# Estimation of Spring Stiffness under Conditions of Uncertainty. Interval Approach

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**Abstract.** The work deals with a problem of estimation of spring stiffness under conditions of uncertainty of its loading and compression measurements. During spring exploitation, the main its parameters *stiffness* and initial length change since of the metal strain aging. At the spring checking, values of its compression are measured in the necessary range of loading. It is used to describe the spring compression by the Hooke's law. In practice, a sample of measurements is very short and measuring is implemented with errors, which probability data are unknown. So, it is difficult to validate application of standard statistical methods to estimation of the spring stiffness. Under such conditions, an alternative to now existing approaches is application of the Interval Analysis methods that do not use any probabilistic properties of the measuring errors. In the work, the Interval Analysis methods are used for constructing the information set of admissible values of the spring parameters (the stiffness and initial length).

**Keywords:** Spring, parameters, stiffness, initial length, estimation, interval analysis methods

## 1 Introduction

During practical exploitation of steel springs, some changing of its material appear. Under this, check (tests) of the spring is performed with prescribed time period.

Usually, for estimation of the spring parameters (the stiffness and initial length), the standard methods to estimation are used that are based on the mathematical statistics ideology [1], [2], [3].

But in practice, a sample of measurements is very short and measuring the loading force and spring compression are implemented with errors, which probability data are unknown. Moreover, the model of the compression process is approximately described by the linear dependence corresponding to the Hooke's law, and the sample of measurements is rather short. So, under these conditions, it is difficult to strictly validate application of standard methods, and in practice, they are often used rather formally. As an alternative, application of the statistical methods can be completed by using the Interval Analysis ones.

For constructing a set of admissible values of the spring parameters, the Interval Analysis methods use only the model description of the process (the Hooke's law) and the bounds on the measuring errors of the loading force and compression value.

Such a set is used to call the *Information Set*. It comprises only such parameters of the model that are consistent with its description, accumulated sample of measurements, and the given bounds onto the mentioned measuring errors. On the basis of the determined Information Set, corresponding *tube* of the admissible dependencies of the spring under compression is formed.

The paper has the following structure. In Section 2, the technical details of the experiment are considered and typical model of the spring compression vs loading is introduced. Section 3 is devoted: to description of the main Interval Analysis procedures used for constructing the Information Set of the spring parameters (the stiffness and initial length) and the problem of investigation is formulated. In Section 4, for illustration of the elaborated estimation algorithms, a model example is investigated on a sample of measurements of a spring compression. Moreover, comparison of this results is made with ones obtained by the standard Least Square Means method (LSQM) [2]. In Section 5, conclusions are given on abilities of the suggested Interval Analysis approach and its applications in addition to known estimation procedures [1], [2], [3].

## 2 Details of experiment for investigation of spring parameters

The following experiment is performed.
1. The spring is put under the loading press.
2. The loading is performed by a collection of given forces $F_n$

$$\{F_n\}, n = \overline{1, N},\ F_n \in [F_1, F_N],\ 0 < F_1, \tag{1}$$

from the working range $[F_1, F_N]$ of the spring.
3. For each force value $F_n$, the spring compression (length) is measured

$$\{s_n\}, n = \overline{1, N}. \tag{2}$$

4. As a result of the test, the following sample is obtained:

$$\{F_n, s_n\}, n = \overline{1, N}. \tag{3}$$

5. Since of the measuring errors both in the loading force and in the spring compression, data (3) are corrupted as follows:

$$\begin{aligned} & n = \overline{1, N}, \\ & F_n = F_n^* + e_n, |e_n| \le e_{\max},\ s_n = s_n^* + b_n, |b_n| \le b_{\max}, \end{aligned} \tag{4}$$

where $F_n$ is the loading measurement; $F_n^*$ is the unknown true value of loading; $e_n$ is the additive measuring error of the loading force; $e_{\max}$ is the bound (on modulus) on the loading measuring error; $s_n$ is the compression measurement; $s_n^*$ is the unknown true value of the compression; $b_n$ is the additive measuring error; $b_{\max}$ is the bound (on modulus) on the compression measuring error. It is assumed that in the neighbor measurements errors are independent.

