Mathematical Model of Microeconomic System with Different Social Responsibilities in Software Module

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Abstract. *Research goals and objectives*: to study of the simplest possible mathematical model of microeconomic system with different social responsibilities of agents in accordance with agent based computational economics paradigm using a desktop application.
Object of research: microeconomics system with heterogeneous agents.
Subject of research: mathematical model of microeconomic system with different social responsibilities, equilibrium and disequilibrium states of the systems using desktop application.
Research methods are: optimization methods, bifurcation analysis, stability analysis, simulation methods, game theoretic approach.
Results of the research: dynamic models of microeconomic system with different social responsibilities (reciprocator and selfish types) were created using specially developed desktop application. Based on software module the conditions of stability, bifurcation and analysis were obtained. As a result of numerical investigation we have found that flip bifurcations occur with increasing of firms' number in the market. If two-thirds of firms use naive expectation, then there appears the state of dynamic chaos.

Keywords: microeconomic system, reciprocity, stability, bifurcation.

Key Terms: DynamicSystem, DesktopApplication, NashEquilibrium, Expectations

1 Introduction

Information technology in the economy made it possible to model artificial societies and study economic models through the computer simulation. This new school in science is called agent based computational economics (ACE) and creates absolutely new possibilities in economic research of microeconomic systems [1]. Now institutional school of economics analyzes microeconomic systems as a result of evolutionary process of participants' interaction.

Evolution appeared due to variation and selection process [2]. In evolutionary microeconomic systems a variation is described by individual learning. Individual learning and adaptation lead to evolutionary stable, self-organized social and economic activities. The evolutionary approach allows us to develop an economic mechanism that could explain why the economic system is sometimes stable, and in other cases - not [3]. The evolutionary process is analogous to social learning. Examples of evolutionary process application are the new pricing mechanisms in auctions and social networks under electronic commerce via the Internet.

Microeconomics has entered the stage of deep transformation of its bases. In recent years, researchers have abandoned the traditional main assumption - the perfect rationality as the basis of unconditional behavior of the economy. Neoclassical "rational man" does not exist in reality, as individuals act according to established rules, do not have full information and do not always maximize benefits [4].

Unlike traditional simultaneous, instantaneous achieving equilibrium by perfectly rational firm in the real economy, "best imperfect decisions" taken by the simple and non-consumable calculations, are well adapted to frequent repetitions in the evolution process. If the system has multiple equilibria, the repetitive interactions, evolutionary dynamics of selection mechanism is a better equilibrium [5]. It means that the process of the real economy is interactive and dynamic.

New paradigm of microeconomics is a combination of the dynamical systems' nonlinear theory and mathematical programming, including game theory and optimal control theory [2, 5]. Simulation modeling and evolutionary approach are the main tools of new microeconomics. Simulation models are grounded on the basis of 3 computer paradigms (object-oriented, dynamic and multi-agent system) that are used to predict the development of economic systems [6].

Example of this new paradigm is an evolutionary model of oligopoly competition where agents can select between different behavioral rules to make decisions on quantity or price settings [7]. In some cases only one behavioral rule survives among other ones and model can explain why the system can be in a state of evolutionary stable strategy [8]. Traditional static models of competition (e.g., Cournot, Bertrand and Stackelberg) were converted in dynamics models which were investigated on existence, stability and local bifurcations of the equilibrium points. Numerical simulations demonstrate that the system with varying model parameters may drive to chaos and the loss of stability may be caused by period doubling bifurcations [9]. One of main task for such models is to keep the system from instability and chaos using feedback parameters. Through local analysis we provide conditions for the stability of the market equilibrium and through global analysis we investigate some bifurcations which cause qualitative changes in the market structure [10].

The traditional method of constructing a scientific theory is first to synthesize and investigate the example of the simplest possible mathematical model of microeconomic system.

These new approaches make a clear explanation for some events in economics rather than traditional mainstream. The evolutionary approach and analysis of the dynamics allow to explain why one type of firm ousts another from the market, why sometimes the economic system is stable, but in other cases is unstable [2, 3]. If the system has multiple equilibria, the dynamics and evolution is the selection mechanism of best equilibrium according to certain criteria [5]. The evolutionary process is analogous to social learning. An example of its application is the pricing mechanisms for auctions that occur in agents' social networks, e-commerce and trade through the Internet [8, 11]. Karl Polanyi identified in reality the alternative economic organization where social norms are not generated by economic self-interest of the individual. This network of reciprocal relations is based on mutual economic cooperation, dominated by cultural norms rather than market laws [12]. Reciprocity implies that the firms are ready to sacrifice some of their own profits for the benefit of consumers without direct compensation for it by the state. Such targets can be stipulated by the firms' desire to get stable profits in the long run rather than maximal short-run profits [13]. Such forward-thinking firms-reciprocators are considered in the model of this paper. Their objective function is a weighted average of the profits and consumer surplus of their market segment.

