hp-Version of Collocation and Least Residuals Method in Mechanics of Laminated Composite Plates

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Abstract. A version of collocations and least residuals method (CLS) based on polynomial approximation of high degree (p-approach) was proposed and implemented. In rectangular domains collocation points are selected using the roots of Chebyshev polynomials and approximate solution is represented in the form of direct products of Chebyshev polynomials series. It was shown that the use of p-approach in the CLS method allows to obtain numerical solutions with high accuracy and to implement complex boundary conditions with no special techniques. The numerical method used to solve a problem of bending of laminated anisotropic rectangular plates within frameworks of classical laminated plate theory, first order shear deformation theory and Grigolyuk-Chulkov's broken line theory. Several specific example problems are solved, including fixed three-ply laminates with transversely isotropic layers under transverse uniform loading.

Keywords: Collocations and least residuals method, Chebyshev polynomials, plate theory, spectral methods, composite materials

1 Introduction

The collocation and least residuals method (CLS) is an efficient method for numerial solution of boundary value problems both for systems of ordinary and partial differential equations. It is based on the collocation method (CM) [1], with approximate solution is represented as a linear combination of basis functions in some functional space. To determine it unknown coefficients in CM residual of equations $R(x)$ vanishes at given points (collocation points)

$$R(x_i^{col}) = 0, \qquad \{x^{col}\} \text{ – collocation points.} \tag{1}$$

The main difference between CLS method and CM is the minimizing technique of $R(x)$. In CLS method we minimize some *functional* of residual in the collocation points [2, 3], instead of the condition (1). The CLS method is used to minimize residual in L_2 norm

$$\sum_i \|R(x_i^{col})\|_2^2 \to \min . \tag{2}$$

In CLS method the number of equations can exceed the number of unknown coefficients in representation of the solution. The solutions of arising overdetermined systems of linear algebraic equations (SLAE) are defined in the sense of (2) (least squares). In comparison with CM (1) the obtained overdetermined SLAE is often better conditioned and leads to less nonphysical oscillations in numerical solutions. Similar regularization approaches are applied in the finite element method (Least squares finite element methods) [4].

In this paper, an approximate solution is represented as a linear combination of polynomials of high degrees (p - approach) that is typical for spectral methods. This allows to obtain numerical solutions of high accuracy at low computational cost. Term (2) makes the implementation of p - approach more convenient in CLS method when compared with CM. This modification based on p - approach is called hp - version of CLS method.

We will demonstrate the application of a method to solving problems of solid mechanics – bending of laminated anisotropic rectangular plates. On a practical level, for calculating the stress and displacement fields of such structures the theories of plates are used. They lead to a smaller computational efforts compared to three-dimentional elasticity formulation.

Boundary value problems arising in a plate theories have a number of features that present difficulties for many well-known numerical methods. First, governing equations of plate theories may contain derivatives of high orders. Second, boundary conditions may be quite complicated, for example, in a form of linear combination of functions and their higher order derivatives. Third, equations of the plate theory may contain small parameters in the derivatives. These features cause serious difficulties for widely used finite differences and finite element methods. The use of both p - approach and term (2) in hp - version of CLS method may resolve these difficulties and obtain high accuracy solutions at relatively low computational efforts.

2 Formulation of Problem and Governing Equations

Let us consider a static bending of laminate composed of 3 layers of constant thickness (Fig. 1). Layers are transversely isotropic with material symmetry axis in the plate's plane. Layers orientation scheme is

$$\theta^1 = \theta^3 = 0, \qquad \theta^2 = \pi/2,$$

where θ^k is an angle measured counterclockwise from the x coordinate axis to the k-th layer material symmetry axis.

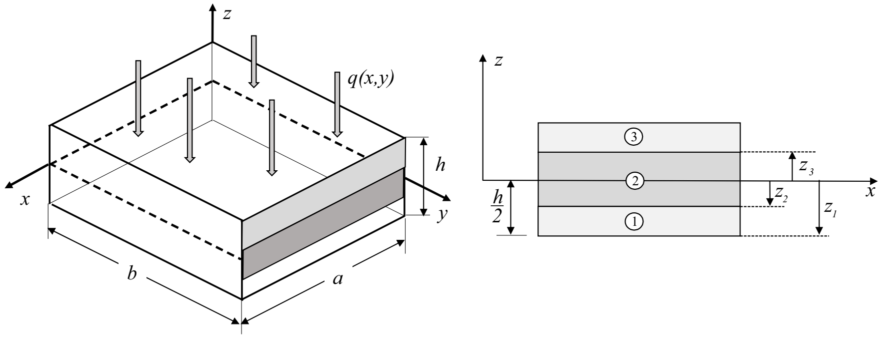

Fig. 1. Rectangular multilayered plate under transverse loading; a, b, h — plate's dimensions in the directions x, y, z respectively, z_k — k-th layer lower surface coordinate, $k = 1, 2, 3$.