**Remark 1.** We suppose that
– measurements (4) are homogeneous, *i.e.*, the bounding values $e_{\max}$ and $b_{\max}$ in (4) are the same in all measuring;
– sample (3) is authentic *i.e.*, it does not contain outliers.

As a rule, actual description of the spring compression (changing its length vs the compression force) is unknown for a researcher (Fig. 1). Here, the true curve is in dashes; true values of the force $F^*$ and compression $s^*$ are marked by circles; the rectangles in dots are regions of possible location of authentic measurements.

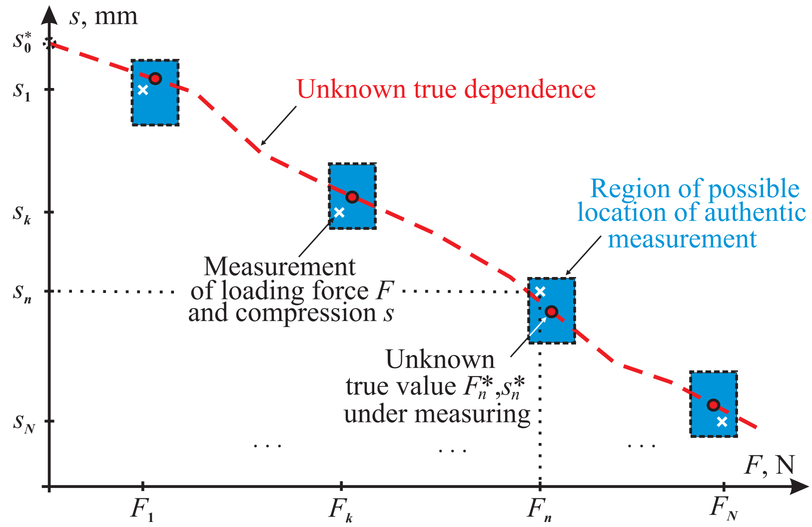


**Fig. 1.** True compression dependence and results of its measuring.

So, for elaboration of algorithms for estimating the compression parameters, the linear model corresponding to the Hooke's law is used in the form

$$S(F, a, s_0) = aF + s_0, \tag{5}$$

where $S(...)$ is the current value of the spring compression (length), mm; $F$ is the loading force, N; $a < 0$ is the stiffness, mm/N; $s_0$ is the spring initial length, mm.

In the linear model (4), its parameters $s_0$ and $a$ are assumed to be constant during the time interval of the spring exploitation.

Often in practice, it is possible to show some useful approximate intervals for the spring parameters in model (5)

$$\boldsymbol{a}_{\mathrm{apr}} = [\underline{\boldsymbol{a}}_{\mathrm{apr}}, \overline{\boldsymbol{a}}_{\mathrm{apr}}], \ \boldsymbol{s}_{0,\mathrm{apr}} = [\underline{\boldsymbol{s}}_{0,\mathrm{apr}}, \overline{\boldsymbol{s}}_{0,\mathrm{apr}}]. \tag{6}$$

(Here and further in the text, we keep at the standard notations accepted for the interval variables [9]).

## 3 Interval approach to estimating the spring parameters. Problem formulation

Foundations of Interval Analysis and its applications to processing observations under uncertainty conditions were developed on the basis of the fundamental pioneering work of L.V. Kantorovich [4].

Nowadays, effective theoretical, applied, and numerical methods of the Interval Analysis have been created both abroad [5], [6] and by Russian researchers [7], [8]. Special interval algorithms and software were developed for solving applied problems of estimation of parameters for experimental chemical processes [10], [11], [12], [13], [14].

