On the estimation of measurement errors in linear dynamical systems

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Abstract

The article is concerned with the problem of linear dynamical system state estimation subject to noise-corrupted observations. In solution of the linear optimal filtering problem under guaranteed statement the problem of measurement noise identification occurs. The approach to adaptive measurement noise estimation is proposed. It is based on statistical processing of the innovation sequence in the Kalman filter. The special case of linear dynamical system with one-dimensional output the state of which is observed only on a short-time interval is considered and it is shown that statistical characteristics of the innovation sequence can be used for measurement noise identification and also for adjustment of the guaranteed estimates. Simulation results are given to confirm the usefulness of the approach.

Keywords: guaranteed estimation; Kalman filter; innovation sequence; short sample of observations.

1 Introduction

Control problems appear in various fields of engineering: aircraft control systems, inertial navigation systems, automatic process control systems, tracking and target acquisition systems [1, 2]. When designing a dynamical object control system, it is necessary to estimate a state of object operating under conditions of a priori uncertainty and incomplete measurement data.

One of the basic requirements to dynamical object control system design is efficient algorithms development for object current state estimation. In its turn, the state estimation is impossible without taking into account the combination of random disturbances that influence an object and measurement errors. Depending on assumptions of the nature of uncontrolled factors, the estimation problem is solved either under stochastic statement [3, 4], assuming a priori knowledge of statistical information, or under guaranteed statement [6, 7, 10, 13, 15, 16, 17, 19, 21], when only possible ranges of uncontrolled factors are known.

A common problem is the state estimation problem of dynamical systems in the presence of disturbances that can be both probabilistic and deterministic [10]. On the one hand, implementation of the Kalman filter (KF) as a probabilistic estimation technique requires complete a priori knowledge of noise statistics. For the only measurement realization noise statistics cannot be obtained, are unknown or only their rough estimates are known. On the other hand, the main problem with guaranteed approach is that the state vector estimate

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obtained as a result of its implementation can be overestimated. This is explained by the fact that a solution of the mininimax filtering problem is chosen taking into account the worst combination of uncertain factors (although those is very unlikely event). Probabilistically guaranteed approach can be used to overcome specified difficulties of the KF and the guaranteed algorithm.

In the present article the filtering problem is considered under probabilistically guaranteed statement when the KF and the guaranteed algorithm are applied together. The estimation problem is solved using the guaranteed algorithm. The Kalman filter implementation is performed for measurement data preprocessing, particularly, for measurement errors estimation. The special case of linear dynamical system with Gaussian random inputs and one-dimensional output is considered.

2 Statement of the problem

Consider the state estimation problem of a discrete linear dynamical system with the state and measurement difference equations:

$$x_{k+1} = Ax_k + \Gamma w_k, \tag{1}$$

$$y_{k+1} = Gx_{k+1} + v_{k+1}, \quad k = 0, 1, \dots, N-1. \tag{2}$$

where k is a discrete-time variable that takes values on a short interval $k = \overline{1, N}$, $N < 30$; $x_k \in R^n$ is the state vector; $w_k \in R^n$ is the process noise vector; $y_k \in R^1$ is the measurement output; $v_k \in R^1$ is the measurement noise; state transition matrix A, constant input matrix Γ and constant output matrix G are known.

A priori information about the initial state x_0, input disturbances w_k and measurement errors v_k is specified by sets of their possible values:

$$x_0 \in X_0, \quad w_k \in W, \quad v_k \in V. \tag{3}$$

The guaranteed (or set-membership, minimax) estimation of the state vector x_k involving the linear model (1), observations (2) and boundary conditions (3) assumes recursive construction of the bounded sets $\bar{X}_{k+1}$ (information sets, feasible sets), $k = 0, 1, \dots, N-1$, i.e. the sets of state vector possible values. The set $\bar{X}_{k+1}$ contains the true value of the state x_{k+1} and its size depends largely on given sets (3):