Engineering constants of transversely isotropic material are [5]

$$\begin{gathered} E_L = 25 \text{ Mpsi}, \qquad E_T = 1 \text{ Mpsi}, \qquad G_{LT} = 0.5 \text{ Mpsi}, \\ G_{TT} = 0.2 \text{ Mpsi}, \qquad \nu_{LT} = \nu_{TT} = 0.25. \end{gathered} \tag{3}$$

Here E, G, ν are elasticity and shear modulus, Poisson ratios. L signifies the material symmetry axe, T the transverse direction. Layers thicknesses are

$$h^1 = h^3 = h/4, \qquad h^2 = h/2.$$

The upper surface of the plate is under uniform transverse load q_0, the lower surface is free, and a continuity condition of displacements u, v, w and stresses σ_{zz}, σ_{xz}, σ_{yz} is used on interface surfaces. The corresponding boundary conditions are defined on the boundary of the plate. The task is to calculate the stress and displacement fields of such plates.

Calculation of thin laminated anisotropic structures within framework of the three-dimensional elasticity is associated with high computational efforts. Therefore, many researchers frequently make use of more robust plate theories, that allows to reduce the dimension of the original problem by excluding the direction of the coordinate z from consideration. We consider three plate theories, where transverse shear stresses are simulated differently: classical laminated plate theory (CLPT), first order shear deformation theory (FSDT) or Timoshenko's plate theory [6] and Grigolyuk-Chulkov's broken line theory (GCT) [7].

Classical laminated plate theory uses the classical Kirchhoff assumption which implies the geometric relationships in the form of

$$\begin{gathered} e_{xx} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2}, \qquad e_{yy} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2}, \qquad e_{zz} = 0, \\ e_{xy} = \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - z\frac{\partial^2 w_0}{\partial x \partial y}, \qquad e_{xz} = 0, \qquad e_{yz} = 0. \end{gathered}$$

Here e_{ij} – strains; $u_0(x,y)$, $v_0(x,y)$, $w_0(x,y)$ – central plane displacements.

Constitutive equations for k-th layer are expressed by

$$\begin{pmatrix}\sigma_{xx}^k\\ \sigma_{yy}^k\\ \sigma_{xy}^k\end{pmatrix} = \begin{pmatrix}Q_{11}^k & Q_{12}^k & Q_{16}^k\\ Q_{12}^k & Q_{22}^k & Q_{26}^k\\ Q_{16}^k & Q_{26}^k & Q_{66}^k\end{pmatrix}\begin{pmatrix}e_{xx}\\ e_{yy}\\ e_{xy}\end{pmatrix}, \tag{4}$$

$$\begin{pmatrix}Q_{11}^k & Q_{12}^k & Q_{16}^k\\ Q_{12}^k & Q_{22}^k & Q_{26}^k\\ Q_{16}^k & Q_{26}^k & Q_{66}^k\end{pmatrix} = D_1^k C_1^k (D_1^k)^T,$$

where

$$C_1^k = \begin{pmatrix}C_{11}^k & C_{12}^k & 0\\ C_{12}^k & C_{22}^k & 0\\ 0 & 0 & C_{66}^k\end{pmatrix}, \quad D_1^k = \begin{pmatrix}\cos^2\theta^k & \sin^2\theta^k & -\sin 2\theta^k\\ \sin^2\theta^k & \cos^2\theta^k & \sin 2\theta^k\\ (\sin 2\theta^k)/2 & -(\sin 2\theta^k)/2 & \cos 2\theta^k\end{pmatrix}$$