Remind that the essence of the interval methods is in estimating the process parameters under conditions of a short measurement sample, uncertainty of the measuring errors probability characteristics and under only interval bounding onto the error values.

Introduce the following necessary definitions (with using the standard on notations in Interval Analysis [9]).

The **uncertainty set** of the spring loading and compression measurement. Since absence of probability characteristics of measuring errors, uncertainty of each measurement $F_n, s_n$ is formalized (Fig. 2) as a rectangle $\boldsymbol{H}_n$ with the left $\underline{\boldsymbol{F}}_n$ and right $\overline{\boldsymbol{F}}_n$, lower $\underline{\boldsymbol{s}}_n$ and upper $\overline{\boldsymbol{s}}_n$ boundaries.

$$\begin{aligned} & n = \overline{1, N}, \ \boldsymbol{H}_n : \\ & \text{7a) } \boldsymbol{F}_n = [\underline{\boldsymbol{F}}_n, \overline{\boldsymbol{F}}_n], \\ & \qquad \text{where } \underline{\boldsymbol{F}}_n = F_n - e_{\max}, \ \overline{\boldsymbol{F}}_n = F_n + e_{\max}; \\ & \text{7b) } \boldsymbol{s}_n = [\underline{\boldsymbol{s}}_n, \overline{\boldsymbol{s}}_n], \\ & \qquad \text{where } \underline{\boldsymbol{s}}_n = s_n - b_{\max}, \ \overline{\boldsymbol{s}}_n = s_n + b_{\max}. \end{aligned} \tag{7}$$

The **admissible value of the parameters vector** $(a, s_0)$ for model (4),(5) is a pair

$$(a, s_0) : \ S(n, a, s_0) \in \boldsymbol{F}_n \times \boldsymbol{s}_n \text{ for all } n = \overline{1, N}, \tag{8}$$

and corresponding dependence $S(n, a, s_0)$ is also called **admissible**.

The **Information Set** (INFS) is a totality of all admissible values of the parameter vector for model (4),(5) satisfying the following system of interval inequalities:

$$\boldsymbol{I}(a, s_0) = \{(a, s_0) : S(n, a, s_0) \in \boldsymbol{F}_n \times \boldsymbol{s}_n, \ n = \overline{1, N}\}. \tag{9}$$

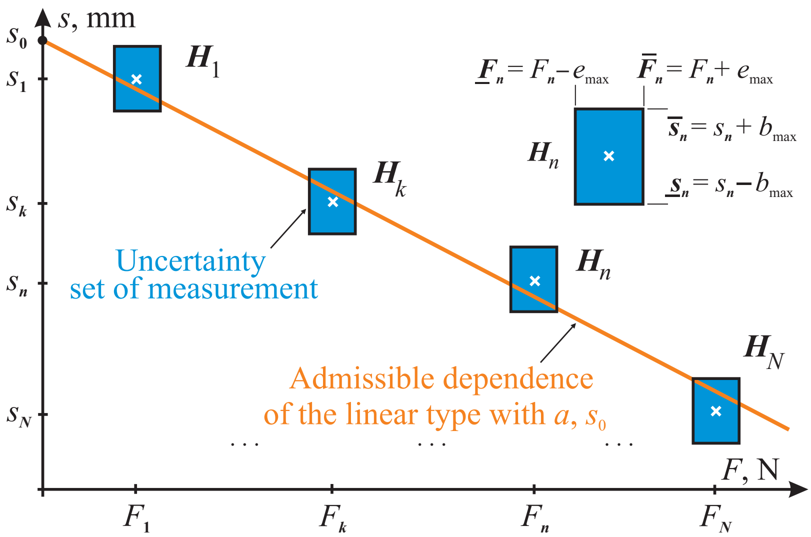


**Fig. 2.** Measurements, their uncertainty sets, and admissible dependence.

The input sample (3) is called **consistent** in the interval sense if by (9) there exists at least one admissible value of the parameter vector and corresponding admissible dependence.

