Congruent Circles Packing and Covering Problems for Multi-Connected Domains with Non-Euclidean Metric, and Their Applications to Logistics

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Abstract. The article is devoted to optimal covering and packing problems for a bounded set in a two-dimensional metric space with a given amount of congruous circles. Such problems are of both theoretical interest and practical relevance. For instance, such statements appear in logistics when one needs to locate a given number of commercial or social facilities. A numerical algorithm based on fundamental physical principles due to Fermat and Huygens is suggested and implemented. It allows us to solve the problems for the cases of non-convex sets and non-Euclidean metrics. The results of numerical experiments are presented and discussed. Calculations show the applicability of the proposed approach its high efficiency for covering of a convex set in the Euclidean space by a sufficiently large amount of circles.

Keywords: circles covering problem, circles packing problem, non-Euclidean metric, optical-geometric approach, logistics, facilities, numerical algorithm, computational experiment.

1 Introduction

The facility location problem is a branch of mathematical modeling concerned with the optimal placement of facilities to minimize various negative factors. The Supply Chain Management Terms and Glossary defines facilities as "An installation, contrivance, or other thing which facilitates something; a place for doing something: Commercial or institutional buildings, including offices, plants and warehouses". There is a number of papers devoted to the problem, see e.g. [11, 12, 37]. However, the known publications are usually concerned with certain particular cases. At the same time, we are aimed at a systemic solution of this problem at the level of regional, national and international transport and logistics systems (TLS).

In connection with the above, we develop a multi-stage technology for studying complex systems. On the first stage, we solve the problem of optimal placement of infrastructure logistic facilities, assumed the absence of these objects in the considered region. For example, there may be cellular towers of a certain operator, ATMs of particular bank, etc. This problem is reduced to special modifications of two well-known mathematical problems: covering of a bounded set in a two-dimensional space with non-Euclidean metric by equal circles. On the second stage, we solve the problem of optimal placement of additional logistics facilities in terms of cooperation and competition. The third stage assumes designing of a proper communication system for the above defined objects. On the final stage, we treat the problem of communications' support in order to keep them in satisfactory conditions. Note that, though the developed approach operates with a variety of mathematical models, the key part is played by covering and packing problems.

The covering problem is to locate congruent geometric objects in a metric space so that its given area lies entirely within their union. This theoretical problem is widely used in solving practical tasks in various fields of human activity. Examples of such tasks are placement of cell towers, rescue points, police stations, ATMs, hospitals, schools [4, 7, 12, 15], designing energy-efficient monitoring of distributed objects by wireless sensor networks [2, 8, 13, 14] etc. Algorithms for covering of simply connected sets by congruent circles employing quasi-differentiability of the objective function are presented in [18], heuristic and metaheuristic methods can be found in [1, 3, 28, 38], algorithms of integer and continuous optimization are proposed by [27, 29, 30]. A modification of feasible directions' method appears in [32], where optimal coverings are given for different $n \leq 100$.

The optimal circle packing problem is to place objects of a prescribed form in a given container. Apparently, the most popular statement here is optimal packing two-dimensional spheres (circles, discs) in a convex set. For example, the authors of [9, 26, 33] consider the following problem: maximize the radius of a given number n of congruent circles packed in a unit square. The number of circles varies between 1 and 200. Papers [16,17,25] address the problem of packing a family of equal circles of unit radius in a great circle. The results for number of packing elements up to 81 are obtained. Birgin and Gentil [5] consider the problem of packing equal circles of unit radius in a variety of containers (circles, squares, rectangles, equilateral triangles and strips of fixed height) in order to minimize the size of the latter.

Note that the most of known results are obtained for the case when covered areas or containers are subsets of the Euclidean plane or a multi-dimensional Euclidean space. In the case of a non-Euclidean metric, covering and packing problems are relatively poorly studied. Here we could mention the works by Coxeter [10] and Boroczky [6], which deal with congruent circles packing problems for multidimensional spaces of a constant curvature (elliptic and hyperbolic cases) and assess the maximum packing density. Besides above, this problem was studied in a series of papers by Szirmai. In [34, 35] we find a method to determine the data and the density of certain optimal ball and horoball packings with Coxeter tiling for hyperbolic 3-, 4- and 5-D spaces, based on a projective interpretation of hyperbolic geometry. The goal of [36] is to extend the problem of finding the densest geodesic ball (or sphere) packing to different 3-D homogeneous geometries.