$$x_{k+1} \in \bar{X}_{k+1} = X_{k+1/k} \cap X[y_{k+1}], \tag{4}$$

where $X_{k+1/k}$ is the predicted state set

$$X_{k+1/k} = A\bar{X}_k + \Gamma W, \tag{5}$$

and $X[y_{k+1}]$ is the measurement consistent set

$$X[y_{k+1}] = \{x \in R^n | Gx = y_{k+1} - v, \forall v \in V\}. \tag{6}$$

All operations (4)–(6) in the guaranteed algorithm are performed on sets: set intersection, linear mapping of sets, Minkowsky sum, which in turn requires more computational power for implementation of the algorithm in real-time mode [13].

When a process is implemented, a priori given set of measurement errors V can be exceeding. For instance, measurement errors can be actually realized from a subset $v_k \in \tilde{V} \subset V$. Therefore, the problem of measurement noise identification in observations (2) occurs. The use of the set $\tilde{V}$ instead of a priori given set in Eq. (6) allows to enhance a solution accuracy of the minimax filtering problem.

3 An adaptive Kalman filter

The stochastic approach to the state vector estimation problem in the system (1), (2) is to assume that the initial state x_0 is a random variable with known mean value $E[x_0] = \hat{x}_0$ and known covariance matrix P_0, the process noise w_k and the measurement noise v_k are uncorrelated white zero mean noise processes with covariance matrices Q and R respectively:

$$x_0 \sim N(0, P_0), \quad w_k \sim N(0, Q), \quad v_k \sim N(0, R). \tag{7}$$

Under these circumstances the estimation problem involving the linear model (1), observations (2) and initial conditions (7) is solved by the KF [3, 4]:

$$\hat{x}_{k+1} = \hat{x}_{k+1/k} + K_{k+1}(y_{k+1} - G\hat{x}_{k+1/k}), \quad \hat{x}_{k+1/k} = A\hat{x}_k, \quad k = 0, 1, \ldots, N-1, \tag{8}$$

where

$$K_{k+1} = P_{k+1/k}G^{\mathrm{T}}[GP_{k+1/k}G^{\mathrm{T}} + R]^{-1}, \tag{9}$$

$$P_{k+1/k} = AP_kA^{\mathrm{T}} + \Gamma Q\Gamma^{\mathrm{T}}, \tag{10}$$

$$P_{k+1} = (I - K_{k+1}G)P_{k+1/k}. \tag{11}$$

Here, $\hat{x}_{k+1}$ is an estimate of the state vector x_{k+1} at the moment of time $k+1$, K_{k+1} is the filter gain, $P_{k+1/k}$ and P_{k+1} are the covariance matrices of predicted and updated filtering error respectively, I is $n \times n$ identity matrix.

The sequence of estimates $\hat{x}_N(\bullet) = \{\hat{x}_1, \ldots, \hat{x}_N\}$ obtained by the filter Eqs. (8)–(11) is optimal in terms of minimum mean square error (MSE) only for a great number of measurement realizations [5, 14]. It is known that the true value of the state vector x_k with required probability will belong to the set

$$E_k = \{x \in R^n | (x_k - \hat{x}_k)^{\mathrm{T}} P_k^{-1}(x_k - \hat{x}_k) \le l^2\}, \tag{12}$$

which is an ellipsoid centered at the point $\hat{x}_k$ (the probability that $x_k \in R^2$ can be found in ellipse with $l = 3$ is 0.989).

Implementation of the KF requires complete a priori knowledge of noise statistics (7) in Eqs. (1), (2) [5, 14]. In certain real systems data processing is performed for the only measurement realization. In this case, statistics of the process and measurement noise cannot be obtained, the process noise covariance matrix Q and the measurement noise covariance matrix R are unknown or only their rough estimates are known. In this case an adaptive Kalman filtering approach can be used to solve the estimation problem [8, 9, 11, 12, 18, 20].