The **tube** of admissible dependencies $\boldsymbol{Tb}(F)$ (Fig. 3) is a totality of all admissible values of the dependence on the loading force $F$. For model (5) and the information set $\boldsymbol{I}(s_0, a)$, the tube boundaries are calculated with taking into account the loading force $F$ interval

$$\begin{aligned} &F \in [F_1, F_N] : \\ &\underline{\boldsymbol{Tb}}(F) = \min_{(a,s_0)\in \boldsymbol{I}(a,s_0)} S(F, a, s_0), \\ &\overline{\boldsymbol{Tb}}(F) = \max_{(a,s_0)\in \boldsymbol{I}(a,s_0)} S(F, a, s_0). \end{aligned} \tag{10}$$

**Problem of estimation** is formulated as follows: *by means of the Interval Analysis methods to construct the Information Set* (9) *of the spring coefficients values consistent with the given data* (3) *and to build the tube* (10) *of the admissible values of the spring compression.*

For the linear model (5), fast procedures for constructing the Information Set (9) with exact description of its boundaries have been elaborated and applied to solving many practical problems [10], [11], [12], [13], [14]. Due to direct using the linearity of model (4), these procedures are more fast and give exact description of the Information Set in comparison with even very powerful procedures of the SIVIA-type [5].

**Remark 2.** As it will be shown below, formal application of the standard Least Square Means method (LSQM) [2] and corresponding point-wise estimate of the parameters (($a_{SQ}, s_{0,SQ}$) for the linear model) demonstrate to be useful for

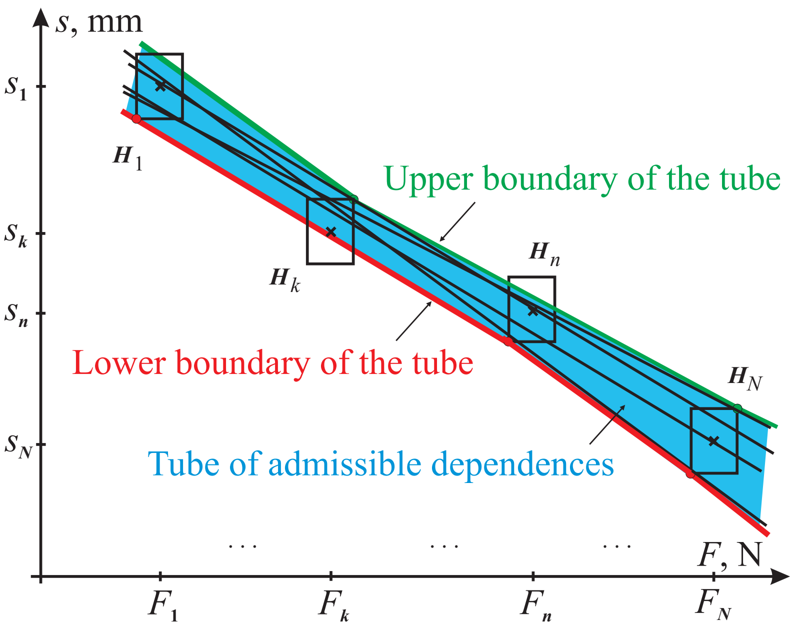


**Fig. 3.** Tube of admissible dependencies.

analysis of the input sample (3) and for qualitative comparison with the results on the basis of Interval Analysis.

### Elaborated engineering interval procedures

For simplicity of narration, consider a practical case when for all $n$

$$\overline{\boldsymbol{F}}_n < \underline{\boldsymbol{F}}_{n+1} \text{ and } \underline{\boldsymbol{s}}_n > \overline{\boldsymbol{s}}_{n+1}. \tag{11}$$

For other mutual location of uncertainty sets, computation formulas are derived in the similar way.

