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USING SINGULAR PERTURBATED SYSTEMS OF DIFFERENCIAL EQUATIONS OF INFINITE ORDER FOR COUNTABLE MARKOV CHAINS ANALYSIS*

ABSTRACT

Tikhonov-type Cauchy problems are investigated for systems of ordinary differential equations of infinite order with a small parameter μ *and initial conditions. It is studying the singular perturbated systems of ordinary differential equations of infinite order of Tikhonov-type* $\dot{x} = f(x(t,g_x), y(t,g_y),t)$, $\mu\dot{y} == F(x(t,g_x), y(t,g_y),t)$ *with the initial conditions* $x(t_0,g_x) = g_x$, $y(t_0,g_y) = g_y$, *where* $x, f \in X$, $X \in R^n$ *are n-dimensional functions;* $y, F \in Y$, $Y \subset l_1$ *are infinite-dimensional functions and* $t \in [t_0,t_1]$ *(*$t_0 < t_1 \le \infty$*),* $t \in T$, $T \in R$*;* $g_x \in X$ *and* $g_y \in Y$ *are given vectors,* $\mu > 0$ *is a small real parameter. The results may be applied to the queueing networks, which arise from the modern telecommunications.*

KEYWORDS

Systems of differential equations of infinite order; singular perturbated systems of differential equations; small parameter; v countable Markov chains.

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ИСПОЛЬЗОВАНИЕ СИНГУЛЯРНО ВОЗМУЩЕННЫХ СИСТЕМ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ДЛЯ АНАЛИЗА СЧЕТНЫХ МАРКОВСКИХ ЦЕПЕЙ

АННОТАЦИЯ

В статье исследованы задачи Коши для систем обыкновенных дифференциальных уравнений бесконечного порядка с малым параметром μ *тихоновского-типа* $\dot{x} = f(x(t,g_x), y(t,g_y),t)$, $\mu\dot{y} == F(x(t,g_x), y(t,g_y),t)$ *с начальными условиями* $x(t_0,g_x) = g_x$, $y(t_0,g_y) = g_y$, *где* $x, f \in X$, $X \in R^n$ *- функции конечного числа измерений;* $y, F \in Y$, $Y \subset l_1$ *- функции бесконечного числа измерений и* $t \in [t_0,t_1]$ *(*$t_0 < t_1 \le \infty$*),* $t \in T$, $T \in R$*;* $g_x \in X$ *, а также* $g_y \in Y$ *заданные векторы,* $\mu > 0$ *- малый параметр. Результаты данной работы могут быть применены для анализа прикладных задачах в теории массового обслуживания.*

КЛЮЧЕВЫЕ СЛОВА

Системы дифференциальных уравнений бесконечного порядка; сингулярно возмущенных системы дифференциальных уравнений; малый параметр; счетные цепи Маркова.

Introduction

The recent research of service networks with complex routing discipline in [16], [17], [18] transport networks [1], [4], [5] faced with the problem of proving the global convergence of the solutions of

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certain infinite systems of ordinary differential equations to a time-independent solution. Scattered results of these studies, however, allow a common approach to their justification. This approach will be expounded here. In work [11] the countable systems of differential equations with bounded Jacobi operators are studied and the sufficient conditions of global stability and global asymptotic stability are obtained. In [10] it was considered finite closed Jackson networks with N first come, first serve nodes and M customers. In the limit $M \to \infty$, $N \to \infty$, $M/N \to \lambda > 0$, it was got conditions when mean queue lengths are uniformly bounded and when there exists a node where the mean queue length tends to infty under the above limit (condensation phenomena, traffic jams), in terms of the limit distribution of the relative utilizations of the nodes. It was deriven asymptotics of the partition function and of correlation functions.

Cauchy problems for the systems of ordinary differential equations of infinite order was investigated A.N. Tihonov [13], K.P. Persidsky [12], O.A. Zhautykov [19], [20], Ju. Korobeinik [7] other researchers.