The adaptive filtering algorithm is based on statistical analysis of the innovation sequence [8, 9, 11, 12]

$$\Lambda_{k+1} = y_{k+1} - G\hat{x}_{k+1/k}, \tag{13}$$

which is a zero mean Gaussian white noise process with correlation properties

$$C_k = \begin{cases} GP'G^{\mathrm{T}} + R, & k = 0 \\ G[A(I - KG)]^{k-1}A[P'G^{\mathrm{T}} - KC_0], & k > 0 \end{cases} \tag{14}$$

where P' is the covariance matrix of predicted filtering error of time invariant filter, K is the Kalman gain.

To verify the linear model (1) and observations (2) (i.e. to check whether the KF constructed using a priori given covariance matrices Q and R is close to optimal or not) it is possible to use correlation analysis methods for processing the sequence $\Lambda_N(\bullet) = \{\Lambda_1, \ldots, \Lambda_N\}$. The estimates of C_k, denoted as $\hat{C}_k$, can be obtained by using the ergodic property of a stationary innovation sequence for lag $l = 1, 2, \ldots, n$, n is the state vector dimension [9]:

$$\hat{C}_k = \frac{1}{N}\sum_{k=l}^{N} \Lambda_k \Lambda_{k-l}^{\mathrm{T}}. \tag{15}$$

The estimates $\hat{C}_k$ can be represented by rewriting (15) explicitly[8]:

$$\begin{aligned} C_1 &= GAP'G^{\mathrm{T}} - GAKC_0; \\ C_2 &= GA^2P'G^{\mathrm{T}} - GAKC_1 - GA^2KC_0; \\ &\ldots \\ C_n &= GA^nP'G^{\mathrm{T}} - GAKC_{n-1} - \ldots - GA^nKC_0. \end{aligned} \tag{16}$$

Using (14)–(16) the correlation of terms of the sequence Λ_k is checked. In other words, according to the Eqs. (14)–(16) an optimality of implemented KF is checked. However, this test requires considerable computing time for the accumulation of innovation sequence statistic, i.e. to obtain $\hat{C}_k$ (15). In general, the test is carried out at the interval of available observations $N \gg 100$ [9]. For an optimal KF, the estimates $\hat{C}_k$ are unbiased and consistent, C_k vanishes for all $k > 1$. In the case of short measurements realization, e.g. when $N < 30$ measurements are taken, the estimates of C_k are inconsistent. If N is small, other tests can be used.

4 An adaptive algorithm of measurement noise estimation

For an optimal filter, when the process noise covariance matrix Q and the measurement noise covariance matrix R used in the filter Eqs. (8)-(11) correspond to the real noises in the system (1), (2), a residual sum of squares (RSS) approaches zero [11, 12]

$$\sum_{k=0}^{N-1} \Lambda_{k+1}^{\mathrm{T}} \Lambda_{k+1} \to 0. \tag{17}$$

The case of biased innovation sequence $E[\Lambda_k] \neq 0$, or if the actual covariance of innovation sequence Λ_k is substantially greater than its expected value $\Lambda_{k+1}^{\mathrm{T}} \Lambda_{k+1} > G P_{k+1/k} G^{\mathrm{T}} + R$, can indicate the suboptimality of the KF [11, 12]. Detection of an innovation sequence bias or deviation of its actual covariance from expected covariance is carried out by averaging over some interval of the innovation sequence $\Lambda_N(\bullet) = \{\Lambda_1, \ldots, \Lambda_N\}$. It is assumed that a number of sample points N may be low ($N = 5 \ldots 10$). The required number of sample points mainly depends on time interval length on which it can be assumed that the filter has reached steady-state conditions.