Consider a pair of uncertainty sets $\boldsymbol{H}_n$ and $\boldsymbol{H}_k$, $n = \overline{2, N}$, $k = n - 1$, with their corner points 1, 2, 3, 4 and 5, 6, 7, and 8 (Fig. 4a). For each pair, the **bunch of admissible dependences** is introduced. The bunch is bounded by the lines $L_{\mathrm{I}}$ and $L_{\mathrm{III}}$ with the extremal values of parameters $a_{min}, s_{0,\max}$ and $a_{\max}, s_{0,\min}$ and by the lines $L_{\mathrm{II}}$ and $L_{\mathrm{IV}}$ with the intermediate values of parameters $a_{\mathrm{II}}, s_{0,\mathrm{II}}$ and $a_{\mathrm{IV}}, s_{0,\mathrm{IV}}$, correspondingly.

Under conditions (10), parameters of these lines are computed as follows:

$$\begin{array}{l}
L_{\mathrm{I}}: \ a_{min} = (\underline{\boldsymbol{s}}_n - \overline{\boldsymbol{s}}_k)/(\underline{\boldsymbol{F}}_n - \overline{\boldsymbol{F}}_k), \ s_{0,\max} = \overline{\boldsymbol{s}}_k + a_{\min}\overline{\boldsymbol{F}}_k; \\
L_{\mathrm{III}}: \ a_{\max} = (\overline{\boldsymbol{s}}_n - \underline{\boldsymbol{s}}_k)/(\overline{\boldsymbol{F}}_n - \underline{\boldsymbol{F}}_k), \ s_{0,\min} = \underline{\boldsymbol{s}}_k + a_{\max}\underline{\boldsymbol{F}}_k; \\
L_{\mathrm{II}}: \ a_{\mathrm{II}} = (\underline{\boldsymbol{s}}_n - \underline{\boldsymbol{s}}_k)/(\underline{\boldsymbol{F}}_n - \underline{\boldsymbol{F}}_k), \ s_{0,\mathrm{II}} = \underline{\boldsymbol{s}}_k + a_{\mathrm{II}}\underline{\boldsymbol{F}}_k; \\
L_{\mathrm{IV}}: \ a_{\mathrm{IV}} = (\overline{\boldsymbol{s}}_n - \overline{\boldsymbol{s}}_k)/(\overline{\boldsymbol{F}}_n - \overline{\boldsymbol{F}}_k), \ s_{0,\mathrm{IV}} = \overline{\boldsymbol{s}}_k + a_{\mathrm{IV}}\overline{\boldsymbol{F}}_k.
\end{array} \tag{12}$$

Note that for homogeneous sample (Remark 1) $a_{\mathrm{II}} = a_{\mathrm{IV}}$.

From Figure 4a it is seen that in the case of mutual location (11) of the pair $\boldsymbol{H}_n$ and $\boldsymbol{H}_k$, the bunch of admissible dependencies is completely determined by the diagonals “2–4” and “6–8”. This allows to use techniques elaborated for one-dimensional uncertainty sets of measurements [10], [13], [14].

For each pair $\boldsymbol{H}_k$ and $\boldsymbol{H}_n$, the **partial information set** $\boldsymbol{G}(a, s_0)_{k,n}$ of parameters is determined by the computed apex points (11). Note (Fig. 4b) that it is a convex four-apex polygon with linear boundaries.

