infty). \tag{10}$$

Let initial condition (8) can be represented as a convergent series

$$u_0(x) = \sum_{n=0}^{\infty} \frac{u_{n0}}{(x^2+1)^n}, \quad u_{n0} = \text{const}. \tag{11}$$

We have

Theorem 1. Let the following conditions are satisfied:

1) let for constant u_{n0}, defining initial data (11), the following inequalities are valid

$$|u_{n0}| \leq \frac{M^n}{2n^3}, \quad n \geq 1, \quad 0 < M < 1,$$

2) for an arbitrary function $f(t) \in C^1[0,\infty)$ conditions

$$f(0) = 1, \quad f(t) \geq 1, \quad |f'(t)| \leq d_1 \exp(d_2 t), \quad t \geq 0, \quad d_1, d_2 > 0$$

are valid.

Then series (9), (10) is a formal solution of equation (7) and there exists a number $b > 0$ such that the series converges uniformly to the solution of Cauchy problem (7), (8), (11) for all $-\infty < x < \infty$, $0 \leq t \leq T = b^{-1} \ln M^{-1}$.

Prof. First, we show that the series (9), (10) is a formal solution of the equation (7). Indeed, for a series (9) the following equalities are valid:

$$\begin{gathered} xu_x = 2\sum_{n=1}^{\infty}[f(n-1)u_{n-1} - nu_n]R_1^n, \\ \frac{u_x}{x} = -2\sum_{n=2}^{\infty}u_{n-1}(n-1)R_1^n, \\ u_x^2 = 4\sum_{n=2}^{\infty}\sum_{m+k=n}[m(k-1)u_m u_{k-1} - f(m-1)(k-1)u_{m-1}u_{k-1}]R_1^n, \\ u_t = \sum_{n=0}^{\infty}[u_n' - (n-1)u_{n-1}R_1^n]. \end{gathered} \tag{12}$$

Therefore, by substituting the series (9), (10) into equation (7), taking into account relations (12) for the coefficients $u_n(t)$ we obtain a sequence of linear first order differential equations

$$u_n' = S_n(t, f, f', u_k(t)), \quad u_n(0) = u_{n0}, \quad n \geq 0, \quad k \leq n. \tag{13}$$

Consequently, the series (9), (10) is a formal solution of the equation (7).

Using an explicit form of solutions of (13) for the coefficients $u_n(t)$ the following estimations are proved by induction:

$$|u_n(t)| \leq \frac{M^n}{n^3}\exp(bnt), \quad n \geq 1, \quad t \geq 0.$$

With these estimations, we can easily prove convergence of series (9), (10) to solution of Cauchy problem (7), (8), (11 for all $-\infty < x < \infty$, $0 \leq t \leq T = b^{-1} \ln M^{-1}$. Theorem 1 is proved.

4 Global convergence of consistent special series with functional arbitrariness

For example of model of nonlinear equations let show that, using the arbitrary function, available in the basic functions, it is possible to construct a consistent series that converges to a solution of a Cauchy problem for all $-\infty < x < \infty$, $t \geq 0$.

For equation

$$u_t = \frac{u_x}{2x} + H(t, u, u_x^2) \tag{14}$$

consider a Cauchy problem with initial condition (11).

Let function H can be represented in the form

$$H(t, u, u_x^2) = \sum_{\nu=2}^{N_1}\sum_{m+k=\nu}\gamma_{mk}(t)u^m u_x^{2k}. \tag{15}$$

Then equation (14) is a special case of equation (7). Consequently, by Theorem 1, the solution can be found in the form of (9), (10). After substitution of consistent special series (9), (10) into equation (14), (15), taking into account relations (12), we obtain for the following sequence of linear differential equations to determ the coefficients $u_n(t)$:

$$u_n'=(f'-1)(n-1)u_{n-1}+\sum_{\nu=2}^{N_1}\sum_{m+k=\nu}\gamma_{mk}(t)\sum_{j_0+j_1=n}\sum_{n_1+\ldots+n_m=j_0}\prod_{i=1}^{m}u_{n_i}$$
$$\times\sum_{l_1+\ldots+l_k=j_1}4^k\prod_{i=1}^{k}\sum_{p+q=l_i}[p(q-1)u_pu_{q-1}-fu_{p-1}u_{q-1}],\quad n\geq 1 \tag{16}$$

with initial data $u_n(0)=u_{n0}$.

