FROM AXIOMS TO STRUCTURAL RULES, THEN ADD QUANTIFIERS.

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We survey recent developments in the program of generating proof calculi for large classes of axiomatic extensions of a non-classical logics by translating each axiom into a set of structural rules, starting from a base calculus. We will introduce three proof formalisms: the sequent calculus, the hypersequent calculus and the display calculus. The calculi that are obtained derive exactly the theorems of the logic and satisfy a subformula property which ensures that a proof of a theorem only contains statements that are 'related to the conclusion'. These calculi can be used as a starting point for developing automated reasoning systems and to facilitate a proof-theoretic investigation of the logic (e.g. to prove interpolation, consistency, decidability, complexity). In the final section we discuss how first-order quantifiers may be added to these propositional calculi.

Much of the content here can be found in an extended form in the survey paper [10]. The main purpose of this abstract is to provide a concise 'hands-on' description of the methods. We present a subjective selection of problems while directing the reader to the references for a more exhaustive exposition.

1. Sequent calculus

Let $\mathcal{L}$ denote the intuitionistic language. The formulae from this language are given by the grammar:

$$A, B := \text{propositional variable } p \ \mid \ \bot \ \mid \ \top \ \mid \ A \wedge B \ \mid \ A \vee B \ \mid \ A \rightarrow B$$

The set of theorems of propositional intuitionistic logic is denoted by Int. Define a sequent to be a tuple (denoted $X \vdash A$) where X (antecedent) is a multiset of formulae and A (succedent) is a formula. A sequent calculus sInt for propositional intuitionistic logic Int is given below.

$$\bot \vdash C \qquad X \vdash \top \qquad p \vdash p \qquad \frac{X \vdash C}{X, Y \vdash C}\,(\mathrm{w}) \qquad \frac{X, Y, Y \vdash C}{X, Y \vdash C}\,(\mathrm{c})$$

$$\frac{X \vdash A \qquad B, Y \vdash C}{A \rightarrow B, X, Y \vdash C}\rightarrow\mathrm{l} \qquad \frac{A, X \vdash B}{X \vdash A \rightarrow B}\rightarrow r \qquad \frac{A_1, A_2, X \vdash C}{A_1 \wedge A_2, X \vdash C}\wedge l$$

$$\frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B}\wedge r \qquad \frac{X \vdash A_i}{X \vdash A_1 \vee A_2}\vee r \qquad \frac{A, X \vdash C \qquad B, X \vdash C}{A \vee B, X \vdash C}\vee l$$

A *derivation* in the sequent calculus is defined recursively in the usual way as either an initial sequent or the object obtained by applying a rule to the sequents concluding a derivation. Given a multiset $A_1, \ldots, A_n$, let $\wedge X$ denote $A_1 \wedge \ldots \wedge A_n$ if $n > 0$ else $\top$. Then the relationship between the sequent calculus and the logic can be described as follows.

Theorem 1 (Gentzen [14]). *Let X be a multiset and A a formula. Then $X \vdash A$ is derivable in* sInt *iff the formula $\wedge X \to A$ is a theorem of* Int. *Also $\vdash A$ is derivable in* sInt *iff A is a theorem of* Int.

The theorem reveals that sequents represent a certain normal form for formulae, specifically conjunction of formulae $\to$ formula. The point of note here is not that we have a proof calculus for the logic. After all, (axiomatic) Hilbert calculi for Int are well-known (see e.g. [5]). Instead, the point is that this proof calculus has the *subformula property*: every formula occurring in the premises occurs as a subformula of some formula in the conclusion. This means that any formula occurring in the derivation must occur in the conclusion.

From an automated reasoning perspective, this immediately suggests a *backward proof-search* procedure. The idea is to repeatedly applying proof-rules backwards. If a derivation is obtained, then the sequent is derivable. The subformula property tells us that the premise(s) are completely determined by the conclusion. Of course, further pruning and optimisation on backward proof-search is required to obtain a terminating efficient reasoning system.