In order to obtain characteristics of the actual estimation, a posteriori value of innovation sequence can be used

$$\Delta_{k+1} = y_{k+1} - G\hat{x}_{k+1}, \tag{18}$$

with covariance

$$var\{\Delta_{k+1}\} = \Sigma_{k+1} = G P_{k+1} G^{\mathrm{T}} + R. \tag{19}$$

Substituting the observations (2) in (18) the expression for the innovation sequence can be rewritten

$$\Delta_{k+1} = G x_{k+1} + v_{k+1} - G\hat{x}_{k+1} = G e_{k+1} + v_{k+1}, \tag{20}$$

where $e_{k+1} = x_{k+1} - \hat{x}_{k+1}$ denotes a vector of estimation errors in the KF. Then the expression for measurement errors v_k can be defined

$$v_{k+1} = \Delta_{k+1} - G e_{k+1}. \tag{21}$$

The covariance matrix P_k is the matrix of an ellipsoid for possible values of an estimation error vector e_k:

$$(x_k - \hat{x}_k)^{\mathrm{T}} P_k^{-1} (x_k - \hat{x}_k) \leq l^2 \quad \text{or} \quad e_k^{\mathrm{T}} P_k^{-1} e_k \leq l^2. \tag{22}$$

Checking an optimality of the KF (17), i.e. computing a posteriori value of innovation sequence Δ_k for each time step k, we can obtain an estimate of the measurement noise covariance $\hat{R}$. For linear dynamical system (1) with one-dimensional measurement output (2), as an estimate $\hat{R}$ it is possible to use a variance of the innovation sequence

$$\hat{\sigma}_{\Delta}^2 = \frac{1}{N} \sum_{k=1}^{N} (\Delta_k - \bar{\Delta})^2, \tag{23}$$

where $\bar{\Delta} = \frac{1}{N} \sum_{k=1}^{N} \Delta_k$ is the innovation sequence mean value.

The obtained estimate $\hat{R} = \hat{\sigma}_{\Delta}^2$ (23) can be used for construction of set $\tilde{V} \subset V$ of measurement errors v_k in the minimax filtering algorithm (6). The set $\tilde{V}$ is defined as follows

$$v_k \in \tilde{V} = [\underline{v}; \quad \overline{v}] = [-l\sqrt{\hat{R}}; \quad +l\sqrt{\hat{R}}], \tag{24}$$

where $\underline{v}$, $\overline{v}$ are the lower and higher values of the measurement errors range. The values of implemented measurement errors v_k with probability 0.997 ($v \in R^1 \quad l = 3$) are in the set (24).

Thus, the adaptive algorithm of one-dimensional measurement noise estimation can be implemented. It is required to

1. Compute RSS (17) of the innovation sequence Δ_k to verify an optimality of the filtering process:
   - if the implemented KF is not optimal, to obtain an estimate of R one can be used the known adaptive filtering algorithm proposed by R.K. Mehra [8];
   - if the implemented KF is optimal, the estimate $\hat{R}$ can be obtained according to the (23).
2. Define the bounded set $\tilde{V}$ of measurement errors in the minimax filtering algorithm according to the (24).

5 Numerical example

Consider implementation of proposed algorithm on an example of linear dynamical system (1), (2). It is assumed that the initial state $x_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$. The process noise w_k and the measurement noise v_k are normally distributed random numbers with zero means and standard deviations $\sigma_w = 0.1$ and $\sigma_v = 0.2$ respectively (Fig. 1, 2). The available $N = 30$ observations y_k are shown in (Fig. 3).

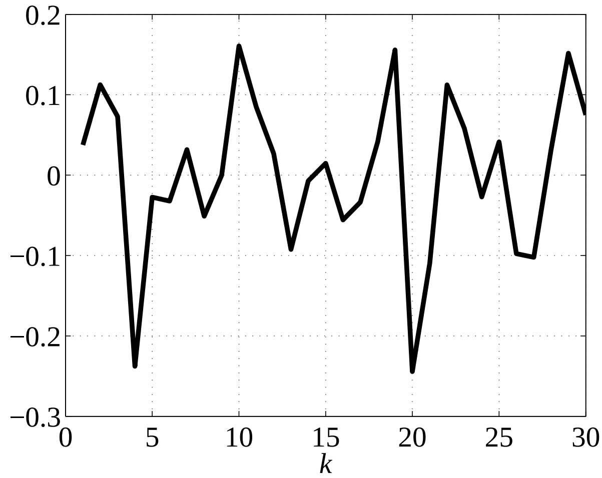


Figure 1: Process noise w_k.

