Topological Foundations for a Formal Theory of Manifolds

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Abstract

Topological manifolds form an important class of topological spaces with applications throughout mathematics. However, the development of this scientific area, even at the initial stage, requires non-trivial Brouwer's theorems: the fixed point theorem, the topological invariance of degree, and the topological invariance of dimension, where each of them is provided for n-dimensional case.

We present a formalization, that is checked in the Mizar system, of several results in algebraic topology that is sufficient to show the basic properties of manifolds with boundary of dimension n.

Keywords: Topological manifolds, Formalization, Mizar.

1 Introduction

The Mizar Mathematical Library (MML) [1] can be considered as a compendium of proven theorems from textbooks and papers, together with the necessary definitions. However, in contrast to the many informal considerations, different approaches to the same theory are generally merged in the MML. It is a consequence of the fact that the overall goal is not only expanding MML by a new formalization – though this still is an important part – but also facilitating the further development of the database based upon the already existing formalizations. Therefore, some theorems or even whole theories contained in the MML are adapted to the needs of new formalizations [4, 6]. To illustrate such situations we can consider the development of topological manifolds in the MML.

2 Development of Topological Manifolds

The first Mizar style [2] formalization of a topological manifold in Mizar was created by M. Riccardi [17, 18]. There, a topological manifold was defined as a topological space that is second-countable Hausdorff and locally Euclidean, i.e. the space resembles locally an open ball of the n-dimensional Euclidean space $\mathcal{E}^n$ near each point for some n. M. Riccardi obtained several basic facts, e.g. if two topological spaces are homeomorphic and one of them is an n-dimensional manifold, then the other one inherits its dimension; the work also showed some examples of topological manifolds: whole planes, spheres in Euclidean space. However, in his approach the parameter n was given a priori. As a justification for this approach we recall that every connected component of a compact manifold is a manifold with a determined dimension. However this fact cannot be clearly formulated in this approach.

To resolve this problem, we can consider a slightly more general definition, where for each point there exist some n and a neighborhood of the point being homeomorphic to an open ball of $\mathcal{E}^n$. Moreover, if we replace an open ball by a closed ball in the definition (see [3]) we can consider a topological manifold with a boundary, but also without it. Such a generalization of the first approach was made by K. Pąk [8]. He proved that every connected component of a compact manifold is a manifold with a determined dimension and also he proved the dependence between interior and boundary points. He showed for an n-dimensional manifold that its interior is a manifold without boundary of dimension n and its boundary is a manifold without boundary of dimension $n-1$. Additionally, he showed that the Cartesian product of manifolds also forms a manifold whose dimension is the sum of the dimensions of its factors and also determined the interior and boundary of this product.

3 Contributions

To establish these results, K. Pąk had to develop mainly the algebraic topology in the MML. It includes theory of simplicial complexes in real linear spaces and its barycenter subdivision. The theory was necessary to provide Brouwer's fixed point theorem based on Sperner's lemma [13, 16]. The theorem has been used in the formalization of the Jordan curve theorem in Mizar [5]. But for this purpose, the 2-dimensional case was sufficient. Therefore, this statement has been provided by A. Korniłowicz, only for this case using basic arguments concerning the fundamental groups of the respective spaces [7]. Note that this approach for higher-dimensional cases requires incomparably more difficult facts about these groups. Another approach based on Sperner's lemma requires only intuitively clear facts about the standard n-dimensional simplex and its arbitrarily small subdivision (see [11, 12]). Additionally, the simplex structure is explored in one of the approaches to prove Brouwer's invariance of the domain theorem that is helpful to distinguish points from the internal and the boundary of a manifold (see [3]).

Obviously the realization of the selected approach required to develop several other areas of topology and algebra. The most important of these are the small inductive dimension of topological spaces [9, 10]; an affine indepedence of points and barycentric coordinates in an affine space including the theory that barycentric coordinates with respect to an affine independence set is continuous [14]; the rotation group of Euclidean topological spaces [15].

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