The Solution to the Problem of Optimal Control in an Unstable Economic System

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Abstract

The article presents a mathematical description of the process of optimal control over an unstable macroeconomic system based on Leontief's input-output model. The optimal equation allows setting a balanced growth rate for a macroeconomic system, which is the main problem in a current development of regional and national economies. Methods of optimal control are generally applicable to the stable systems. This article shows that a developing macroeconomic system is unstable and thus, optimal control over it has its peculiarities. An unstable macrosystem divides into two subsystems: a stable multidimensional and an unstable dimensional. The stable system is optimized via standard methods, where the growth rate of the entire system is set by a single growing exponent from the second unstable system. In order to divide the system, the author suggests using homothetic transformation; and calculating parameters of optimal control is achieved by solving a Riccati equation. Results of solving a matrix of factors determine the cost of restructuring unstable macroeconomic systems with a balanced growth rate. Knowing the cost of optimal control and restructuring creates prerequisites for a more effective process of managing socio-economic politics in the region and the whole country. These results play a vital role in decision-making processes of management and administrative bodies concerning statistical analyses and managing the economic situation. Results are based on the hypothesis that dynamic models of macroeconomic systems are linear. In practice though actual economic systems are subject to various effects like synergy and self-organization, which cannot be described under the linearity hypothesis. Elaboration upon the problems of optimal control over nonlinear and unstable economic systems is required in future research.

Keywords: macroeconomic systems; economic growth; mathematical modeling; consumption; gross output; optimal control, restructuring.

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1 Problem Statement

Ensuring optimal control over macroeconomic systemsis is a serious problem, solving which will allow radical restructuring [1, 2] of the economy at a macro- and meso-level and at minimal cost. Restructuring is essential in achieving a balanced growth of gross output, GDP and other macroeconomic indicators [3, 4, 5, 6].

It is an accepted fact that optimal control is only achievable in stable systems. A question about the stability of certain economic systems, e.g. a firm or a country is frequently raised in the economics community. There are a variety of methods of achieving a stable growth [7], but the problem lies in the contradiction in the definition of a «stable growth» itself.

On the one hand, if a system is growing, then its parameters increase, i.e. grow. It is desirable to have an economy that is constantly growing. However, on the other hand, systems with an indefinite increase in any parameters are unstable; hence a macroeconomic system with constantly growing parameters is unstable. Achieving a stable growth by definition unstable system is difficult but possible. One approach to solving this problem presents in this article.

At the core of optimal control lies a process of restructuring the macroeconomic system that would follow a certain plan in order set a balanced growth rate of the system while maintaining set proportions for material [8], capital, labour and other costs. Every system was put in place to achieve some sort of goal that is why for future discussion we should introduce a concept of an ideal macroeconomic system where all of its cost proportions are balanced. Let's call a system with balanced development trajectories an ideal model. This article presents a method of forming ideal trajectories, to which aspires every macroeconomic system in order to achieve desired growth rates and proportions. An important part in that is creating optimal criteria for control, signifying actual optimal control over an economic system. Both ideal and growing systems are unstable.

2 Problem Solution

Regarding theoretical grounds of the article should be mentioned that modern theories on economic growth are based on two sources: the neoclassical theory conceived by J.B. Say and fully formed in the works of J.B. Clark (1847-1938) and the Keynesian theory of macroeconomic equilibrium [9].

Out of all the models of economic dynamics, the multitude of those which are able to most fully demonstrate the transient processes and control over them, the structural shifts and statistical stability, we will point out the Neumann model and the Leontief's model in dynamic contrast [10, 11]. One of the most useful properties of these models is their ability to be presented in a form of differential equations which describe the dynamic economic systems.

Balanced trajectories with the maximum growth rate are called turnpikes — a term suggested by a Nobel Prize winner, Paul Samuelson. The first turnpike model was created in the 1930s by John von Neumann. His model of an expanding economy had a deep impact on the making of mathematical economics [12]. Theoretical principles of the turnpike were summarized in the Gale model, of which, as shown in [13], the Leontief model is a special case.