Microeconomic system consists of two types of agents with heterogeneous responsibilities, such as selfish and reciprocator firms. Firms' social responsibility implies that they have not only selfish goal of increasing their own profits immediately, but are also willing to sacrifice a part of their short-run profits and to save consumer surplus in return for stable nonmaximal long-run profit. In other words reason of reciprocator firms' appearance is their desire to obtain stable profit in long-run period instead of short-run maximal profit.

The **purpose** of the paper is a study of the dynamic microeconomic system through synthesis of the simplest possible mathematical model according to agent based computational economics paradigm using our specially developed software module.

This paper is a direct continuation of research [14], where our model was introduced. Our next task is a numerical investigation of the model using software module developed by us.

The paper is organized as follows: part 2 describes the simplest mathematical model of microeconomic system according to new paradigm; part 3 demonstrates desktop C#-application for numerical experiments; part 4 includes numerical investigations of microeconomic system using this application; part 5 concludes.

2 The Simplest Model of Dynamical Microeconomic System

First of all we show that our mathematical model is the simplest one in the ACE paradigm. In general, almost any microeconomic market model is constructed as follows: 1) n firms operate in the market (to simplify the notation suppose $n=2$); 2) these firms produce homogeneous products in quantities $x_1(t)$ and $x_2(t)$ in time period t; 3) they use adaptive approach, i.e. they try to predict the quantity of their competitor in the next time period $t+1$ where $x_j^e(t+1)$ be expected quantity of rival j by a firm i in period t. Then under planning of its quantity $x_i(t+1)$ in the next period the firms decide the following optimization problem:

$$Max\Pi_1(x_1(t+1); x_2^e(t+1)),\ Max\Pi_2(x_1^e(t+1); x_2(t+1)),$$

where Π_i, $i=1,2$ is a profit function of firm *i*. The assumption about unchangeable quantity of the competitor (i.e. firm *i* will use $x_j(t)$ instead of $x_j^e(t+1)$ when it solves the optimization problem) is an example of imperfect, bounded rationality in firm's strategies; it is called naive expectations. As a rule these two approaches (adaptive and naive) coexist in the market with a certain probability. Our model is based on this assumption.

We consider a market of homogeneous product, where n firms operate, among them are k identical reciprocator firms with the same output x and $n-k$ identical selfish firms with the same output y. Thus the industry output of the two types of firms is $Q = k \cdot x + (n-k) \cdot y$. Product price P is given by the inverse market demand function $P = P(Q) = \dfrac{b}{Q}$ ($b > 0$). This is simplest demand function leads to a non-linear dynamics. Alternative demand function is linear $P = P(Q) = b - c \cdot Q$ ($b, c > 0$, wherefrom $Q = k \cdot x + (n-k) \cdot y \le \dfrac{b}{c}$) is used to test the general model's properties. This model is uniquely defined by objective functions of firms and types of their expectations. It does not use any additional assumptions or restrictions.

The objective function of selfish firm-egoist is a profit $\pi_Y = (P - v) \cdot y$, where v is the firm's cost per unit in the market. Reciprocator firm maximizes both its own profit $\pi_X = (P - v) \cdot x$ and consumer surplus CS of its market segment (loyal consumers) $CS = \dfrac{\gamma}{k}\left(\int_{\varepsilon}^{Q} P(q)dq - P \cdot Q \right)$, where parameter γ specifies the segment of the market, which the reciprocator firmr believes its own and optimizes it ($0 < \gamma \le k$); ε is the minimal technologically possible product quantity. Then $CS = \dfrac{\gamma}{k}\left(b \cdot \ln(\dfrac{Q}{\varepsilon}) - \dfrac{b}{Q} \cdot Q \right) = \dfrac{b\gamma}{k} \cdot \left(\ln(\dfrac{Q}{\varepsilon}) - 1 \right) = \dfrac{b\gamma}{k} \cdot \ln \dfrac{Q}{\hat{\varepsilon}}$, where $\hat{\varepsilon} = \varepsilon \cdot e$ (specific choice of ε does not affect the model dynamics). Using CS (difference between price which consumer can pay and real price) profit function of reciprocator firm is:

$$\Pi_X = \alpha \cdot (P - v) \cdot x + (1-\alpha) \cdot CS = \alpha \cdot (P - v) \cdot x + (1-\alpha) \cdot \frac{b\gamma}{k} \cdot \ln \frac{Q}{\hat{\varepsilon}},$$

where α is share of own profit π_X in the objective function. In other words Π_X is a weighted average short-run profit π_X and expected factor of stable long-run profit CS from loyal consumers.