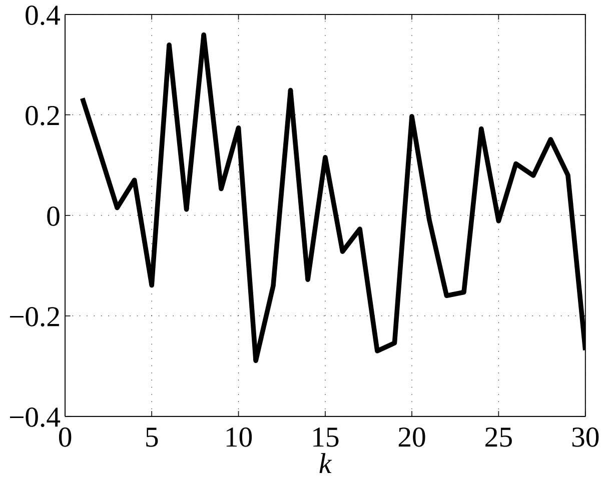


Figure 2: Measurement noise v_k.

System matrices are

$$A = \begin{pmatrix} 0.8 & -0.7 \\ 0.4 & 0.8 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{25}$$

The set X_0 is specified by square

$$X_0 = \{x \in R^2 | -1 \le x(1) \le 1, \quad -1 \le x(2) \le 1\}. \tag{26}$$

The sets W_k and V_k are specified by intervals

$$W = \{w \in R^1 | -0.3 \le w \le 0.3\}, \quad V = \{v \in R^1 | -0.6 \le v \le 0.6\}. \tag{27}$$

The covariance matrix P_0 is defined in a way that the value of state vector x_0 at a three sigma level is in the set X_0. Then the initial conditions for the KF are defined as follows:

$$\hat{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad P_0 = \frac{R^2}{9} I, \quad q = \sigma_w^2, \quad r = \sigma_v^2, \tag{28}$$

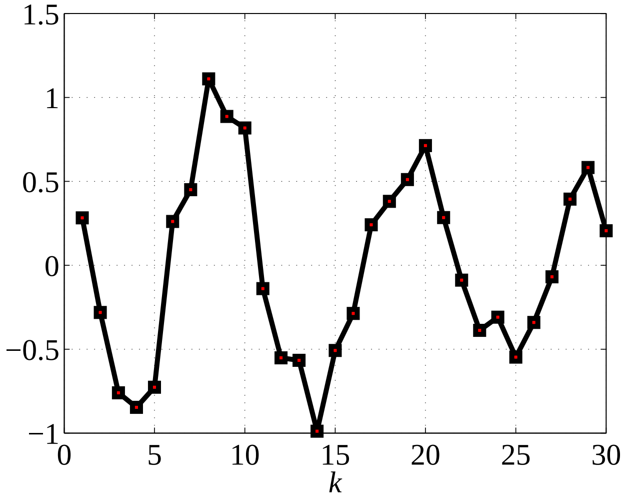


Figure 3: Noise-corrupted observations y_k.

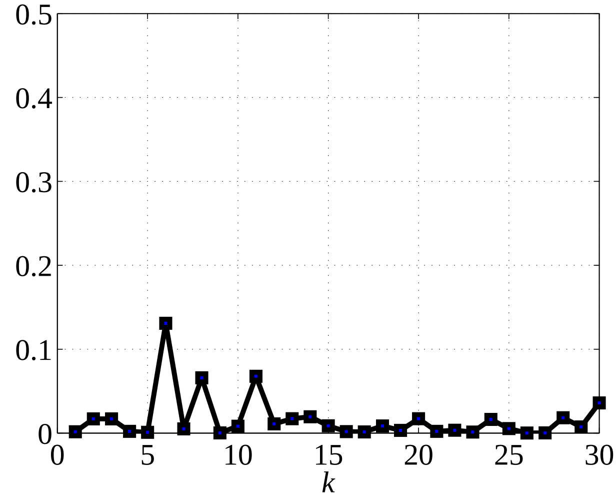


Figure 4: RSS processing result.

where $R = \sqrt{2}$ is a radius of circle circumscribed around the set X_0, $P_0 = diag\left(0.2222, \quad 0.2222\right)$.