A dynamic variation of the Leontief model [14] is a system of inhomogeneous linear differential equations:

$$X(t) = AX(t) + B\dot{X}(t) + Y(t) \quad \text{or} \quad \dot{X}(t) = B^{-1}(E\text{-}A)X(t) - B^{-1}Y(t) \tag{1}$$

$X(t)$ – gross output; A – matrix of coefficient of direct costs, B – capital cost matrix; $Y(t)$ – vector-function of final demand; E – Unit matrix, the point above $\dot{X}(t)$ denotes the operation of differentiation.

The formal solution to the system (1) has two parts – a free $X_{fr}(t)$ and a forced $X_{for}(t)$:

$$X(t) = X_{fr}(t) + X_{for}(t) \tag{2}$$

or

$$X(t) = e^{B^{-1}(E-A)t}X(0) - e^{B^{-1}(E-A)t}\int_0^t e^{-B^{-1}(E-A)\tau}B^{-1}Y(\tau)d\tau \tag{3}$$

where $e^{B^{-1}(E-A)t}$ is a matrix exponent.

The equation (3) is greatly simplified, if you assume that there is a connection between the end product and gross output, by introducing a norms of consumption matrix Q:

$$Y(t) = QX(t) \tag{4}$$

This assumption can be considered valid because, gross output for consumption will be constant for rather large intervals of time. The simplified system will have a consumption loop and will look like this:

$$\dot{X}(t) = GX(t) \tag{5}$$

Matrix $G=B^{-1}(E\text{-}A\text{-}Q)$ is a homogenous matrix. The solution to this matrix will not be so complex, in fact, it will be quite compact:

$$X(t) = e^{Gt} X(0) \qquad (6)$$

where $X(0)$ is a starting value of the system, representing the level of gross output for the current year.
Using the classic method of calculating transient processes, we get a solution to (5) that looks like this:

$$X(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \ldots + C_n e^{\lambda_n t} \qquad (7)$$

where C_1, C_2, …, C_n are integration constants; λ_n are eigenvalues of matrix G, that define the unique dynamic properties (UDP) of a socio-economic system [15].
In accordance with the system of national accounts, production accounting is done for 17 types of economic activities in Russia. In order to predict the growth of gross output we need to solve a system of differential equations with a degree of 17. Solving such a multi-dimensional problem is best done through a matrix using homothetic transformation. In this case, our model can be presented as a state space model:

$$\dot{X}(t) = \bar{A}X(t) + \bar{B}Y(t) \qquad (8)$$

where $\bar{A} = B^{-1}(E - A)$ is a main matrix, and $\bar{B} = -B^{-1}$ is a matrix of external influences.
Solving the system of differential equations (8) will allow us to determine the expected values of gross output of a country or its regions. Naturally, disregarding the effects of external influences from the government or ineffective production will make the resulting values unbalanced. These points out an important issue of balancing the main macroeconomic factors for every type of economic activity. To do this, we need to establish such a level of socio-economic consumption that would let the system be in a constant state of balanced expansion. Classical economists call this expansion a turnpike development or Neumann ray [16]. This problem should be solved by using Pontryagin's maximum principle from the optimal control theory.
The difficulty for direct decision-making lies in the need to know the optimal level of social consumption that takes into account all of the socio-economic capacity. The problem boils down to defining matrix Z that connects the end product $Y(t)$ with the gross output:

$$Y(t) = ZX(t) \qquad (9)$$

Statement (9) lets us present the model with a consumption loop like this:

$$\dot{X}(t) = (\bar{A} + \bar{B}Z)X(t) \qquad (10)$$

Information about optimal control over the system consists in matrix $\bar{B}Z$. This is the value at which we have to change the coefficients of matrix $\bar{A}$ to achieve balanced function as a result of optimal control.
We have the system now that has positive feedback. It is unstable. Methods for determining matrix Z, which contains information about socio-economic norms and costs are developed for stable systems. The problem of separating the generally unstable system (10) into subsystems rises now. One of which would be stable and multidimensional, and the other – unstable and one dimensional. Such division can be achieved by using homothetic transformation which would outline n new phase variables $\tilde{X}_h$ by using:

$$X_i = \sum_{h=1}^{n} t_{ih} \tilde{X}_h \quad or \quad X = T\tilde{X} \qquad (11)$$