Decidability is immediate: because of the weakening and contraction rules, we can replace the multisets in the antecedent with sets. Then every sequent appearing in backward proof search from $\vdash A$ has the form $Y \vdash B$ where $Y \in \mathcal{P}(\Omega)$ and $B \in \Omega$ where Ω is the set of all subformulae in $X \vdash A$ and $\mathcal{P}$ is the powerset operator. There are $|\mathcal{P}(\Omega)| \cdot |\Omega|$ possibilities. Now enumerate the trees with nodes labelled by such sequents (each node is permitted one or two children) such that the height of the tree is $\leq |\mathcal{P}(\Omega)| \cdot |\Omega|$. If one of these trees is a derivation of $\vdash A$ then A is a theorem. If not, there cannot possibly be a derivation of $\vdash A$ so A is not a theorem. Moreover, it is possible to amend the rules such that every premise antecedent is $\supseteq$ the conclusion ancetedent. This corresponds to a pruning of the backward proof search procedure and leads to the tight PSPACE upper-bound for Int.

Now suppose we wish to obtain a sequent calculus for the axiomatic extension of Int with the axiom $(p \to q) \vee (q \to p)$ (denoted $\mathsf{Int} + (p \to q) \vee (q \to p)$). This formula is not a theorem of Int (e.g. argue that $\vdash (p \to q) \vee (q \to p)$ is not derivable).

To obtain a sequent calculus for $\mathsf{Int} + (p \to q) \vee (q \to p)$, it would be tempting to add the sequent $\vdash (p \to q) \vee (q \to p)$ to sInt. Unfortunately not every theorem of $\mathsf{Int} + (p \to q) \vee (q \to p)$ is derivable using $\mathsf{sInt} + \vdash (p \to q) \vee (q \to p)$. The addition of the (cut) rule below rectifies this: $\vdash A$ derivable in $\mathsf{sInt} + (cut) + \vdash (p \to q) \vee (q \to p)$ iff A is a theorem of $\mathsf{Int} + (p \to q) \vee (q \to p)$.

$$\frac{X \vdash A \qquad A, Y \vdash B}{X, Y \vdash B}\ (\text{cut})$$

Unfortunately the calculus $\mathsf{sInt} + (cut) + \vdash (p \to q) \vee (q \to p)$ lacks the subformula property since the cut-formula A in (cut) need not appear in the conclusion. The key to applications is having a calculus with the subformula property so we need another solution.

2. Hypersequents

Theorem 1 reveals that every sequent can be read as a formula of the form conjunction of formulae $\to$ formula. Let us call such a formula an *implicational formula*. It turns out that the obstacle for constructing a calculus for $\mathsf{Int} + (p \to q) \vee (q \to p)$ is that the sequent-representation is too restrictive. A solution is to move from an implicational formula to a *finite disjunction of implicational formulae*.

A *hypersequent* [1, 23] is a non-empty multiset of sequents denoted as below. In particular, each X_i is a multiset of formulae and A_i is a formula.

$$X_1 \vdash A_1 \;\mid\; X_2 \vdash A_2 \;\mid\; \dots \;\mid\; X_{n+1} \vdash A_{n+1} \tag{1}$$

A sequent in the hypersequent is called a *component*. The notion of derivability of a hypersequent is defined analogously to the sequent case. The rules of the hypersequent calculus hInt are obtained from sInt by appending $g \mid$ to each sequent and adding the rules below left and centre. The g is a schematic variable that can be omitted or instantiated with a hypersequent. Define the (cut) rule for the hypersequent calculus as below right.

$$\frac{g \mid X \vdash B}{g \mid X \vdash B \mid Y \vdash A}\,\text{ew} \qquad \frac{g \mid X \vdash B \mid X \vdash B}{g \mid X \vdash B}\,\text{ec} \qquad \frac{g \mid X_1 \vdash A \qquad h \mid A, Y \vdash B}{g \mid h \mid X, Y \vdash B}\,(\text{cut})$$

Theorem 2. *The hypersequent h in (1) is derivable in $\mathsf{hInt} + (cut)$ iff the formula $(\wedge X_1 \to A_1) \vee \dots \vee (\wedge X_{n+1} \to A_{n+1})$ (the* interpretation *h^I of h) is a theorem of Int. Also $\vdash A$ is derivable in $\mathsf{hInt} + (cut)$ iff A is a theorem of Int.*

2.1. **Translating suitable axioms into rules.** The method introduced in [7] transforms suitable axiom into a structural rule (i.e. a rule which contains no logical connectives) and ultimately yields a calculus with the subformula property.