Fig. (4) shows computational results of RSS.

The results of the guaranteed algorithm are reported in Fig. (6): the sets $X[y_k]$ and $X_{k+1/k}$ are obtained as a result of set operations (4)–(6), when as a set of measurement errors v_k (3) the set V is used; the sets $X^{'}[y_k]$ and $X^{'}_{k+1/k}$ are obtained as a result of set operations (4)–(6), when as a set of measurement errors v_k (3) the set $\tilde{V}$ is used.

As it is seen from Fig. (6), the information sets obtained as intersection $X^{'}_{k+1/k} \cap X^{'}[y_{k+1}]$ are smaller than the information sets obtained by $X_{k+1/k} \cap X[y_{k+1}]$. This confirms that the use of the set $\tilde{V}$ instead of a priori given set in Eq. (6) allows to enhance a solution accuracy of the minimax filtering problem.

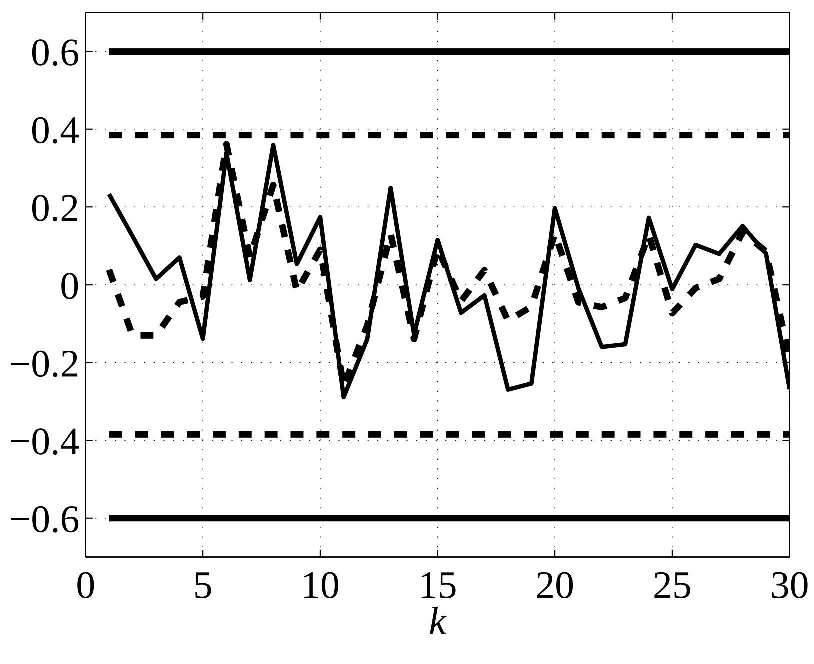
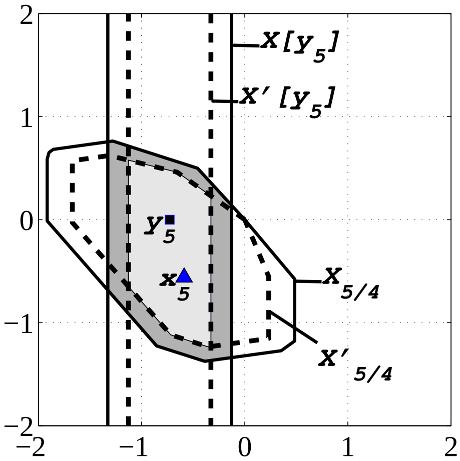
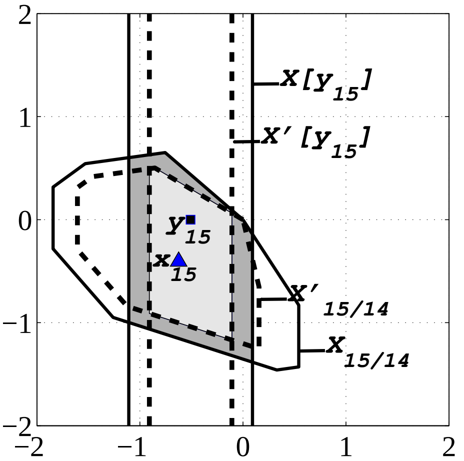