As a result, system

$$\left.\begin{aligned} \dot{\tilde{X}}(t) &= \tilde{G}\tilde{X}(t), & \tilde{X}(0) &= \tilde{X}_0 \\ where\ \tilde{G} &\equiv T^{-1}GT, & \tilde{X}_0 &\equiv T^{-1}X_0 \end{aligned}\right\} \qquad (12)$$

will contain matrix $\tilde{G}$, the structure of which is far simpler, than the initial one.
To use homothetic transformation (11) to transform matrix G into a diagonal matrix, the initial system can be transformed to a system with separated variables by using coefficients $\tilde{X}_h$:

$$\frac{d\tilde{X}_h}{dt} = \lambda_h \tilde{X}_h \qquad (13)$$

The solution to such system will look like this:

$$\tilde{X}_h = \tilde{X}_{h0} e^{\lambda_h t} \quad (h = 1,2,\ldots,n) \qquad (14)$$

The final solution to the system using the homothetic transformation method will contain a diagonal matrix *diag($e^{\lambda t}$)*:

$$X(t) = T \cdot diag(e^{\lambda t}) \cdot T^{-1} \cdot X(0) \tag{15}$$

where λ and T are eigenvalues and eigenvector of matrix G.
Using homothetic transformation lets us transform the matrix into a diagonal one where it can be divided into subsystems. These systems can be connected parallel; the body of mathematics for parallel system connection has been developed in control engineering. Homothetic transformation is applicable not only to closed-loop systems but also to open-loop ones. In this case we need to do the following action on the transformed (converted) matrix:

$$\tilde{A} = T^{-1}\overline{A}T, \quad \tilde{B} = T^{-1}\overline{B} \tag{16}$$

The dynamic properties of the converted system and that of the initial system are identical, because they have the same spectrum of eigenvalues. Main matrix of the converted system is diagonal and thus can be divided into parallel subsystems. In order to do that we shall use the Perron–Frobenius theorem, which states that in a model of a macroeconomic balance system, among positive eigenvalues will surely be a minimal number, which would correspond to the entire positive eigenvector. Finding the subsystem with the lowest eigenvalue is not a difficult task. It will be one dimensional, and presence of a positive number in the index of an exponent will signify a constant growth, which in turn would make it one of the unstable systems. The other subsystem will be stable. There is a possible to synthesize optimal control for it.
Let's present the converted system in the following form:

$$\begin{pmatrix} \dot{\tilde{X}}1 \\ \dot{\tilde{X}}2 \end{pmatrix} = \begin{pmatrix} \tilde{A}1 & \tilde{A}2 \\ \tilde{A}3 & \tilde{A}4 \end{pmatrix} \begin{pmatrix} \tilde{X}1 \\ \tilde{X}2 \end{pmatrix} + \begin{pmatrix} \tilde{B}1 & \tilde{B}2 \\ \tilde{B}3 & \tilde{B}4 \end{pmatrix} \begin{pmatrix} \tilde{Y}1 \\ \tilde{Y}2 \end{pmatrix} \tag{17}$$

$$\tilde{X} = \begin{pmatrix} \tilde{X}1 \\ \tilde{X}2 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} \tilde{A}1 & \tilde{A}2 \\ \tilde{A}3 & \tilde{A}4 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{B}1 & \tilde{B}2 \\ \tilde{B}3 & \tilde{B}4 \end{pmatrix}$$

This would let us to divide the matrices and vectors of the initial system into subparts by these dimensions:

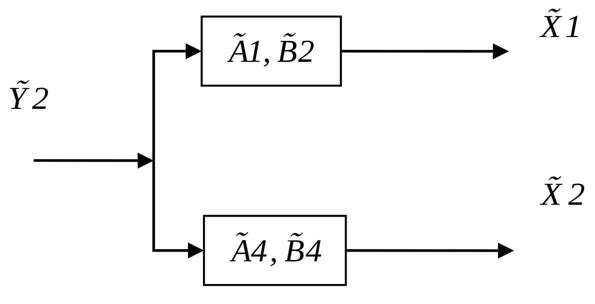
$$\tilde{X}1[1], \tilde{X}2[n-1], \tilde{A}1[1], \tilde{A}2[1,n-1], \tilde{A}3[n-1,1], \tilde{A}4[n-1,n-1]$$

Dimensions of submatrices in matrices $\tilde{A}$ and $\tilde{B}$ are identical. The matrix of the converted system is diagonal and that means that the coefficients of submatrices $\tilde{A}2$ and $\tilde{A}3$ contain zeros which would let us present system (17) as a parallel connection of two subsystems:

$$\dot{\tilde{X}}1(t) = \tilde{A}1\tilde{X}1(t) + \tilde{B}2\tilde{Y}1(t), \tag{18}$$

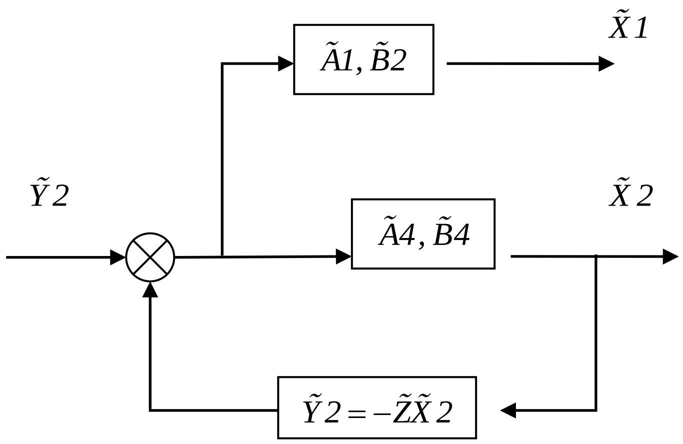
$$\dot{\tilde{X}}2(t) = \tilde{A}4\tilde{X}2(t) + \tilde{B}4\tilde{Y}2(t). \tag{19}$$

This can also be presented graphically, as shown on Picture 1.



Picture 1 – The parallel connection of two subsystems

Entrance $\tilde{Y}2$ has an effect on both subsystems. It can be optimized by optimal synthesis of the linear-quadratic regulator. Based on the structure of the system (17) the same entrance will influence an unstable system. Of course, this effect will be suboptimal; but a whole system will perform more effectively because one of subsystems would be optimized. Graphically, this situation is shown on Picture 2. The second system is controlled by feedback from the linear-quadratic regulator and thus can be considered optimal.



Picture 2 – The connection of subsystems with feedback

In order to determine $\tilde{Z}$ in a chain of negative feedback $\tilde{Y}2 = -\tilde{Z}\tilde{X}2$ we need to minimize the square functional:

$$J(X) = \int_0^\infty (\tilde{X}2^T Q\tilde{X}2 + \tilde{Y}2^T R\tilde{Y}2)dt \tag{20}$$

here Q and R are matrices of the weight coefficient. These matrices set the ratio of quality of the economic process of management to the cost of management.

Functional (20) lets us to optimize management in the system whereas spending minimal amounts of effort managing the dynamics of exit $\tilde{X}2$ by means of entrance $\tilde{Y}2$. To solve the minimization problem of (20) we will use the classic method of calculus variations. To do that let's introduce an auxiliary functional:

$$J(X) = \int_0^\infty \left[(\tilde{X}2^T R\tilde{X}2 + \tilde{Y}2^T Q\tilde{Y}2) - 2\lambda^T(\dot{\tilde{X}}2 - \tilde{A}4\tilde{X}2 - \tilde{B}4\tilde{Y}2) \right] dt \tag{21}$$

where λ - *(n-1)* is a dimensional vector of Lagrange multipliers.