Coefficients C_{ij}^k express in terms of the engineering constant as follows:

$$C_{11}^k = \frac{E_L^k}{1-\nu_{LT}^k\nu_{TL}^k}, \qquad C_{22}^k = \frac{E_T^k}{1-\nu_{LT}^k\nu_{TL}^k},$$
$$C_{12}^k = \frac{\nu_{LT}^k E_T^k}{1-\nu_{LT}^k\nu_{TL}^k}, \qquad C_{66}^k = G_{LT}.$$

It is convenient to define the following quantities

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{3}\int_{z_k}^{z_{k+1}} Q_{ij}^k\, dz = \sum_{k=1}^{3} Q_{ij}^k (z_{k+1}-z_k),\\ D_{ij} &= \sum_{k=1}^{3}\int_{z_k}^{z_{k+1}} Q_{ij}^k\, z^2\, dz = \frac{1}{3}\sum_{k=1}^{3} Q_{ij}^k (z_{k+1}^3 - z_k^3), \end{aligned} \tag{5}$$

where z_k – coordinates of layers lower surface (Fig. 1). Finaly governing equations for considered problem within CLPT framework are given by

$$\begin{aligned} (A_{12}+A_{66})\frac{\partial^2 v_0}{\partial x\,\partial y} + A_{11}\frac{\partial^2 u_0}{\partial x^2} + A_{66}\frac{\partial^2 u_0}{\partial y^2} &= 0,\\ (A_{12}+A_{66})\frac{\partial^2 u_0}{\partial x\,\partial y} + A_{22}\frac{\partial^2 v_0}{\partial y^2} + A_{66}\frac{\partial^2 v_0}{\partial x^2} &= 0,\\ (2D_{12}+4D_{66})\frac{\partial^4 w_0}{\partial x^2\,\partial y^2} + D_{11}\frac{\partial^4 w_0}{\partial x^4} + D_{22}\frac{\partial^4 w_0}{\partial y^4} &= q_0. \end{aligned} \tag{6}$$

We consider two kinds of boundary conditions:

– clamped

$$
\begin{aligned}
&x = 0: && u_0 = 0, \quad v_0 = 0, \quad w_0 = 0, \quad \frac{\partial w_0}{\partial x} = 0;\\
&x = a: && u_0 = 0, \quad v_0 = 0, \quad w_0 = 0, \quad \frac{\partial w_0}{\partial x} = 0;\\
&y = 0: && u_0 = 0, \quad v_0 = 0, \quad w_0 = 0, \quad \frac{\partial w_0}{\partial y} = 0;\\
&y = a: && u_0 = 0, \quad v_0 = 0, \quad w_0 = 0, \quad \frac{\partial w_0}{\partial y} = 0;
\end{aligned}
$$

– simply-supported

$$
\begin{aligned}
&x = 0: && \frac{\partial u_0}{\partial x} = 0, \quad v_0 = 0, \quad w_0 = 0, \quad \frac{\partial^2 w_0}{\partial x^2} = 0;\\
&x = a: && \frac{\partial u_0}{\partial x} = 0, \quad v_0 = 0, \quad w_0 = 0, \quad \frac{\partial^2 w_0}{\partial x^2} = 0;\\
&y = 0: && u_0 = 0, \quad \frac{\partial v_0}{\partial y} = 0, \quad w_0 = 0, \quad \frac{\partial^2 w_0}{\partial y^2} = 0;\\
&y = a: && u_0 = 0, \quad \frac{\partial v_0}{\partial y} = 0, \quad w_0 = 0, \quad \frac{\partial^2 w_0}{\partial y^2} = 0.
\end{aligned}
$$

First order shear deformation theory allows transverse shear in a first approximation by defining independent function of rotation of the transverse normal about central surface: $\phi_x(x, y)$ and $\phi_y(x, y)$. The strains are obtained by

$$
e_{xx} = \frac{\partial u_0}{\partial x} + z\frac{\partial \phi_x}{\partial x}, \qquad e_{yy} = \frac{\partial v_0}{\partial y} + z\frac{\partial \phi_y}{\partial y},
$$
$$
e_{xy} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right),
$$
$$
e_{xz} = \frac{\partial w_0}{\partial x} + \phi_x, \qquad e_{yz} = \frac{\partial w_0}{\partial y} + \phi_y, \qquad e_{zz} = 0.
$$