We show that by choosing an arbitrary function $f(t)$ can be construct a series with recurrently computed coefficients, converging to the solution for all x and $t\geq 0$. We have

Theorem 2. Suppose that the following conditions are valid:

1) $\gamma_{mk}(t)\in C[0,\infty)$, $|\gamma_{mk}(t)|\leq\gamma_0$, $t\geq 0$, $\gamma_0\geq 0$,

2) $u_{00}=0$, and for the constant u_{n0}, $n\geq 1$ the following inequalities are satisfied

$$|u_{n0}|\leq\frac{M}{2n^3},\quad M,M_0=\text{const}\quad 0<M\leq M_0=M_0(\gamma_0,N_1),$$

3) as a function $f(t)$ we consider

$$f(t)\equiv 1+t.$$

Then consistent special series (9), (10) converges uniformly to the solution of problem (14), (15),(11) for all $-\infty<x<\infty$ and $t\geq 0$.

Proof. Since $f(t)=1+t$, then the solution of equation (16) may be written in the form

$$u_n=u_{n0}+\int_0^t\sum_{\nu=2}^{N_1}\sum_{m+k=\nu}\gamma_{mk}(\tau)\sum_{j_0+j_1=n}\sum_{n_1+\ldots+n_m=j_0}\prod_{i=1}^{m}u_{n_i}$$
$$\times\sum_{l_1+\ldots+l_k=j_1}4^k\prod_{i=1}^{k}\sum_{p+q=l_i}[p(q-1)u_pu_{q-1}-fu_{p-1}u_{q-1}]d\tau,\quad n\geq 1. \tag{17}$$

By function $f(t)$ it is possible to equal the expression with coefficient u_{n-1} to zero in the right side of equation (16). I.e., the following equality is valid

$$(f'-1)(n-1)u_{n-1}\equiv 0.$$

This is one of the essential factors, taking into account the specifics of the equation, and thus it allows to prove global convergence of the consistent series.

By the method of mathematical induction using direct evaluation of the right side of formula (17), we prove inequality

$$|u_n(t)|\leq\frac{M}{n^3}(1+t)^{n-1},\quad t\geq 0,\quad n\geq 1. \tag{18}$$

If $n=1$, $u_1(t)\equiv u_{10}$ and inequality (18) is valid. Assuming the validity of estimates (18) with $n\leq N$, we carry out the proof for $n=N+1$.

To simplify the calculations we carry out the proof of estimates (18) for the following equation:

$$u_t=\frac{u_x}{2x}+u^2, \tag{19}$$

which is a particular case of equation (14). For $n=2$

$$|u_2|=|u_{20}+u_{10}^2t|\leq|u_{20}|+|u_{10}^2t|\leq\frac{M}{2^4}+\frac{M^2}{4}t\leq\frac{M}{2^3}\Big(\frac{1}{2}+2M_0t\Big)\leq\frac{M}{2^3}(1+t).$$

Here we assume that $M_0\leq\frac{1}{2}$. I.e., inequality (18) is true. If $n=N+1$ for equation (18) formula (17) has the form

$$u_{N+1}=u_{N+1,0}+\int_0^t\sum_{k+m=N+1}u_k(\tau)u_m(\tau)d\tau.$$

For $|a_{N+1}|$ the following inequalities are valid

$$|u_{N+1}| \le |u_{N+1,0}| + \int_0^t \sum_{k+m=N+1} |u_k(\tau)||u_m(\tau)| d\tau$$

$$\le \frac{M}{2n^3} + M^2 \int_0^t \sum_{k+m=N+1} \frac{(1+\tau)^{k-1}}{k^3} \frac{(1+\tau)^{m-1}}{m^3} d\tau$$

$$\le \frac{M}{(N+1)^3}\Big(\frac{1}{2} + M_0(1+t)^N \frac{4^3\pi^2}{3}\Big) \le \frac{M}{(N+1)^3}(1+t)^N.$$