A detailed description of the proposed research technology for transport and logistics systems (including the developed software) is the subject of an extra publication. In the present paper, we are focused on mathematical modeling and their numerical implementation. The study follows our previous works on mathematical apparatus for problems of domestic logistics. In particular, in [19–23] we elaborate numerical algorithms based on optical-geometric analogy.

2 Mathematical models

Assume we are given a bounded domain containing a collection of disjoint “prohibited” areas, i.e. subdomains, where any activity (including passing through them) is banned. Suppose, the number of consumers is large enough, so that we can regard them as continuously distributed over the domain.

It is required to locate a given number of logistic centers so that, at first, they can serve the maximum possible proportion of the domain, at second, their service areas do not overlap, and, finally, the maximum time of delivery to the mostly distant consumer coincides for all logistics centers.

A mathematical model of the logistic problem is as follows.

Assume we are given a metric space X, a bounded domain $D \subset X$, compact sets $B_k \subset D, k = 1, ..., m$ (prohibited domains), and n of logistic centers $S_n = \{s_i\}$ with coordinates $s_i = (x_i, y_i)$, $i = 1, ..., n$. Let $0 \leq f(x, y) \leq \beta$ be a continuous function, which makes sense of the instantaneous speed of movement at every point of D. Note that $f(x, y) = 0 \Leftrightarrow (x, y) \in B_k$, $k = 1, ..., m$. Then, instead of D, we can consider closed multiply-connected set P:

$$P = \mathrm{cl}\left(D \setminus \bigcup_{k=1}^{m} B_k\right) \subset X \subseteq \mathbb{R}^2 . \tag{1}$$

Here cl is the closure operator.

The distance in space X is determined as follows:

$$\rho(a, b) = \min_{\Gamma \in G(a,b)} \int\limits_{\Gamma} \frac{d\Gamma}{f(x, y)} , \tag{2}$$

where $G(a, b)$ is the set of all continuous curves, which belong to X and connect the points a and b. In other words, the shortest route between two points is a curve, that requires to spend the least time.

We are to find a location $S_n^* = \{s_i^*\}$, which brings maximum to the expression

$$R^* = \min_{i=\overline{1,n}} \min \left\{ \frac{\rho\left(s_i, (S_n \backslash \{s_i\})\right)}{2}, \rho(s_i, \partial P) \right\} . \tag{3}$$

Here, ∂P is the boundary of the set P and $\rho(s_i, \partial P)$ is the distance from a point to a closed set,

$$\rho(s_i, \partial P) = \min_{x \in \partial P} \rho(s_i, x) \ . \tag{4}$$

One can reformulate (3) as follows: maximize the radius of equal circles, which can be located so that they overlap each other and the boundary of the set P only at their boundary points. In other words, a solution to the logistic problem is equivalent to an optimal packing circles of equal radius in a multiply-connected set with metric (2).

Along with (3), we consider another problem: place a predetermined number of logistic centers so that, at first, all consumers are serviced, secondly, the maximum time of delivery to the mostly distant consumer is the same for all logistic centers, and the time of delivery is minimal.

Let $M \subset X$ be a given bounded set with continuous boundary, m be an amount of logistic centers $P_m = \{O_k\}$, and $O_k = (x_k, y_k)$, $k = 1, ..., m$, be their coordinates.

Our second goal is to find a partition of M on m segments M_k, $k = 1, ..., m$, and the location of the centers $P_m^* = \{O_k^*\}$, which provide minimum for

$$R_* = \max_{k=\overline{1,m}} \rho\left(O_k, \partial M_k\right) \ . \tag{5}$$

The formulated logistic problem is equivalent to optimal covering of the set M with the metric (2) by circles of equal radia.