Starting from Theorem 2 it may be argued that h is derivable in $\mathsf{hInt}{+}(cut){+}g \mid \;\vdash p \to q \mid \;\vdash q \to p$ iff $h^{\mathbf{I}}$ is a theorem of $\mathsf{Int} + (p \to q) \vee (q \to p)$ (**replace top-level disjunction symbols with $\mid$ and add context $g \mid$**).

The new aim is thus to transform $\mathsf{hInt} + (cut) + g \mid \;\vdash p \to q \mid \;\vdash q \to p$ into a hypersequent calculus with the subformula property which derives the same hypersequents. Certainly we can simplify $g \mid \;\vdash p \to q \mid \;\vdash q \to p$ to $g \mid p \vdash q \mid q \vdash p$ by **repeated application of the invertible rules** ($\to$r). *Invertible rules* are rules which preserve derivability upwards. In other words, the premises are derivable whenever the conclusion is derivable. To deal with $\mathsf{hInt} + (cut) + g \mid p \vdash q \mid q \vdash p$ we use the proof-theoretic form of Ackermann's lemma [7, 11]:

Lemma 2.1 (Ackermann's lemma)**.** *Let $\mathcal{C}$ be a hypersequent calculus extending $\mathsf{sInt}{+}(cut)$. Let $r_1, \dots, r_4$ be the rules defined below where $\mathcal{S}$ is a set of hypersequents and the variables Y and Π do not appear other than where indicated. Then $\mathcal{C} + r_1$ and $\mathcal{C} + r_2$ (also $\mathcal{C} + r_3$ and $\mathcal{C} + r_4$) derive the same hypersequents.*

$$\frac{\mathcal{S}}{g \mid X \vdash A}\,r_1 \qquad \frac{\mathcal{S} \quad g \mid A, Y \vdash B}{g \mid X, Y \vdash B}\,r_2$$

$$\frac{\mathcal{S}}{g \mid A, X \vdash B}\,l_1 \qquad \frac{\mathcal{S} \quad g \mid Y \vdash A}{g \mid X, Y \vdash B}\,l_2$$

Setting $g \mid p \vdash q \mid q \vdash p$ as the zero-premise rule ρ_0, a single application of **Ackermann's lemma** tells us that $\mathsf{hInt} + (cut) + \rho_0$ and $\mathsf{hInt} + (cut) + \rho_1$ derive the same hypersequents. Three further applications of Ackermann's lemma yield that $\mathsf{hInt} + (cut) + \rho_4$ also derives the same hypersequents (*equivalent calculus*).

$$\frac{g \mid X_1 \vdash p}{g \mid X_1 \vdash q \mid q \vdash p}\,\rho_1 \qquad \frac{g \mid X_1 \vdash p \quad g \mid X_2 \vdash q}{g \mid X_1 \vdash q \mid X_2 \vdash p}\,\rho_2$$

$$\frac{g \mid X_1 \vdash p \quad g \mid X_2 \vdash q \quad g \mid Y_1, q \vdash B_1}{g \mid Y_1, X_1 \vdash B_1 \mid X_2 \vdash p}\,\rho_3 \qquad \frac{g \mid X_1 \vdash p \quad g \mid X_2 \vdash q \quad g \mid Y_1, q \vdash B_1 \quad Y_2, p \vdash B_2}{g \mid Y_1, X_1 \vdash B_1 \mid Y_2, X_2 \vdash B_2}\,\rho_4$$

The hypersequent calculus $\mathsf{hInt} + (cut) + \rho_4$ does not have the subformula property because the propositional variables p and q in the premise do not appear in the conclusion. To rectify this there is one final step: take the **cut-closure** on the premises of the rule. In effect we delete those premises which contain propositional variables which appear in either the antecedent or succedent but not both, and we apply cut in all possible ways to the remaining the premises. This operation may not terminate in general (consider the situation when the same propositional variable appears in the same component of the antecedent and succedent). In the case of ρ_4 it does terminate to yield the rule

$$\frac{g \,|\, X_1, Y_2 \vdash B_2 \qquad g \,|\, X_2, Y_1 \vdash B_1}{g \,|\, Y_1, X_1 \vdash B_1 \,|\, Y_2, X_2 \vdash B_2} \text{ (com)}$$

When cut-closure terminates it can be shown that it yields an equivalent structural rule. I.e. $\mathsf{hInt} + (cut) + (com)$ and $\mathsf{hInt} + (cut) + \rho_4$ are equivalent. It remains to show that $\mathsf{hInt} + (cut) + (com)$ has cut-elimination i.e. $\mathsf{hInt} + (com)$ is an equivalent calculus. In turns out that the structural rules obtained by the above procedure satisfies sufficient conditions for cut-elimination (such rules are called *analytic rules*) so we are done.