It was studied the singular perturbated systems of ordinary differential equations by A.N. Tihonov [14], A.B. Vasil'eva [15], S.A. Lomov [9] other researchers.

A particular our interest is the synthesis all these methods and its applications in telecommunications. In this paper we apply methods from [11] for the singular perturbated systems of ordinary differential equations of infinite order of Tikhonov-type.

TIKHONOV-TYPE CAUCHY PROBLEMS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS OF INFINITE ORDER WITH A SMALL PARAMETER

Let us consider Tikhonov-type Cauchy problems for systems of ordinary differential equations of infinite order with a small parameter μ and initial conditions:

$$\begin{aligned} \dot{x} &= f(x(t,g_x), y(t,g_y), t), \\ \mu\dot{y} &= F(x(t,g_x), y(t,g_y), t); \end{aligned} \tag{1}$$

$$\begin{aligned} x(t_0,g_x) &= g_x, \\ y(t_0,g_y) &= g_y, \end{aligned} \tag{2}$$

where $x, f \in X$, $X \in R^n$ are n-dimensional functions; $y, F \in Y$, $Y \subset l_1$ are infinite-dimensional functions and $t \in [t_0, t_1]$ ($t_0 < t_1 \le \infty$), $t \in T$, $T \in R$; $g_x \in X$ and $g_y \in Y$ are given vectors, $\mu > 0$ is a small real parameter; $x(t,g_x)$ and $y(t,g_y)$ are solutions of (1)-(2). Given functions $f(x(t,g_x), y(t,g_y), t)$ and $F(x(t,g_x), y(t,g_y), t)$ are continuous functions for all variables. Let S is an integral manifold of the system (1)-(2) in $X \times Y \times T$. If any point $t^* \in [t_0, t_1]$ $(x(t^*), y(t^*), t^*) \in S$ of trajectory of this system has at least one common point on S this trajectory $(x(t,G), y(t,g), t) \in S$ belongs the integral manifold S totally. If we assume in (1)-(2) that $\mu = 0$ than we have a degenerate system of the ordinary differential equations and a problem of singular perturbations

$$\begin{aligned} \dot{x} &= f(x(t,g_x), y(t), t), \\ 0 &= F(x(t,g_x), y(t), t); \\ x(t_0,g_x) &= g_x, \end{aligned} \tag{3}$$

where the dimension of this system is less than the dimension of the system (1)-(2), since the relations $F(x(t), y(t), t) = 0$ in the system (3) are the algebraic equations (not differential equations). Thus for the system (3) we can use limited number of the initial conditions then for system (1)-(2). Most natural for this case we can use the initial conditions $x(t_0,g_x) = g_x$ for the system (3) and the initial conditions $y(t_0,g_y) = g_y$ disregard otherwise we get the overdefined system. We can solve the system (3) if the equation $F(x(t), y(t), t) = 0$ could be solved. If it is possible to solve we can find a finite set or countable set of the roots $y_q(t,g_x) = u_q(x(t,g_x), t)$ where $q \in N$.

If the implicit function $F(x(t), y(t), t) = 0$ has not simple structure we must investigate the question about the choice of roots. Hence we can use the roots $y_q(t,g_x) = u_q(x(t,g_x), t)$ ($q \in N$) in (3) and solve the degenerate system

$$\begin{aligned} &\dot{x}_d = f(x_d(t,g_x),u_q(x_d(t,g_x),t),t); \\ &y_d(t_0,g_x) = g_x. \end{aligned} \tag{4}$$

Since it is not assumed that the roots $y_q(t,g_x) = u_q(x(t,g_x),t)$ satisfy the initial conditions of the Cauchy problem (1)-(2) ($y_q(t_0) \neq g_x$, $q \in N$), the solutions $y(t,g_y)$ (1)-(2) and $y_q(t,g_x)$ do not close to each other at the initial moments of time $t > 0$. Also there is a very interesting question about behaviors of the solutions $x(t,g_x)$ of the singular perturbated problem (1)-(2) and the solutions $x_d(t,g_x)$ of the degenerate problem (4). When $t = 0$ we have $x(t_0,g_x) = x_d(t_0,g_x)$. Do these solutions close to each other when $t \in (t_0,t_1]$? The answer to this question depends on using roots $y_q(t,g_x) = u_q(x(t,g_x),t)$ and the initial conditions which we apply for the systems (1)-(2) and (3).