3 Numerical methods

In this section, the authors propose methods for solving problems (2),(3) and (2),(5), based on the analogy between the propagation of the light wave and finding the minimum of the functional integral (2). This analogy is a consequence of physical laws of Fermat and Huygens. The first principle says that the light in its movement chooses the route that requires to spend a minimum of time. The second one states that each point reached by the light wave, becomes a secondary light source. This approach is described more detail in [19, 20, 22, 23].

The essence of the algorithm is as follows. We consistently divide the given set (P or M) into segments with respect to the randomly generated initial set of circles centers based on Voronoi diagrams; then find the best center covering or packaging circle for each segment; finally construct segmentation for the found centers.

An algorithm for circles covering constructing

1. Randomly generate an initial coordinates of the circles centers O_k, $k = \overline{1,m}$. Coordinate coincidences are not allowed.
2. From $O_k, k = \overline{1,m}$ we initiate the light waves using the algorithm [19]. It allows us to divide set M on m segments M_k and to find their boundaries ∂M_k, $k = \overline{1,m}$.

3. Boundary ∂M_k of segment M_k is approximated by the closed polygonal line with nodes at the points $A_i, i = \overline{1,q}$.
4. From $A_i, i = \overline{1,q}$ we initiate the light waves using the algorithm [19] as well.
5. Every point $(x, y) \in M_k$, first reached by one of the light waves is marked (here and further we assume using an analytical grid for x and y). We memorize time $T(x, y)$ which is required to reach (x, y).
6. Find $\bar{O}_k = \arg\max\limits_{(x,y)\in M_k} T(x, y)$. Then, the minimum radius of circle which covers M_k, is given by

$$R_{k\,\min} = \max_{i=\overline{1,q}} \rho(\bar{O}_k, A_i)\ .$$

Steps 3–6 are carried out independently for each segment $M_k, k = \overline{1,m}$.

7. Find $R_{\min} = \max\limits_{k=1,..m} R_{k\,\min}$. Then go to step 2 with $O_k = \bar{O}_k$, $k = \overline{1,m}$.

Steps 2–7 are being carried out while $R_{\min}$ is decreasing, then the current covering

$$P_m = \bigcup_{k=1}^{m} C_k(\bar{O}_k, R_{\min})$$

is memorized as a solution.

8. The counter of an initial coordinates generations $Iter$ is incremented. If $Iter$ becomes equal some preassigned value, then the algorithm is terminated. Otherwise, go to step 1.

An algorithm for circles packing constructing

1. Randomly generate an initial coordinates of the circles centers s_i, $s_i \in P$, $i = \overline{1,n}$. Radius R is assumed to be zero.
2. Domain P is divided on segments P_i, $i = 1, ..., n$, as well as in the algorithm above.
3. We initiate the light waves propagating from the boundary of ∂P_i of every segment P_i in the inner area and construct the wave fronts until until they converge at a point. Denote this point by $\bar{s}_i$ and calculate $r_i = \rho(\bar{s}_i, \partial P_i)$ by (4), $i = \overline{1,n}$.
4. Calculate $R = \min\limits_{i=1,...,n} r_i$.

Steps 2-4 are being carried out until R is increasing, then the current vector $\bar{S}_n = \{\bar{s}_i\}$ is memorized as a solution.

5. The counter of an initial coordinates generations $Iter$ is incremented. If $Iter$ becomes equal some preassigned value, then the algorithm is terminated. Otherwise, go to step 1.

4 Computational experiment

Example 1. This example illustrates algorithm for circles covering constructing in the case of the Euclidean metric $f(x, y) \equiv 1$. We solve the equal circle covering problem in unit square. The number of circles is given and we maximize the radius. The results are presented in table 1. Here $R_{\min}$ is the best radius of covering obtained by the presented algorithm for circles packing constructing, $\Delta R = R_{Known} - R_{\min}$, t is time of calculation, $Iter = 25$.

Note, that the R_{Known} results were obtained from [31].