The solution of the minimization problem (21) for subsystem (19) yields the following system:

$$\begin{cases} \dot{\tilde{X}}2 = \tilde{A}4\tilde{X}2 + \tilde{B}4\tilde{Y}2 \\ \dot{\lambda} = -Q\tilde{X}2 - \tilde{A}4^T\lambda \\ \tilde{Y}2 = -R^{-1}\tilde{B}4^T\lambda \end{cases} \tag{22}$$

By substituting value $\tilde{Y}2$ into the first equation of system (22) we get:

$$\begin{cases} \dot{\tilde{X}}2 = \tilde{A}4\tilde{X}2 - \tilde{B}4R^{-1}\tilde{B}4^T\lambda \\ \dot{\lambda} = -Q\tilde{X}2 - \tilde{A}4^T\lambda \end{cases} \tag{23}$$

In order to solve this system we need to substitute the corresponding variables:

$$\lambda = P\tilde{Y}2 \tag{24}$$

Multiplying the left part of the first equation in system (23) by matrix P and subtracting from it the second equation of the system will lead us to:

$$P\tilde{A}4 + \tilde{A}4^T P - P\tilde{B}4R^{-1}\tilde{B}4^T P + Q = 0 \tag{25}$$

Equation (25) is the Riccati algebraic matrix equation [17] which comes as a result of Riccati's differential equation being set in conditions of $t \to \infty$. Solving this equation is a difficult task; however it is standardized and has solutions in some cases, which grants us the possibility of determining the coefficients of matrix *P*. By substituting statement (24) into the last equation of system (23), we get the desired equation of optimal control:

$$\begin{gathered} \tilde{Y}2 = -R^{-1}(\tilde{B}4)^T P\tilde{X}2 = -\tilde{Z}\tilde{X}2, \\ \tilde{Z} = R^{-1}(\tilde{B}4)^T P \end{gathered} \tag{26}$$

The closed-loop matrix of the second subsystem with the linear-quadratic regulator $\tilde{Z}$ will be determined by formula:

$$\tilde{G}4 = \tilde{A}4 - \tilde{B}4 \cdot \tilde{Z} \tag{27}$$

Then the converted (and already optimized) system (17) will look like this:

$$\begin{pmatrix} \dot{\tilde{X}}1 \\ \dot{\tilde{X}}2 \end{pmatrix} = \begin{pmatrix} \tilde{A}1 & \tilde{A}2 \\ \tilde{A}3 & \tilde{G}4 \end{pmatrix} \begin{pmatrix} \tilde{X}1 \\ \tilde{X}2 \end{pmatrix} \tag{28}$$

Or in its condensed form:

$$\dot{\tilde{X}}(t) = \tilde{A}_{opt}\tilde{X}(t) \tag{29}$$

where $\tilde{A}_{opt}$ is a matrix of optimized closed-loop converted system coefficients.

3 Results

Determining the close-loop matrix of the macrosystem's coefficients is achieved by inversing homothetic transformation:

$$\bar{A}_{opt} = T\tilde{A}_{opt}T^{-1} \tag{30}$$

This matrix is necessary to calculate the addition to the coefficients of that first unstable system so we can get the optimal equation:

$$\bar{B}Z = \bar{A} - \bar{A}_{opt} \tag{31}$$

With the help of equation (12) we can evaluate the optimal level of the end product, accounting for the costs from socio-economic transformations of the macrosystem.

4 Conclusion

As we can see, dividing an unstable macroeconomic system into subsystems creates a possibility to determine the optimal level of expenses for the system, which creates prerequisites for a more effective management of socio-economic policies inside a region or an entire country.

5 Directions For Future Research

Results are based on the hypothesis that dynamic models of macroeconomic systems are linear. In practice though actual economic systems are subject to various effects like synergy and self-organization, [18, 19] which cannot be described under the linearity hypothesis. Elaboration upon the problems of optimal control over nonlinear and unstable economic systems is required in future research.

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