LOCAL EXISTENCE THEOREM FOR CAUCHY PROBLEMS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS OF INFINITE ORDER

Let Tikhonov-type Cauchy problems for systems of ordinary differential equations of infinite order with a small parameter $\mu > 0$ and initial conditions (1)-(2) has a form:

$$\begin{aligned} &\dot{z} = P(z(t,G,\mu),t,\mu); z(t_0,G,\mu) = G, \\ &z = (x_1,x_2,...,x_n,y_1,y_2,...)^T, \\ &P(z(t,G,\mu),t,\mu) = (f_1,f_2,...,f_n,\mu^{-1}F_1,\mu^{-1}F_2,...)^T, \\ &G = (g_{x1},g_{x2},...,g_{xn},g_{y1},g_{y2},...)^T \end{aligned} \tag{5}$$

where $P(z(t,G,\mu),t,\mu)$ is the infinite-dimensional function; G is the given vector; $t \in [t_0,t_1]$ ($t_0 < t_1 \leq \infty$).

Let $z(t,G,\mu)$ be a continuously differentiable solution of the Cauchy problems (5) then there are $\Phi(t,G,\mu) = \partial z(t,G,\mu)/\partial G$, $\Psi(t,G,\mu) = \partial z(t,G,\mu)/\partial \mu$ where $\Phi(t,G,\mu)$ and $\Psi(t,G,\mu)$ satisfy of the system of ordinary differential equations in variations:

$$\begin{aligned} &\dot{z} = P(z(t,G,\mu),t,\mu), \\ &\dot{\Phi}(t,G,\mu) = J_z(t,G,\mu)\Phi(t,G,\mu), \\ &\dot{\Psi}(t,G,\mu) = J_z(t,G,\mu)\Psi(t,G,\mu) + \Lambda_\mu(t,G,\mu); \\ &z(t_0,G,\mu) = G, \Phi(t_0,G,\mu) = I, \Psi(t_0,G,\mu) = 0, t_0 \in T, \end{aligned} \tag{6}$$

where $J_z(t,G,\mu) = (\partial P_i/\partial z_j)_{i,j=1}^{\infty}$ is Jacobi matrix, I is an identity operator and $\Lambda_\mu(t,G,\mu) = (\partial P_i/\partial \mu)_{i=1}^{\infty}$ is a vector.

Theorem 1 (local existence theorem). *Let $P(z(t,G,\mu),t,\mu)$, $J_z(t,G,\mu)$, $\Lambda_\mu(t,G,\mu)$ be continuous and meet Gelder's local condition with $z \in U_\varepsilon(G)$ then the system (6) has only one solution, which meet the conditions $z(t_0,G,\mu) = G$, $z(t,G,\mu) \in U_\varepsilon(G)$. Thus $z(t,G,\mu)$ continuously differentiable with respect to the initial condition, and its derivative meet the equation (6).*

Proof. This statement is following from [3] (theorem 3.4.4) when the unlimited operator be $A = 0$. *End proof.*

The behavior of the solution $z(t,G,\mu)$ (5) and the nonnegative condition for the off-diagonal elements of the matrix $J_z(t,G,\mu)$ is demonstrated by the following theorem.