Table 1. Comparison of the results of covering equal of circles in the unit square

n	R_{Known}	$R_{\min}$	ΔR	t(sec)
10	0.218233512793	0.218233693441	0.000000180648	43.477
15	0.179661759933	0.180281054179	0.000619294246	47.611
20	0.152246811233	0.152426892598	0.000180081365	51.683
25	0.133548706561	0.134470667521	0.000921960960	54.835
30	0.122036868819	0.123001449585	0.000964580766	58.984
40		0.108376286825		67.221
50		0.095904051463		73.508
75		0.078877824148		88.141
100		0.068332659403		128.342
150		0.05554107666		144.831
500		0.03082452774		408.707

Blank lines in table 1 means no known results for the corresponding n. It is easily seen that in comparison with known results, the results obtained by the authors, a little bit worse, but the deviation of circles radius does not exceed 0.1%. The total time for solving the problem is relatively small even for $n = 500$. It can be concluded that the proposed algorithm, despite the fact that it is, strictly speaking, not directly suitable for the covering problem in the Euclidean metric, shows reasonably good results here.

Example 2. This example shows a comparison of the results of the authors with the results from [24] and [31] for the packing of equal of circles in the unit square in the Euclidean metric $f(x, y) \equiv 1$ (table 2).

It is easy to see that, as in example 1, the results obtained by the authors, is slightly worse, but the deviation of radius of packed circles from the optimal is low. Furthermore, when $n = 1,500$ we found a solution which improves the known one. At the same time, the total time for solving the problem is relatively small even for $n = 3000$ (for example, compared with the FSS-algorithm [24]).

Example 3. Let now $M = \{(x, y) \colon (x-6)^2 + (y-6)^2 \leq 4^2\}$ and

$$f(x, y) = \frac{(x-4.5)^2 + (y-6)^2}{(x-4.5)^2 + (y-6)^2) + 1} + 0.5.$$

It is required to find the covering P_n^* with minimal radius R and $n = 8$.

The resulting approximation of coordinate of covering circles centers is following

$$S_8 \approx \{(3.610, 4.375), (3.725, 7.750), (5.603, 9.748), (6.0, 8.745)\ ,$$

Table 2. Comparison of the results of packing equal of circles in the unit square

n	Packomania	FSS-Algorithm			Proposed Algorithm		
	R_{Known}	$R_{\max}$	ΔR	t	$R_{\max}$	ΔR	t
50	0,071377104	0,071376623	0,000000481	276	0,070578606	0,000798498	88
75	0,058494535	0,058091304	0,000403232	621	0,057954653	0,000539883	217
100	0,051401072	0,051272763	0,000128308	1317	0,050269024	0,001132048	376
150	0,042145465	0,041976579	0,000168887	4107	0,041309389	0,000836076	895
200	0,036612799	0,025722283	0,010890516	8790	0,035969127	0,000643672	1457
250	0,032876318	0,030028915	0,002847403	18111	0,032102759	0,000773559	2246
300	0,030219556				0,029447787	0,000771768	3254
500	0,023455498	0,000974943	0,022480556	133443	0,022846434	0,000609065	8725
1500	0,013157896				0,013163195	-0,000005299	31713
3000	0,009674511				0,009243172	0,000431339	217489

$$(6.115, 7.125), (6.918, 3.375), (7.628, 8.875), (9.156, 6.0)\} .$$

Radius $R_{min} \approx 1.8134$. Set M (bold line), covering P_8^* (thin line) and coordinates of circles centers S_8 (dots) are shown at fig. 1.

Example 4. Here set M and function $f(x, y)$ are the same as in the example 3. It is required to find the packing U_n^* with maximum radius R and $n = 8$. The resulting approximation of circles centers coordinate is following

$$O_8 \approx \{(3.7997, 5.852), (6.547, 6.4971), (4.802, 6.2009), (8.5778, 5.132),$$

$$(3.7201, 7.6607), (6.7970, 3.3827), (4.9825, 4.8146), (5.7846, 8.7356)\}.$$

Radius $R_{max} \approx 0.8787$. Set M (bold line), packing U_8^* (thin line) and coordinates of circles centers O_8 (dots) are shown at fig. 2.