Now let us consider the dynamics of this model with discrete time $t = 0, 1, \dots$. Let $x_i(t)$, $y_j(t)$ be the outputs at time t of reciprocator ($i = 1, \dots, k$) and selfish firms ($j = 1, \dots, n-k$), respectively. On the basis of these values at time t each firm finds the optimal value for its own quantity setting in the next moment $t+1$, maximizing its objective function.

Quantity setting strategy of firms with naive expectations.

Each reciprocator firm i ($i = 1, \dots, k$) is looking for such value of $x_i(t+1)$ at which it maximizes its own profit function, suggesting that all other firms leave their quantities $x_{-i}(t)$, $y_j(t)$ unchanged: $x_s^e(t+1) = x_s(t)$, $y_j^e(t+1) = y_j(t)$:

$$Max\Pi_i(x_1^e(t+1), \dots, x_{i-1}^e(t+1), x_i(t+1), x_{i+1}^e(t+1), \dots, x_k^e(t+1); y_1^e(t+1), \dots, y_{n-k}^e(t+1)) =$$
$$Max\Pi_i(x_1(t), \dots, x_{i-1}(t), x_i(t+1), x_{i+1}(t), \dots, x_k(t); y_1(t), \dots, y_{n-k}(t))$$

Similarly each selfish firm j ($j=1,...,n-k$) is looking correspondingly for such value of $y_j(t+1)$ at which it maximizes its profit π_j, suggesting that all other firms leave their quantities $x_i(t)$, $y_{-j}(t)$ unchanged:

$$Max\pi_j(x_1^e(t+1),...,x_k^e(t+1);y_1^e(t+1),...,y_{j-1}^e(t+1),y_j(t+1),y_{j+1}^e(t+1)...,y_{n-k}^e(t+1)) =$$
$$Max\pi_j(x_1(t),...,x_k(t);y_1(t),...,y_{j-1}(t),y_j(t+1),y_{j+1}(t),...,y_{n-k}(t)).$$

Hence, in view of [14], we obtain a dynamic system model of firms' reaction functions:

$$\begin{cases} x_i(t+1)=\sqrt{\frac{b}{v}((k-1)x_i(t)+(n-k)y_j(t))+(\frac{1}{2}\frac{1-\alpha}{\alpha}\frac{b}{vk})^2}-((k-1)x_i(t)+(n-k)y_j(t))+\frac{1}{2}\frac{1-\alpha}{\alpha}\frac{b}{vk}, \\ y_j(t+1)=\sqrt{\frac{b}{v}(kx_i(t)+(n-k-1)y_j(t)}-(kx_i(t)+(n-k-1)y_j(t)), \\ x_1(t)=...=x_k(t), i=1,...,k, \\ y_1(t)=...=y_{n-k}(t), j=1,...,n-k. \end{cases} \quad (1)$$

The last equations of this system mean that k reciprocator firms and $n-k$ selfish firms are identical for all t.

Quantity setting strategy of firms with adaptive expectations

Since all selfish firms and all reciprocator firms are assumed as identical and they have the same strategies at moment t so it is natural to suggest that their production quantities will be equal at next moment $t+1$ too. In accordance with such expectations each reciprocator firm under quantity setting assumes that $x_i(t+1)=x_1^e(t+1)=...=x_k^e(t+1)$. Therefore, to determine its quantity in the next period this firm solves the following optimization problem:

$$Max\Pi_i(x_1^e(t+1),...,x_{i-1}^e(t+1),x_i(t+1),x_{i+1}^e(t+1),...,x_k^e(t+1);y_1^e(t+1),...,y_{n-k}^e(t+1)) =$$
$$Max\Pi_i(x_i(t+1),...,x_i(t+1),x_i(t+1),x_i(t+1),...,x_k(t+1);y_1(t),...,y_{n-k}(t)).$$

Similarly each selfish firm j in accordance with common sense believes under quantity setting that $y_j(t+1)=y_1^e(t+1),...,y_{n-k}^e(t+1)$. So this firm solves the following optimization task:

$$Max\pi_j(x_1^e(t+1),...,x_k^e(t+1);y_1^e(t+1),...,y_{j-1}^e(t+1),y_j(t+1),y_{j+1}^e(t+1),...,y_{n-k}^e(t+1)) =$$
$$Max\pi_j(x_1(t),...,x_k(t);y_1(t+1),...,y_j(t+1),y_j(t+1),y_j(t+1)...,y_j(t+1)).$$

It leads to such dynamic system [14] of firms' reaction functions:

$$\begin{cases} kx_i(t+1)=\sqrt{\frac{b}{v}(n-k)y_j(t)+(\frac{1}{2}\frac{1-\alpha}{\alpha}\frac{b\gamma}{v})^2}-(n-k)y_j(t)+\frac{1}{2}\frac{1-\alpha}{\alpha}\frac{b\gamma}{v}, \\ (n-k)y_j(t+1)=\sqrt{\frac{b}{v}kx_i(t)}-kx_i(t), \\ x_1(t)=...=x_k(t), i=1,...,k, \\ y_1(t)=...=y_{n-k}(t), j=1,...,n-k. \end{cases} \quad (2)$$

As above, the last equations of this system means the identity of all reciprocator and selfish firms.