The four steps are summarised below.

1. Given the axiom $\vdash A_1 \vee \ldots \vee A_{n+1}$ apply all possible invertible rules backwards to the hypersequent $g \,|\, \vdash A_1 \;\;|\;\; \ldots \;\;|\;\; \vdash A_{n+1}$
2. Use Ackermann's lemma on each formula in the conclusion in order to obtain a rule where the conclusion contains no formulae.
3. Apply all possible invertible rules backwards to the premises of the rule. (Fail if one of the resulting hypersequents contain a compound formula which cannot be made propositional by the invertible rules.)
4. Delete premises containing propositional variables appearing only on one side. Apply all possible cuts to the remaining premises. (Fail if this step does not terminate.)

As we would expect, not all axioms can be handled by this method. If the nesting depth of logical connectives invertible in the antecedent/succedent is too great then item 3 will fail. In the context of intuitionistic logic cut-closure always terminates due to the presence of weakening and contraction. However this is not always the case in substructural logics.

2.2. **Some open problems.** Hypersequents in the calculus hInt were built using components $X \vdash A$ where X is a multiset. If we take X as a list of formulae, then the exchange rule (ex) below left, which states that formulae in the list can be permuted, must be stated explicitly. Deleting structural properties of the comma such as exchange (ex), weakening (w), contraction (c) and associativity lead to hypersequent calculi for *substructural logics*. The substructural logics typically contain additional language connectives. For example, in the absence of (w) and (c), a commutative operator $\otimes$ distinct from $\wedge$ can be defined using the rules below centre and right. An identity $\mathbf{1}$ for the $\otimes$ operator may also be defined.

$$\frac{g \,|\, X, A, B, Y \vdash C}{g \,|\, X, B, A, Y \vdash C} \text{ (ex)} \qquad \frac{g \,|\, X, A, B \vdash C}{g \,|\, X, A \otimes B \vdash C} \otimes\text{l} \qquad \frac{g \,|\, X \vdash A \quad h \,|\, Y \vdash B}{g \,|\, h \,|\, X, Y \vdash A \otimes B} \otimes\text{r}$$

Let $\mathsf{hFL_e}$ denote the substructural hypersequent calculus lacking the weakening and contraction properties (and containing the rules for $\otimes$). The corresponding logic is the Full Lambek calculus with exchange (denoted $\mathsf{FL_e}$).

- A hypersequent for the fuzzy logic MTL [13] (monoidal T-norm based logic) can be constructed by the procedure above: add the (w) and (com) rules to $\mathsf{hFL_e}$. This logic is known to be decidable via a semantic proof. It would be interesting to obtain a proof by arguing directly on the hypersequent calculus. This is turn would yield an upper bound on the logical complexity. Meanwhile it is unknown if Involutive MTL is decidable. A hypersequent for this calculus can be obtained by amending the rules of the calculus to use a list of formulae in the succedent rather than a single formula. A syntactic proof of decidability for MTL may provide a pathway for obtaining a proof of decidability for IMTL.
- A hypersequent calculus for IUL [20] (involutive uninorm logic) can be obtained by the addition of (com) to $\mathsf{hFL_e}$. A result of interest to the fuzzy logic community is if this logic is standard complete [16]. This in turn would follow by showing that whenever $g \mid X \vdash Y, p \mid p, U \vdash V$ is derivable (propositional variable p occurs only where indicated) then so is $g \mid X, U \vdash Y, V$. Such an argument is called *density elimination* [20, 8]. Some automated solutions to this problem are described in [3].

See [21] for further details on these two problems. A more general open problem concerns the handling of axioms which fail this methodology. A solution is to venture to a more expressive proof formalism as described in the following section. In the context of modal logics, alternative approaches [19] to generating hypersequent calculi from axioms have also been investigated.