Theorem 2. *Let the solution z of (5) be $z(t,G,\mu) \in l_1$ for any $t \geq 0$, $G \in l_1$ and μ. The following claims are equal: (i) the off-diagonal elements $J_z(t,G,\mu)$ are non-negative for any G; (ii) for any G and any vector $h \in l_1, h \geq 0, z(t,G+h,\mu) \geq z(t,G,\mu)$.*

Proof. Let us examine a convex set Z, and $z(t,G,\mu) \in Z$ for any $G \in Z$, derivative $\Phi(t,G,\mu)$

of function $z(t,G,\mu)$ can be specify by simultaneous equations (6). In that case the following formula is fair for any $G^0, G^1 \in Z$:

$$\begin{aligned} &z(t,G^1,\mu) - z(t,G^0,\mu) = \\ &= \int_0^1 \Phi(t,\gamma(s),\mu)(G^1 - G^0)ds \end{aligned}, \tag{7}$$

where $\gamma(s) = (1-s)G^0 + sG^1, 0 \le s \le 1$.

In fact the function $z(t,G,\mu)$ transfer the segment $\gamma(s)$ into the curve $z(t,\gamma(s),\mu)$ in (7). The following formula is fair because of the continuous differentiability of function $z(t,G,\mu)$

$$z(t,\gamma(\tau),\mu) = z(t,G^0,\mu) + \int_0^\tau \frac{\partial z(t,\gamma(s),\mu)}{\partial s} ds.$$

By the formula of complex derivative

$$\frac{\partial z(t,\gamma(s),\mu)}{\partial s} = \frac{\partial z}{\partial G}(\gamma(s))\gamma'(s).$$

Recalling that $\partial z / \partial G = \Phi$ and $\gamma'(s) = G^1 - G^0$, with $\tau = 1$ we get (7). Let us suppose that statement (i) is fair. So because of (7)

$$z(t,G+h,\mu) - z(t,G,\mu) = \int_0^1 \Phi(t,\gamma(s),\mu)h ds,$$

where $\gamma(s) = G + sh, 0 \le s \le 1$. Because of non-negativeness of function $J_z(t,G,\mu)$ outside of diagonal from (7) we get $\Phi(t,\gamma(s),\mu) \ge 0$, so $\Phi(t,\gamma(s),\mu)h \ge 0$ whence we get statement (ii).

Let us suppose that (ii) is fair. Under the conditions of Theorem 1 P, J_z with $z \in U_\varepsilon(G)$ be continuous and meet Gelder's local condition. Let Gelder's local condition be $\|P\| < M_0, \|J\| < M_1$, and there are numbers $\delta > 0, \delta = min(\varepsilon / M_0, 1/M_1)$. Let $z(t,G,\mu) = G + z^*(t,G,\mu)$ be a solution of (7), where $z^*(t,G,\mu)$ is a fixed point of Picard's mapping $(\prod \theta)(t) = \int_{t_0}^t P(G + \theta(\tau))d\tau$ under conditions $t \in [t_0 - \delta_1, t_0 + \delta_1], \delta_1 < \delta$. Mapping $\prod$ is contraction with coefficient $\lambda = \delta_1 M_1 < 1$. Consider the approximation to solution $\tilde{z}(t,G,\mu) = G + \tilde{z}^*(t,G,\mu) = G + (t-t_0)P(z(t,G,\mu),t,\mu)$. We can see that

$$\begin{gathered} \|\tilde{z}(t,G,\mu) - z(t,G,\mu)\| = \\ = \|\tilde{z}^*(t,G,\mu) - z^*(t,G,\mu)\| \le \\ \le \frac{1}{1-\lambda} \left\|\prod \tilde{z}(t,G,\mu) - \tilde{z}(t,G,\mu)\right\|, \\ \prod \tilde{z}(t,G,\mu) - \tilde{z}(t,G,\mu) = \\ = \int_{t_0}^t P(G + (\tau - t_0)P)d\tau - \int_{t_0}^t P d\tau = \\ = \int_{t_0}^t (P(G + (\tau - t_0)P) - P)d\tau = D. \end{gathered}$$