Constitutive equations of FSDT are obtained by adding to (4) expressions for the shear stresses

$$
\begin{pmatrix} \sigma^k_{yz} \\ \sigma^k_{xz} \end{pmatrix} = \begin{pmatrix} Q^k_{44} & Q^k_{45} \\ Q^k_{45} & Q^k_{55} \end{pmatrix} \begin{pmatrix} e_{yz} \\ e_{xz} \end{pmatrix},
$$

$$
\begin{pmatrix} Q^k_{44} & Q^k_{45} \\ Q^k_{45} & Q^k_{55} \end{pmatrix} = D^k_2 C^k_2 (D^k_2)^T, \quad C^k_2 = \begin{pmatrix} C^k_{44} & 0 \\ 0 & C^k_{55} \end{pmatrix}, \quad D^k_2 = \begin{pmatrix} \cos\theta^k & \sin\theta^k \\ -\sin\theta^k & \cos\theta^k \end{pmatrix},
$$

where stiffness coefficients are express by engineering constants

$$
C^k_{44} = G_{TT}, \quad C^k_{55} = G_{LT}.
$$

Governing equations for FSDT can be written as

$$(A_{12}+A_{66})\frac{\partial^2 v_0}{\partial x\,\partial y}+A_{11}\frac{\partial^2 u_0}{\partial x^2}+A_{66}\frac{\partial^2 u_0}{\partial y^2}=0,$$

$$(A_{12}+A_{66})\frac{\partial^2 u_0}{\partial x\,\partial y}+A_{22}\frac{\partial^2 v_0}{\partial y^2}+A_{66}\frac{\partial^2 v_0}{\partial x^2}=0,$$

$$-A_{44}\frac{\partial \phi_y}{\partial y}-A_{44}\frac{\partial^2 w_0}{\partial y^2}-A_{55}\frac{\partial \phi_x}{\partial x}-A_{55}\frac{\partial^2 w_0}{\partial x^2}=q_0,$$

$$(D_{12}+D_{66})\frac{\partial^2 \phi_y}{\partial x\,\partial y}-A_{55}\phi_x-A_{55}\frac{\partial w_0}{\partial x}+D_{11}\frac{\partial^2 \phi_x}{\partial x^2}+D_{66}\frac{\partial^2 \phi_x}{\partial y^2}=0,$$

$$(D_{12}+D_{66})\frac{\partial^2 \phi_x}{\partial x\,\partial y}-A_{44}\phi_y-A_{44}\frac{\partial w_0}{\partial y}+D_{22}\frac{\partial^2 \phi_y}{\partial y^2}+D_{66}\frac{\partial^2 \phi_y}{\partial x^2}=0,$$

where coefficients are defined by (5).

Boundary conditions for clamped edges in FSDT are written as follows:

$$u_0=0,\quad v_0=0,\quad w_0=0,\quad \phi_x=0,\quad \phi_y=0,\qquad (x,y)\in\partial\Omega.$$

More details of CLPT and FSDT plate theories are described in [6].

Grigolyuk-Chulkov's theory is layerwise theory, where mechanical properties of each layer are considered separately. For this purpose, in each layer the rotations of transverse normal about central surface $\phi_x^k(x,y)$ and $\phi_y^k(x,y)$ are defined. It can be assumed that the GCT is a generalization of the FSDT that takes into account transverse shear stresses in each layer separately.

Expressions for geometrical equations in GCT have the form

$$e_{xx}^k=\frac{\partial u_0}{\partial x}+\sum_{i=1}^{3}P_{ki}\,\frac{\partial \phi_x^i}{\partial x}+(z-z_{k-1})\,\frac{\partial \phi_x^k}{\partial x},$$

$$e_{yy}^k=\frac{\partial v_0}{\partial y}+\sum_{i=1}^{3}P_{ki}\,\frac{\partial \phi_y^i}{\partial y}+(z-z_{k-1})\,\frac{\partial \phi_y^k}{\partial y},$$

$$e_{xy}^k=\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)+\sum_{i=1}^{3}P_{ki}\left(\frac{\partial \phi_y^i}{\partial x}+\frac{\partial \phi_x^i}{\partial y}\right)+(z-z_{k-1})\left(\frac{\partial \phi_y^k}{\partial x}+\frac{\partial \phi_x^k}{\partial y}\right),$$

$$e_{xz}^k=\phi_x^k+\frac{\partial w}{\partial x},\quad e_{yz}^k=\phi_y^k+\frac{\partial w}{\partial y},\quad e_{zz}^k=0,$$

where

$$P=\begin{pmatrix}0&0&0\\ h^1&0&0\\ h^1&h^2&0\end{pmatrix}.$$

Boundary conditions for clamped edges in GCT are $(k=1,2,3)$

$$u_0=0,\quad v_0=0,\quad w_0=0,\quad \phi_x^k=0,\quad \phi_y^k=0,\qquad (x,y)\in\partial\Omega.$$