3. Display Calculus

Since item 3 above fails when the invertible rules cannot reduce compound formulae to propositional variables, it follows that the more invertible rules in the calculus, the more axioms that can be presented. Essentially, the hypersequent calculus was able to invert top-level disjunctions. The display calculus formalism [4] extends [24] the hypersequent formalism: the formula corresponding to a display sequent has a normal form which is broader and also more general in the sense that the normal form is based on the algebraic semantics of the logic.

Below we introduce the display calculus $\delta\mathsf{BiInt}$ [28] for bi-intuitionistic logic BiInt. The language of BiInt extends the intuitionistic language with the coimplication $\leftarrow_d$. This is forced because the display calculus formalism requires that the logical connectives come in residuated pairs. To make the disjunction invertible on the right we added the semicolon on the right. Residuation then necessitated the addition of the structural connective $<$ standing for $\leftarrow_d$. An attractive feature of the display calculus is the general sufficient conditions [4] for cut-elimination.

$$p \vdash p \qquad \frac{\mathbf{I} \vdash X}{\top \vdash X}\,\top\mathrm{l} \qquad \frac{X \vdash \mathbf{I}}{X \vdash \bot}\,\bot\mathrm{r} \qquad \frac{}{\mathbf{I} \vdash \top}\,\top\mathrm{r} \qquad \frac{}{\bot \vdash \mathbf{I}}\,\bot\mathrm{l}$$

$$\frac{A, B \vdash Y}{A \wedge B \vdash Y}\,\wedge\mathrm{l} \qquad \frac{X \vdash A > B}{X \vdash A \to B}\,\to\mathrm{r} \qquad \frac{B < A \vdash Y}{B \leftarrow_d A \vdash Y}\,\leftarrow_d\mathrm{l} \qquad \frac{X \vdash A ; B}{X \vdash A \vee B}\,\vee\mathrm{r}$$

$$\frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B}\,\wedge\mathrm{r} \qquad \frac{X \vdash A \qquad B \vdash Y}{A \to B \vdash X > Y}\,\to\mathrm{l} \qquad \frac{X \vdash B \qquad A \vdash Y}{X < Y \vdash B \leftarrow_d A}\,\leftarrow_d\mathrm{r}$$

$$\frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash Y} \vee l \qquad \frac{\frac{X \vdash Y > Z}{X, Y \vdash Z}}{Y \vdash X > Z} \qquad \frac{\frac{X < Y \vdash Z}{X \vdash Y ; Z}}{X < Z \vdash Y} \qquad \frac{\frac{\mathbf{I}, X \vdash Y}{X \vdash Y}}{X, \mathbf{I} \vdash Y} \qquad \frac{\frac{X \vdash Y ; \mathbf{I}}{X \vdash Y}}{X \vdash \mathbf{I} ; Y}$$

$$\frac{X \vdash Y}{X \vdash Y ; Z} \text{(wr)} \qquad \frac{X \vdash Y}{X, Z \vdash Y} \text{(wl)} \qquad \frac{X \vdash Y ; Z}{X \vdash Z ; Y} \text{(ex-r)} \qquad \frac{X, Z \vdash Y}{Z, X \vdash Y} \text{(ex-l)}$$

$$\frac{X \vdash Y ; Y}{X \vdash Y} \text{(c-r)} \qquad \frac{X, X \vdash Y}{X \vdash Y} \text{(c-l)} \qquad \frac{X \vdash (Y ; Z) ; U}{X \vdash Y ; (Z ; U)} \qquad \frac{(X, Y), Z \vdash U}{X, (Y, Z) \vdash U}$$

The double lines indicate rules that can be applied in both directions. This calculus has the subformula property. The large number of structural connectives "$\mathbf{I}$", ",", ">", ";" and "<" reflect the flexible normal form of the corresponding formulae.