5 Conclusions

We raised two classical problems of continuous optimization: optimal covering and optimal packing with equal 2-D spheres (circles, discs) for a bounded subset of a metric space. Practically, the addressed issues appear in logistics (so-called "facility location problems"), communication, security, energy management etc. The developed numerical algorithms, based on fundamental physical principles by Fermat and Huygens (an "optical-geometrical approach"), prove themselves rather efficient for covering and packing problems with non-convex sets, even if the metric is non-Euclidean: numerical simulation confirms that the designed approach is relevant. At the same time, in the case of Euclidean metric and a sufficiently large number of covering circles, our algorithms are shown to be competitive, compared to known approaches. The latter observation was pleasantly unexpected.

The obtained results could be further extended to multiply covering problems. This would be a subject of our future study.

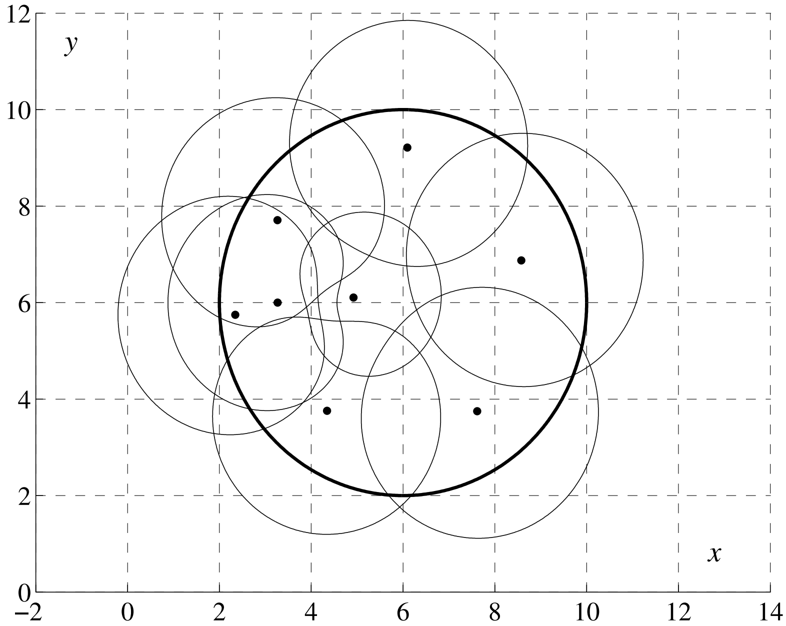

Fig. 1. Covering by 8 circles

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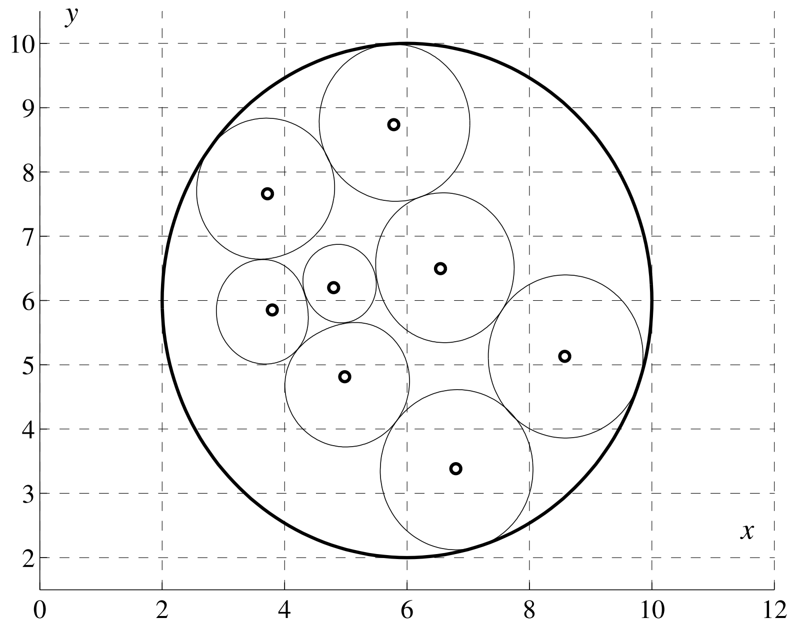

Fig. 2. Packing of 8 circles

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