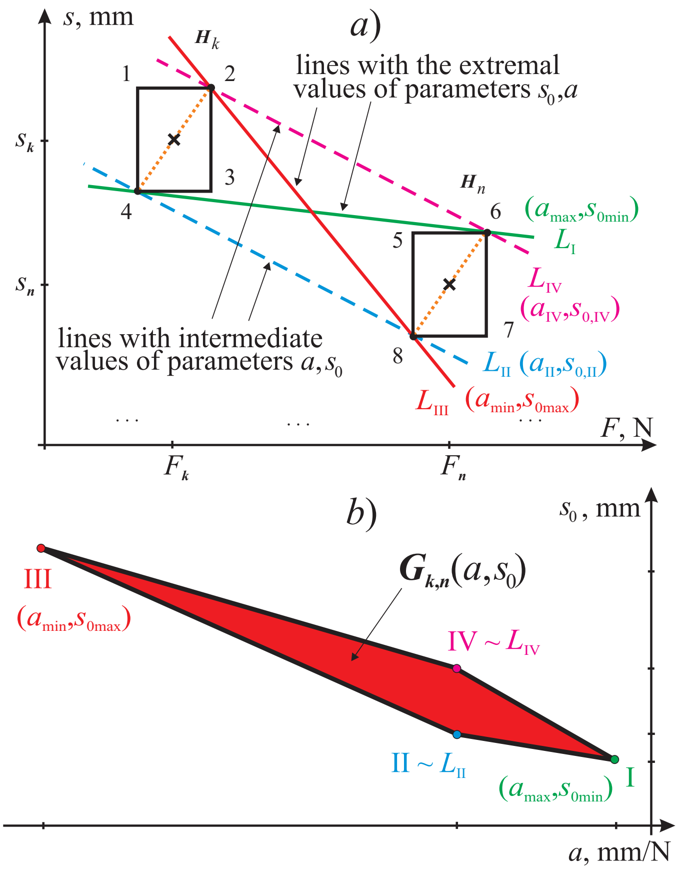


**Fig. 4.** Constructing the partial information set; a) a pair of uncertainty sets; b) partial information set of possible values of compression parameters.

Having constructed the collection of partial information sets $\{\boldsymbol{G}_{k,n}(a, s_0)\}$, $n = \overline{2, N}$, $k = n - 1$, it becomes possible to calculate the desirable information set $\boldsymbol{I}(a, s_0)$ (9) of admissible values of the compression parameters for the whole sample (3) of measurements

$$\boldsymbol{I}(a, s_0) = \bigcap\nolimits_{n=\overline{2,N},\ k=n-1} \boldsymbol{G}_{k,n}(a, s_0). \tag{13}$$

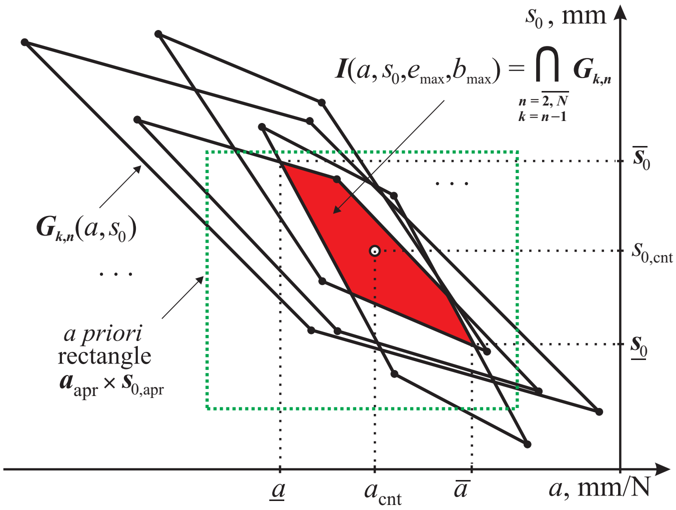


**Fig. 5.** Output information set $\boldsymbol{I}(a, s_0)$ of admissible values of compression parameters $a, s_0$; *a priori* data rectangle (in dots).

This procedure is implemented by some appropriate standard program of intersection of convex polygons. Image of the information set $\boldsymbol{I}(a, s_0)$ is presented in Fig. 5 (shadowed by red). It is determined by the apex points I, II, III, and IV

$$(a, s_0)_{\mathrm{I}},\ \ldots,\ (a, s_0)_{\mathrm{IV}}. \tag{14}$$

The information set (12) can be compared with the *a priori* data rectangle $\boldsymbol{a}_{\mathrm{apr}} \times \boldsymbol{s}_{0,\mathrm{apr}}$ (6), and there is possibility of its enhancing (Fig. 5).