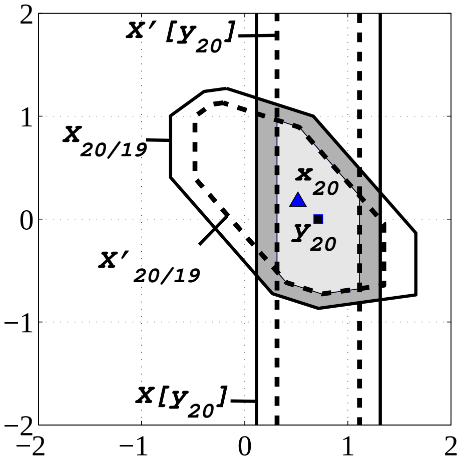
Figure 5: Measurement noise v_k and innovation sequence Δ_k (the solid lines denote v_k and boundaries of the set V, the dashed lines denote Δ_k and boundaries of the set $\tilde{V}$).



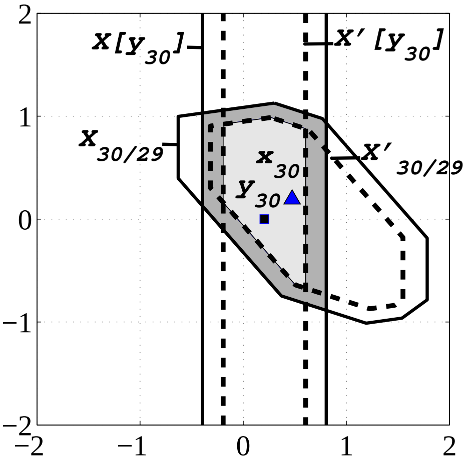
a) The sets $X[y_{k+1}]$, $X_{k+1/k}$, $X^{'}[y_{k+1}]$, $X^{'}_{k+1/k}$
at the time step $k = 5$



b) The sets $X[y_{k+1}]$, $X_{k+1/k}$, $X^{'}[y_{k+1}]$, $X^{'}_{k+1/k}$
at the time step $k = 15$



c) The sets $X[y_{k+1}]$, $X_{k+1/k}$, $X^{'}[y_{k+1}]$, $X^{'}_{k+1/k}$
at the time step $k = 20$



d) The sets $X[y_{k+1}]$, $X_{k+1/k}$, $X^{'}[y_{k+1}]$, $X^{'}_{k+1/k}$
at the time step $k = 30$

Figure 6: The adaptive minimax filtering results. The solid line denotes the sets $X[y_{k+1}]$ and $X_{k+1/k}$. The dashed line denotes the sets $X^{'}[y_{k+1}]$ and $X^{'}_{k+1/k}$.

6 Summary and conclusions

The adaptive algorithm of one-dimensional measurement noise estimation is proposed. It is not necessary to assume the model of measurement errors. The algorithm is based on statistical analysis of the innovation sequence values in the Kalman filter. The state vector estimation problem is considered on an example of linear dynamical system the state of which is observed only on a short-time interval. The estimation problem is solved under probabilistically guaranteed statement. It is assumed that the Kalman filter and the guaranteed algorithm are applied together. The Kalman filter implementation is performed for measurement data preprocessing. A numerical example is given to illustrate the results of proposed algorithm.

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