Because of awkward form, governing equations of GCT are not presented. Detailed description of the theory can be found in [7]. It is necessary to note, that plate theories are approximations to elasticity theory and bring about their own errors which are important to estimate.

3 *hp* - Version of CLS Method

Implementation aspects of CLS method are similar to the CM. Consider a general boundary value problem for a linear elliptic system in a rectangular domain $\Omega = [0, a] \times [0, b]$:

$$Lu(x,y) = f(x,y), \qquad (x,y) \in \Omega,$$
$$L_{bnd}u(x,y) = g(x,y), \qquad (x,y) \in \partial\Omega.$$

Let us define a grid with non-overlapping rectangular cells Ω^k $(k = 1, \dots K)$. In each cell Ω^k we introduce local variables (α_1^k, α_2^k), which are associated with global variables (x, y) in the Cartesian coordinate system by

$$\alpha_1^k = \frac{x - x^{*k}}{d_1^k}, \quad \alpha_2^k = \frac{y - y^{*k}}{d_2^k},$$

where $2d_1^k$, $2d_2^k$ — sizes of cell in x and y directions, (x^{*k}, y^{*k}) — the coordinates of the cell centers. Local variables are varying in canonical interval $\alpha_1^k \in [-1, 1]$, $\alpha_2^k \in [-1, 1]$. The upper index k, that indicates the cell's number, will be omitted further.

In this version of CLS method approximate solution in the cell is represented in form of direct product of single variable basis functions :

$$u(\alpha_1, \alpha_2) = \sum_{i_1=0}^{N_1-1} \sum_{i_2=0}^{N_2-1} c_{i_1 i_2} \phi_{i_1}(\alpha_1) \phi_{i_2}(\alpha_2). \tag{7}$$

Functions ϕ_i are chosen as the Chebyshev polynomials of the first kind T_n

$$\phi_{i_1}(\alpha_1) = T_{i_1}(\alpha_1), \qquad \phi_{i_2}(\alpha_2) = T_{i_2}(\alpha_2).$$

In previous paper [8] we used cardinal functions in Lagrange-like form

$$\phi_{i_1}(\alpha_1) = \prod_{\substack{m=0 \\ m \neq i_1}}^{N_1-1} \frac{\alpha_1 - (\alpha_1)_m^{col}}{(\alpha_1)_{i_1}^{col} - (\alpha_1)_m^{col}}, \qquad \phi_{i_2}(\alpha_2) = \prod_{\substack{l=0 \\ l \neq i_2}}^{N_2-1} \frac{\alpha_2 - (\alpha_2)_l^{col}}{(\alpha_2)_{i_2}^{col} - (\alpha_2)_l^{col}}, \tag{8}$$

where $\left((\alpha_1)_m^{col}, (\alpha_2)_m^{col}\right)$ are local coordinates of collocation points. But in practice polynomials (8) require a large number of arithmetic operations and lead to complex expressions when differentiating. In this sense, Chebyshev polynomials are more convenient choice.

To determine the unknown coefficients in representation (7) for each cell let us write down the equation of three types

– collocation equations at the collocation point $\left(\alpha_1^{col}, \alpha_2^{col}\right)$

$$Lu(\alpha_1^{col}, \alpha_2^{col}) = f(\alpha_1^{col}, \alpha_2^{col}); \tag{9}$$

– boundary equations at given point $\left(\alpha_1^{bnd}, \alpha_2^{bnd}\right)$, on boundary $\partial\Omega^k$, adjacent to $\partial\Omega$

$$L_{bnd}u(\alpha_1^{bnd}, \alpha_2^{bnd}) = g(\alpha_1^{bnd}, \alpha_2^{bnd}); \tag{10}$$

– matching conditions on interface between neighbour cells at given points $(\alpha_1^{mat}, \alpha_2^{mat})$

$$L_{mat}u(\alpha_1^{mat}, \alpha_2^{mat}) = L_{mat}u^{adj}(\alpha_1^{mat}, \alpha_2^{mat}), \tag{11}$$

Here u^{adj} – solution defined in neighbour cell Ω^{adj}.