For practical using, the following parameters' values are given to a researcher:

$$\begin{aligned}
&- \text{exact description of the information set } \boldsymbol{I}(a, s_0);\\
&- \text{unconditional interval in parameter } a:\ [\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}}];\\
&- \text{unconditional interval in parameter } s_0:\ [\underline{\boldsymbol{s}}_0, \overline{\boldsymbol{s}}_0];\\
&- \text{central point } a_{\mathrm{cnt}} = 0.5(\overline{\boldsymbol{a}} + \underline{\boldsymbol{a}}),\ s_{0,\mathrm{cnt}} = 0.5(\overline{\boldsymbol{s}}_0 + \underline{\boldsymbol{s}}_0).
\end{aligned} \tag{15}$$

## 4 Model example. Results of estimation of spring parameters

For computations, the following input model data are given:
– the initial true length of the spring $s_0^* = 120$ mm;
– the stiffness true value $a^* = -0.1020$ mm/N;
– number of measurements $N = 5$;
– the true loading forces are $\{F_n^*\} = \{196, 392, 588, 784, 980\}$ N;
– the true values of compression are $\{s_n^*\} = \{100, 80, 60, 40, 20\}$ mm;
– the bound on the loading force measurement $e_{\max} = 40$ N;
– the bound on the compression measuring $b_{\max} = 2$ mm;
– corrupted measured values of the loading forces are $\{F_n\} = \{221.9, 364.2,$

$611.4, 778.2, 983.5\}$ N;
– corrupted measured values of compression are $\{s_n\} = \{98.28, 78.22, 61.34, 41.90, 18.02\}$ mm.

After procession, the resultants information set $\boldsymbol{I}(a, s_0)$ is presented in Fig.6. It is a convex polygon with five apices. It has sizes: $\underline{\boldsymbol{a}} = -0.1189$ mm/N, $\overline{\boldsymbol{a}} = -0.0905$ mm/N, $\underline{\boldsymbol{s}}_0 = 112.76$ mm, $\overline{\boldsymbol{s}}_0 = 128.317$ mm.

The information set central point is $a_{\text{cnt}} = -0.1047$ mm/N, $s_{0,\text{cnt}} = 120.53$ mm.

Formal estimation of the input corrupted sample by the Least Squares Means method gave the point $a_{\text{SQ}} = -0.1009$ mm/N, $s_{0,\text{SQ}} = 119.31$ mm; and estimation of the standard deviation is $\sigma = 3.12$ mm.

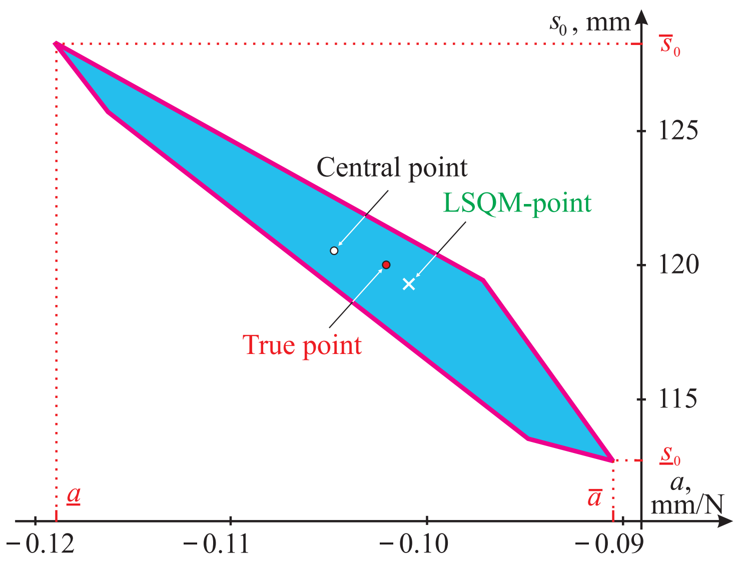


**Fig. 6.** Model example; resultant information set.

It is seen (Fig. 6) that since of the outliers absence, both the true point and the LSQM one are inside the information set.