Because of the derivative of the function P is limited and P meet Gelder's local condition with the constant M_1, where $\|P(G + (\tau - t_0)P(G)) - P(G)\| \le M_1 \|(\tau - t_0)P(G)\| \le M_0 M_1 |\tau - t_0|$, so $\|D\| \le M_0 M_1 (t-t_0)^2 / 2(1-\lambda)$, or $\|\tilde{z}(t,G,\mu) - z(t,G,\mu)\| \le M_0 M_1 (t-t_0)^2 / 2(1-\lambda)$. Using this estimation and for all small $\zeta > 0$ we have that

$$0 \le z(t, G + \zeta e_j, \mu) - z(t,G,\mu) = \zeta e_j + (t-t_0)[P(G + \zeta e_j) - P(G)] + \gamma(G,t),$$

where $\|\gamma(G,t)\| \le M_0 M_1 (t-t_0)^2 / 2(1-\lambda)$ and e_j is a vector, which has all coordinates equal to 0 but j-th coordinate equal to 1. Component $i \ne j$ of this inequality is given by $0 \le (t-t_0)[P_i(G + \zeta e_j) - P_i(G)] + \gamma_i(G,t)$. Dividing by $t - t_0 > 0$ and directing $t \to t_0$ on the right, considering $\gamma_i(G,t)/(t-t_0)\to 0$ we get $0\le P(G+\zeta e_j)-P(G)$. Let us divide last expression by ζ and direct $\zeta\to 0$

$$0\le \lim_{\zeta\to 0+}\frac{P(G+\zeta e_j)-P(G)}{\zeta}=\frac{\partial P_i}{\partial G_i}=J_{ij},$$

what is mean the fairing of statement (i). *End proof.*

Theorem 3. *Let* Φ *be Markovian mapping and* $G^0,G^1\in X$, $t\ge 0$, $\mu>0$ *than* $\|z(t,G^1,\mu)-z(t,G^0,\mu)\|\le\|G^1-G^0\|$.

Proof. Using (6) from the proofing of theorem 4 we have

$$\|z(t,G^1,\mu)-z(t,G^0,\mu)\|\le\int_0^1\|\Phi(t,\gamma(s))(G^1-G^0)\|ds. \tag{8}$$

Let function $\Phi(t,\gamma(s))$ is Markovian mapping for any

$$\forall t\ge 0, s\in[0,1], \|\Phi(t,\gamma(s))(G^1-G^0)\|\le\|G^1-G^0\|.$$

Estimating the integral, considering this inequality, we get required. *End proof.*

This theorem shows us the following sufficient condition for the boundedness of the norm-solution $z(t,G,\mu)$.

Corollary fact from theorem 3. *Let* $\exists G^*\in X: z(t,G^*,\mu)=G^*$. *Then* $\|z(t,G,\mu)-G^*\|\le\|G-G^*\|$ *with* $t\ge 0, G\in X$.

This fact we can use for solutions analysis of the systems (5).

Conclusions

The boundaries of applications and possible generalizations. Some works in the routing disciplines. All systems can be analyzed for the global stability but with some condition that the convergence to the steady-state solution will not coordinate-wise, but the norm. We have seen that the most serious constraints of our methods are non-negativity of the Jacobi matrix off-diagonal elements and the availability of the first integral, which equal to the sum of the components. It would be interesting to understand the physical meaning of these conditions. It is necessary to remember that such systems describes the behavior of the queue lengths on the devices. Roughly speaking, z_k is the proportion of units in the queue for a service, to which there is at least k requests (including requests, which are serviced at the moment). Non-negative elements of the Jacobi matrix indicate that the rate of change of z_k (i.e., the time derivative of z_k) can only grow at the expense of z_j with $j\ne k$. It can be reduced (or decrease) only due to u_k. Thus, with the increase of the portion of queues with a minimum number of requests j in the system, the percentage change in intensity with the minimum number of queues requests $k\ne j$ can only increase.

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