Theorem 3 ([9])**.** *Let $\mathcal{C}$ be a display calculus for the logic L satisfying the conditions in* [9]*. Also suppose that $\{r_i\}_{i \in I}$ are the analytic structural rules computed from the finite set $\{\alpha_j\}_{j \in J}$ of axioms using the analogous procedure to the one in Section 2. Then $\mathcal{C} + \{r_i\}_{i \in I}$ is a display calculus with the subformula property for $L + \{\alpha_j\}_{j \in J}$.*

For example, the logic $\mathsf{BiInt} + (p \to \bot) \vee ((p \to \bot) \to \bot)$ can be presented in this manner. Since $\mathsf{BiInt} + (p \to \bot) \vee ((p \to \bot) \to \bot)$ is conservative over $\mathsf{Int} + (p \to \bot) \vee ((p \to \bot) \to \bot)$ (argue via the Kripke or algebraic semantics), we can obtain a display calculus for the latter by deleting the logical rules (not the structural rules!) introducing $\leftarrow_d$. Incidentally, proving the conservativity directly on the display calculus appears to be surprisingly difficult. The issue is with the interaction of the display rules introducing $<$ and contraction.

It should be noted that a version of the method of extracting analytic structural rules from axioms appeared rather early on [17] in the context of display calculi for tense logics. Recently, the tools of unified correspondence theory have been applied [15] to extend that work in order to provide a new and uniform perspective on the axioms to rules paradigm.

4. First-order quantifiers

A cut-free hypersequent calculus hInt^{fo} for first-order intuitionistic logic Int^{fo} is obtained by adding to hInt the following rules for quantifiers [2, 6, 22]:

$$\frac{g \mid A(t), X \vdash B}{g \mid \forall x A(x), X \vdash B} \forall l \qquad \frac{g \mid \Gamma \vdash A(a)}{g \mid X \vdash \forall x A(x)} \forall r$$

$$\frac{g \mid A(a), X \vdash B}{g \mid \exists x A(x), X \vdash B} \exists l \qquad \frac{g \mid X \vdash A(t)}{g \mid X \vdash \exists x A(x)} \exists r$$

where the rules $(\forall r)$, $(\exists l)$ have an eigenvariable condition: the free variable a must not occur in the lower *hypersequent*.

The addition of quantifiers to the hypersequent calculus can have some unexpected effects. For example, if we add the (com) rule to hInt^{fo} then $\vdash \forall x(A(x) \vee B) \to (\forall x A(x) \vee B)$ is derivable whenever x is not free in B. The corresponding formula is known as the quantifier-shift or constant-domains formula and is not a theorem of $\mathsf{Int}^{fo} + (p \to q) \vee (q \to p)$! A non-standard hypersequent calculus for the latter logic (known as Corsi's logic [12] or Gödel-Dummett logic with non-constant domains) has been introduced [25] where the eigenvariables are incorporated into the hypersequent syntax. These hypersequents have no formula-interpretation in the logic and it is not clear how to generalise the calculus to capture other logics.

4.1. **Some open problems.** The hypersequent calculus $\mathsf{hInt}^{fo} + (com)$ presents first-order Gödel logic. The question of interpolation for this logic is open. It is conceivable to investigate this problem via the hypersequent proof-theory although it has defied all such attempts thus far.

Meanwhile, the interaction between the (com) rule and the first-order quantifiers witnessed in first-order Gödel logic can be generalised to a new question: let L be a propositional axiomatic extension of Int and suppose that $\mathcal{C}$ is the hypersequent calculus for it, obtained by the methods of the previous section. Let L_1 be the first-order axiomatic extension of L_1 and let L_2 consist of those formulae A such that $\vdash A$ is derivable in the extension of $\mathcal{C}$ by the first-order quantifier rules above. What then is the relationship between L_1 and L_2?

The only investigation of first-quantifiers for display calculi appears in [27]. Adding the obvious correspondents of the quantifier rules above to $\delta\mathsf{BiInt}$ derives the quantifier-shift formula. Meanwhile classical predicate logic may be treated as a kind of propositional tense logic [26, 18], where the quantifiers $\exists$ and $\forall$ are treated as a residuated pair analogous to $\blacklozenge$ and $\Box$ (see [17]). However the resulting display calculus fails cut-elimination due to the presence of certain rules. A decidable minimal first-order system may be obtained by dropping these rules.

References

[1] A. Avron. A constructive analysis of RM. *J. of Symbolic Logic*, 52(4):939–951, 1987.