Matching conditions L_{mat} usually require the continuity of the solutions and the necessary number of its derivatives along the normal to the boundary of the cell.

In this version of CLS method the local coordinates of collocation points are roots of Chebyshev polynomials ($i_1 = 1, \dots, N_1$, $i_2 = 1, \dots, N_2$)

$$\left((\alpha_1)_{i_1}^{col}, (\alpha_2)_{i_2}^{col}\right) = (t_1^{i_1}, t_2^{i_2}),$$

where $t_1^{i_1}$ and $t_2^{i_2}$ – roots of Chebyshev polynomials of N_1 and N_2 degree respectively. By the same way we define $\left(\alpha_1^{bnd}, \alpha_2^{bnd}\right)$ and $(\alpha_1^{mat}, \alpha_2^{mat})$ on cell boundaries ($i_1 = 1, \dots, N_1$, $i_2 = 1, \dots, N_2$)

$$(-1, t_2^{i_2}), \qquad (1, t_2^{i_2}), \qquad (t_1^{i_1}, -1), \qquad (t_1^{i_1}, 1).$$

Thus, for N_1N_2 unknown coefficients in cell we use N_1N_2 collocation equations appended by equations on cell boundary. Thus, in hp-version of CLS method corresponding SLAE becomes overdetermined. In this version of method approximate solution form does not satisfy to boundary and matching conditions identically, so residual $R(x)$ (2) must contain not only collocation equations, but boundary conditions (10) and matching conditions (11) too. In particular, this allows us to consider the boundary conditions in complex form.

To solve the overdetermined linear systems in the least squares sense (2) we use QR factorization of its matrix, implemented by Householder method. In the case of large linear systems we use domain decomposition method [9]. This allows to reduce the solution of the problem in whole region to iterative process through subdomains with computational complexity is much smaller than the original problem for the region. For linear systems in a subdomain we use Householder method again. In this case special matching conditions between subdomains are used. For example, the continuity of function and its first derivative at the boundary of the cell Ω^k can be written as

$$u + p_1\frac{\partial u}{\partial n} = u^{adj} + p_1\frac{\partial u^{adj}}{\partial n},$$

where u is solution in the cell at the current iteration; u^{adj} — solution in the neighbor cell; n — the outer normal to the boundary Ω^k. For plate theories, that may contain derivatives up to the 4th order, we can additionally require the continuity of a linear combination of the second and third derivatives of the approximate solution:

$$\frac{\partial^2 u}{\partial n^2} + p_2 \frac{\partial^3 u}{\partial n^3} = \frac{\partial^2 u^{adj}}{\partial n^2} + p_2 \frac{\partial^3 u^{adj}}{\partial n^3}.$$

The choice of weights p_1, p_2 can affect the properties of the numerical solutions and speed of convergence of the iterative process.

4 Numerical Experiments

In all numerical experiments a single cell that coincides with the entire domain is used. Further we will consider only square plates $a = b = 1\,\mathrm{m}$.

To demonstrate the capabilities of hp - version of CLS method let us consider the problem with the known exact solution. The last equation of the system (6) is similar to the Kirchhoff-Love plate theory equation for bending of a homogeneous orthotropic plates

$$D_{11}\frac{\partial^4 w_0}{\partial x^4} + (2D_{12} + 4D_{66})\frac{\partial^4 w_0}{\partial x^2\, \partial y^2} + D_{22}\frac{\partial^4 w_0}{\partial y^4} = q_0.$$

Consider 3-ply simply-supported square laminate under uniform transverse load q_0 with stiffness coefficients defined by (3). This problem can be solved by Fourier method [10]. In this case maximum deflection is observed in the center of the plate and if $h = 0.01\,\mathrm{m}$, then the deflection value for Fourier method solution (sum of first 2500 members) is

$$w^* = \frac{w_0(0.5, 0.5)}{q_0} 10^8 = 9.8577127.$$

Deflections in the center of the plate are calculated by $hp-$version of CLS method are shown in Table 1.