Picture of the compression process and results of procession are also shown in Fig. 7. Note that formal application of the standard rule "$\pm 3\sigma$" gives approximate very wide "corridor" around the LSQM-line.

In a contrast, application of the described interval approach allows one to build significantly narrow tube of admissible dependences. Moreover, it is seen that uncertainty sets of measurements have been enhanced: their inadmissible parts are cut off by the tube.

Underline that due to admissibility of the true and the LSQM points (Fig. 6), corresponding line dependencies are whole inside the tube (Fig. 7).

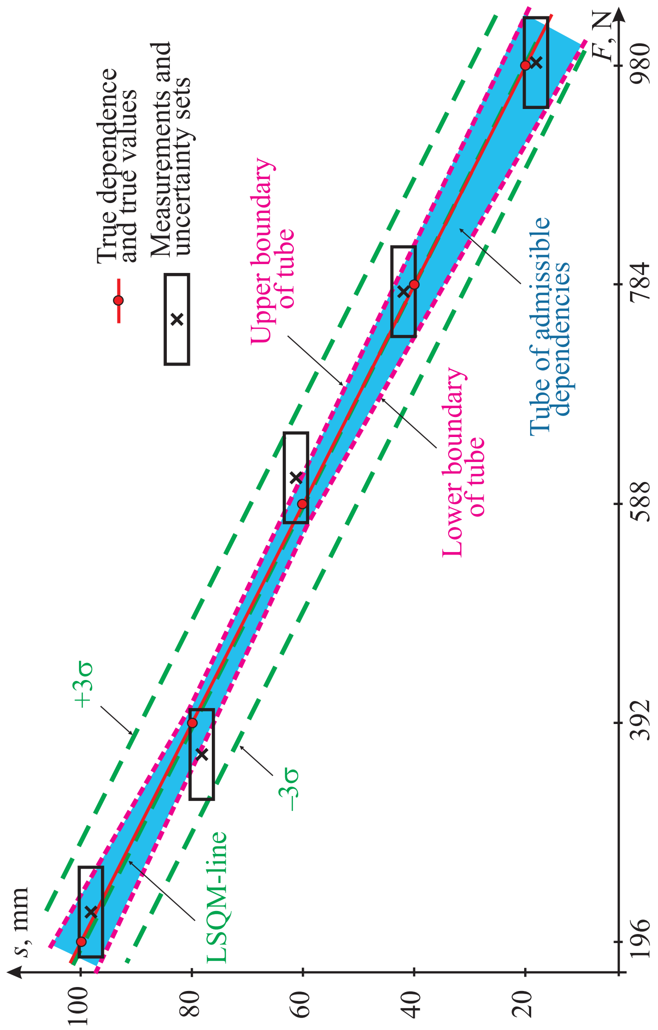


**Fig. 7.** Model example; results of processing.

## 5 Conclusions

The Interval Analysis methods was applied to estimation of spring parameters in the compression process under conditions of absence of probability data for the measuring errors.

Important case was investigated when errors are **both** in the loading force measurements **and** in ones of the spring compression (length).

Investigations are fulfilled on the basis of the wide used Hooke's law with the linear dependence of spring compression vs the loading force.

It was shown that under mentioned conditions Interval Analysis approach gives guaranteed estimation of the process parameters and better estimation of the tube of admissible dependencies.

Moreover, simulation results show that using simultaneously, the interval and standard statistical approaches complement each other; and this allows one to perform more detailed analysis and qualitative comparison of the estimation results.

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