[2] M. Baaz and R. Zach. Hypersequents and the proof theory of intuitionistic fuzzy logic. In *Computer science logic (Fischbachau, 2000)*, volume 1862 of *Lecture Notes in Comput. Sci.*, pages 187–201. Springer, Berlin, 2000.

[3] P. Baldi, A. Ciabattoni, and L. Spendier. Standard completeness for extensions of MTL: an automated approach. In *WOLLIC 2012*, volume 7456 of *LNCS*, pages 154–167. Springer, 2012.

[4] N. D. Belnap, Jr. Display logic. *J. Philos. Logic*, 11(4):375–417, 1982.

[5] A. Chagrov and M. Zakharyashchev. Modal companions of intermediate propositional logics. *Studia Logica*, 51(1):49–82, 1992.

[6] A. Ciabattoni. A proof-theoretical investigation of global intuitionistic (fuzzy) logic. *Archive of mathematical Logic*, 44:435–457, 2005.

[7] A. Ciabattoni, N. Galatos, and K. Terui. From axioms to analytic rules in nonclassical logics. In *LICS 2008*, pages 229 – 240, 2008.

[8] A. Ciabattoni and G. Metcalfe. Density elimination. *Theor. Comput. Sci.*, 403(2-3):328–346, 2008.

[9] A. Ciabattoni and R. Ramanayake. Power and limits of structural display rules. *ACM Trans. Comput. Logic*, 17(3):17:1–17:39, February 2016.

[10] A. Ciabattoni, R. Ramanayake, and H. Wansing. Hypersequent and display calculi – a unified perspective. *Studia Logica*, 102(6):1245–1294, 2014.

[11] W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for distributive modal logic. *Annals of Pure and Applied Logic*, 163(3):338–376, 2012.

[12] G. Corsi. A cut-free calculus for Dummett's LC quantified. *Mathematical Logic Quarterly*, 35(4):289–301, 1989.

[13] F. Esteva and L. Godo. Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124:271–288, 2001.

[14] G. Gentzen. *The collected papers of Gerhard Gentzen.* Edited by M. E. Szabo. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1969.

[15] G. Greco, M. Ma, A. Palmigiano, A. Tzimoulis, and Z. Zhao. Unified correspondence as a proof-theoretic tool. to appear. *Journal of Logic and Computation*, 2016.

[16] P. Hájek. *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht, 1998.

[17] M. Kracht. Power and weakness of the modal display calculus. In *Proof theory of modal logic (Hamburg, 1993)*, volume 2 of *Appl. Log. Ser.*, pages 93–121. Kluwer Acad. Publ., Dordrecht, 1996.

[18] D. Leivant. Quantifiers as modal operators. *Studia Logica*, 39:145–158, 1980.
[19] B. Lellmann. Hypersequent rules with restricted contexts for propositional modal logics. *Theoretical Computer Science*, 656, Part A:76 – 105, 2016.
[20] G. Metcalfe and F. Montagna. Substructural fuzzy logics. *J. of Symbolic Logic*, 72(3):834–864, 2007.
[21] G. Metcalfe, N. Olivetti, and D. Gabbay. *Proof Theory for Fuzzy Logics*, volume 39 of *Springer Series in Applied Logic*. Springer, 2009.
[22] G. Metcalfe, N. Olivetti, and D. Gabbay. *Proof Theory for Fuzzy Logics*, volume 39 of *Springer Series in Applied Logic*. Springer, 2009.
[23] G. Pottinger. Uniform, cut-free formulations of T, S4 and S5 (abstract). *J. of Symbolic Logic*, 48(3):900, 1983.
[24] R. Ramanayake. Embedding the hypersequent calculus in the display calculus. *Journal of Logic and Computation*, 25(3):921–942, 2015.
[25] A. Tiu. A hypersequent system for Gödel-Dummett logic with non-constant domains. In *Tableaux 2011. LNAI*, pages 248–262. Springer, 2011.
[26] J. van Benthem. Modal foundations of predicate logic. *Log. J. IGPL*, 5:259–286, 1997.
[27] H. Wansing. Predicate logics on display. *Studia Logica*, 62(1):49–75, 1999.
[28] H. Wansing. Constructive negation, implication, and co-implication. *J. Appl. Non-Classical Logics*, 18(2-3):341–364, 2008.