Here we assume that $M_0 \le \frac{1}{2}\frac{3}{4^3\pi^2}$. Consequently, inequality (18) is valid for any n.
These estimates allow us to prove convergence of series (9), (10)

$$|u(x,t)| \le \frac{M}{x^2+1+t} \sum_{n=1}^{\infty} \frac{(1+t)^{n-1}}{n^3(x^2+1+t)^{n-1}} \le \frac{M}{(x^2+1+t)} \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{M\pi^2}{6(x^2+1+t)}.$$

Similarly we can prove convergence of the series corresponding u_t, u_x. Theorem 2 is proved.

Remark 1. Convergence of (9), (10) can be proved under condition that

$$|f'(t) - 1| \le M_2 = M_2(\gamma_0, N_1), \qquad M_2 = \text{const}, \quad t \ge 0. \tag{20}$$

Otherwise, the series can diverge.

Let show that if condition (20) is not satisfied, then series (9), (10) diverges when $t \ge 1$. Let $f(t) \equiv 1$, i.e. the basic function is $R_1(x,t) \equiv (x^2+1)^{-1}$. Consider for linear equation

$$u_t = \frac{u_x}{2x} \tag{21}$$

a Cauchy problem

$$u(x,0) = \frac{M}{x^2+1}. \tag{22}$$

Then the coefficients of series (9), (10) satisfy equations

$$u_n' = -(n-1)u_{n-1}, \quad n \ge 1, \quad u_1(0) = M, \quad u_n(0) = 0$$

and the solution of problem (21) (22) has the form

$$u(x,t) = M \sum_{n=1}^{\infty} \frac{(-1)^{n+1} t^{n-1}}{(x^2+1)^n}.$$

Thus

$$u(0,t) = M \sum_{n=1}^{\infty} (-1)^{n+1} t^{n-1}$$

and for $t \ge 1$ the series diverges. Hence, we have shown that by choosing function $f(t)$, which is increased in a certain way, we can prove global convergence of series (9), (10). Otherwise, the series can diverge.

Remark 2. Constructing global solutions for a class of nonlinear equations in the form of consistent special series, not only the form of these equations is important, but also the magnitude of the initial data is important too.

In the case of initial value problem (22) for linear equation (21) only selected function $f(t)$ is important, and the value of the constant M could be arbitrary, but for nonlinear equations the value of M is essential in constructing global solutions.

Consider for nonlinear equation (19) initial problem (22). The solution will be constructed in the form of consistent special series (9), (10) with function $f(t) \equiv 1 + t$. Then the coefficients $u_n(t)$ are determined from equations

$$u_1 = M, \qquad u_n' = \sum_{k+m=n} u_k u_m, \quad u_n(0) = 0, \quad n \geq 2$$

and can be found explicitly

$$u_n = M^n t^{n-1}.$$

The final solution of problem (19), (22) can be represented in the form of consistent series

$$u(x,t) = \sum_{n=1}^{\infty} \frac{M^n t^{n-1}}{(x^2 + 1 + t)^n} \tag{23}$$

equals to this sum

$$s(x,t) = \frac{M}{x^2 + 1 + t(1 - M)}. \tag{24}$$

Expression (24) is an exact solution of (19), (22) and $M > 1$ in solution $s(x,t)$ a peculiarity occurs with increasing t, and series (23) will also diverge. For $0 < M \leq 1$ we have an exact global solution $s(x,t)$ of the problem and series (23), which converges for all x and $t \geq 0$ to this solution.

5 Conclusion

Thus, a positive answer to the A.F. Sidorov's question about the possibility of using the arbitrary function is given, This function included to the basic functions of special series is used to prove convergence of the constructed series to the solution of nonlinear partial differential equations. It was shown that the choice of an arbitrary function allows to prove convergence of special series to the solution of the equation for all x and $t \geq 0$. In addition, it was found that the magnitude of the initial data is significantly affected on the region of convergence of special series.

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