Table 1. Deflections at the center of the plate obtained by the $hp-$version of CLS method, $h = 0.01$.

$N_1 \times N_2$	$w_0(0.5, 0.5)/q_0 \cdot 10^8$
10×10	**9.857**9
20×20	**9.8577**3
30×30	**9.85771**5

Table 1 demonstrates high accuracy of numerical results, obtained by $hp-$ version of CLS method even for the differential equation with derivatives of

4-th order of the unknown functions. Thus, $hp-$version of CLS method has no difficulties when working with differential equations containing derivatives of high orders like (3).

Consider another formulation of the problem. Let us use the free edge conditions on the one of the edges ($x = 1$) in the previous problem.

$$D_{11}\frac{\partial^2 w_0}{\partial x^2} + D_{12}\frac{\partial^2 w_0}{\partial y^2} = 0,$$
$$D_{11}\frac{\partial^3 w_0}{\partial x^3} + (D_{12} + 2D_{66})\frac{\partial^3 w_0}{\partial x^2 \partial y} = 0.$$

These boundary conditions by the use of term (2) implements with no additional effort in the CLS method. Fig. 2 shows the deformed shape of the plate with the free edge.

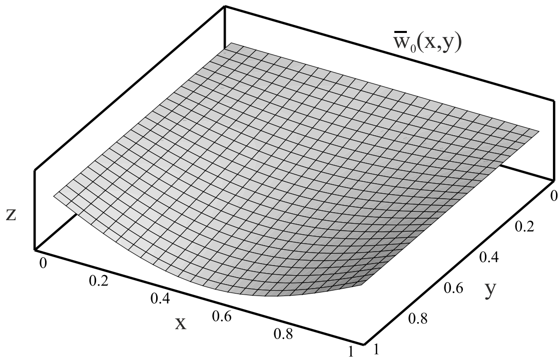

Fig. 2. Simply-supported plate with free edge.

Now let us consider the bending of clamped 3-ply laminates with different relative thickness h/a, as described in Section 2. Stress and displacement fields calculation for such plates will be carried out within framework of three theories described above.

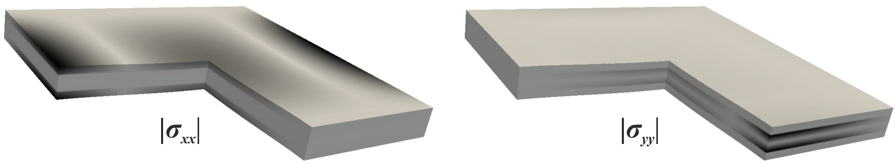

Fig. 3. Stress fields for absolute value of σ_{xx} and σ_{yy} in 3-ply laminate for h/a=0.02.

Brief analysis of solution shows that σ_{xx} prevails in the stress state of outer layers (Fig. 3). And maximum absolute values are observed in the vicinity of the clamped edges ($x = 0, 1$). Similar conclusion are true for the middle layer of the plate and component σ_{yy}, which is associated with an orientation of transversely isotropic material.

Further we will use the following normalized quantities

$$\bar{\sigma}_{xx}^k = \frac{\sigma_{xx}^k}{q_0 S^2}, \qquad \bar{\sigma}_{yy}^k = \frac{\sigma_{yy}^k}{q_0 S^2}, \qquad \bar{\sigma}_{xy}^k = \frac{\sigma_{xx}^k}{q_0 S^2} 10^{-2},$$

$$\bar{w}_0 = \frac{\sigma_{xx}^k}{q_0} 10^{-9}, \qquad \bar{z} = \frac{z}{h}.$$

Table 2. Stresses and deflection in 3-ply laminate. The results of calculations carried out in the framework of the CLPT, FSDT and GCT plate theories. Sign (%) is used for relative percentage deviation from GCT.

h/a	GCT	FSDT	CLPT	FSDT (%)	CLPT (%)
			$\bar{\sigma}_{xx}^3(a, 0, h/2)$		
0.1	0.975	0.475	0.579	51.2	40.6
0.05	0.569	0.545	0.579	4.28	1.61
0.02	0.576	0.574	0.578	0.30	0.42
0.01	0.578	0.578	0.578	0.05	0.13
			$\bar{\sigma}_{yy}^2(0, a, h/4)$		
0.1	0.636	0.662	0.456	4.21	28.2
0.05	0.605	0.536	0.456	11.5	24.6
0.02	0.483	0.471	0.456	2.65	5.72
0.01	0.463	0.460	0.455	0.64	1.82
			$\bar{\sigma}_{xy}^3(3/4a, 3/4a, h/2)$		
0.1	0.983	0.675	0.006	31.3	38.4
0.05	0.632	0.647	0.006	2.28	4.20
0.02	0.616	0.616	0.006	0.04	1.74
0.01	0.609	0.609	0.006	0.02	0.62
			$\bar{w}(a/2, a/2, h/2)$		
0.1	–0.751	–0.603	–0.208	31.30	38.36
0.05	–2.650	–2.538	–1.667	2.28	4.20
0.02	–28.523	–28.335	–26.044	0.04	1.74
0.01	–213.312	–212.974	–208.354	0.02	0.62

Table 2 shows calculated deflections and stresses in vicinities of their maximum absolute values. We assume GCT as the most accurate among considered theory, because its hypothesis is the most suitable in given structure [8, 7]. Therefore, we will treat it as a reference.

Despite the simplicity, the CLPT may be used for very thin laminated plates. If the 5% deviations from GCL are admissible, CLPT can be applied for $h/a < 0.02$ case. More accurate results are obtained within the framework of FSDT. It can be used for plates with aspect ratio $h/a < 0.05$. For thicker plates GCT theory differs significantly.

In presented calculations all functions are approximated with $N_1 = N_2 = 17$ that allowed to get three guaranteed digits for the maximum deflection. In this case, the CLPT is required to determine $17 \cdot 17 \cdot 3 = 867$ unknown coefficients, FSDT – $17 \cdot 17 \cdot 5 = 1445$, and GCT – 2601. It is interesting that for $h/a < 0.02$ CLPT results do not differ significantly from those of FSDT. In particular, it means that for plates with $h/a < 0.02$ CLPT is preferable, because of lower computational efforts.

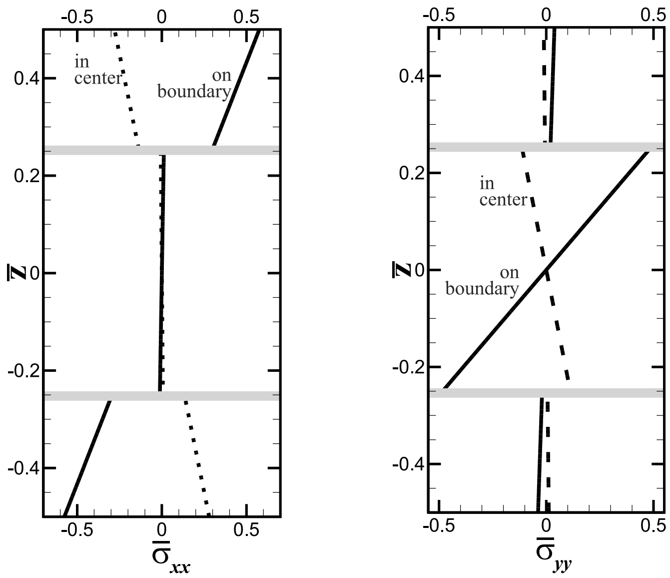

Fig. 4. Stresses distribution along z coordinate $\bar{\sigma}_{xx}(a, 0, \bar{z})$, $\bar{\sigma}_{xx}(a/2, a/2, \bar{z})$ and $\bar{\sigma}_{yy}(0, b, \bar{z})$, $\bar{\sigma}_{yy}(a/2, a/2, \bar{z})$ in 3-ply laminate for $h/a = 0.02$.

Fig. 4 shows the distribution of stresses σ_{xx} and σ_{yy} along the z coordinate (thickness) for different points (x, y): in the center of the plate and on the border where they reach the maximum values. The absolute maximum stresses values are observed at the outer sutfaces of layers. And the absolute values at the edges of the plate exceed values at the center of the plate a few times, that is true for both the stress tensor components.

Presented numerical results shows that hp - version of CLS method can be successfully applied to problems of mechanics of laminates anisotropic rectangular plates within framework the various plates theories. Term (2) allows us to consider a wide class of boundary value problems including complex boundary conditions. Moreover p-approach allows to obtain high accuracy of numerical solutions at low computational efforts.

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