Quantity setting strategy in general case

In real life both decision making approaches (adaptive and naive) coexist in the market with a certain probability p for adaptive and correspondingly $q = 1 - p$ for naïve expectations. According to such expectations typical (representative) reciprocator firm i suggests that production quantities of its rival j will be equal to $x_j^e(t+1) = p \cdot x_j(t) + q \cdot x_i(t+1)$ ($j = 1,...,k$, $j \neq i$). Typical reciprocator firm i ($i = 1,...,k$) resolves following optimization problem:

$$Max\Pi_i(x_1^e(t+1),...,x_i(t+1),...,x_k^e(t+1); y_1^e(t+1),...,y_{n-k}^e(t+1)) =$$
$$Max\Pi_i(px_1(t) + qx_i(t),...,x_i(t+1),...,px_k(t) + qx_i(t+1); y_1(t),...,y_{n-k}(t)).$$

Similarly typical selfish firm j ($j = 1,...,n-k$) solves following optimization problem:

$$Max\pi_j(x_1^e(t+1),...,x_k^e(t+1); y_1^e(t+1),...,y_j(t+1),...,y_{n-k}^e(t+1)) =$$
$$Max\pi_j(x_1(t),...,x_k(t); py_1(t) + qy_j(t+1),...,y_j(t+1),...,py_{n-k}(t) + qy_j(t+1)).$$

This hybrid case leads to following dynamics [14]:

$$\begin{cases} (1 + p(k-1))x_i(t+1) = \sqrt{\frac{b}{v}w_x + d^2} - w_x + d, \\ (1 + p(n-k-1))y_j(t+1) = \sqrt{\frac{b}{v}w_y} - w_y, \\ x_1(t) = ... = x_k(t), i = 1,...,k, \\ y_1(t) = ... = y_{n-k}(t), j = 1,...,n-k, \end{cases} \quad (3)$$

where $d = \frac{1}{2}\frac{1-\alpha}{\alpha}\frac{b(1 + p(k-1))}{vk}$ and $\begin{cases} w_x = q(k-1)x_i(t) + (n-k)y_j(t), \\ w_y = kx_i(t) + q(n-k-1)y_j(t). \end{cases}$

3 Desktop C#-application Model for Numerical Investigation

For our research we developed desktop application *Model* for numerical experiments with dynamical systems on two-dimensional phase space. The main purpose of the application is to provide the best service for research cycle: hypothesis $\rightarrow$ computing experiment $\rightarrow$ hypothesis. For a given differential equations system and parameters set of the model the application immediately generates a window of this model. It makes it easy to specify and modify the considered model. Window tools allow us to obtain trajectories, phase curves, bifurcation diagrams and its animation after setting of the initial parameters.

Note that C#-application is created on GUI-based C # System.DrawingiSystem.Windows. All calculations concerned with a model, are localized in method Calc of application Model which allows us to modify easily the equations of the model, or switch to other models. To work with continuous models, i.e. with differential equations, the program uses OpenMaple interface access to the Maple computational kernel from various programming languages, such as C#, Java, VisialBasic etc. The program also uses namespace System.Runtime.InteropServices, allowing to make reference to the dynamic assembling of Maple kernel - maplec.dll. The application window for the model of this paper is shown in the following fig.1.

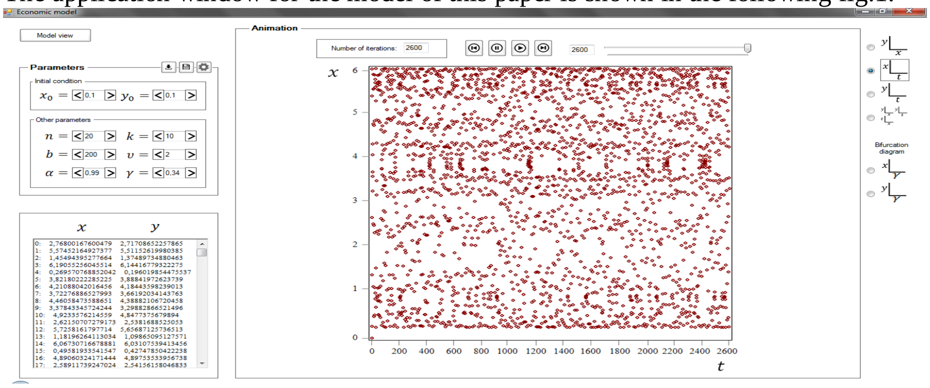

Fig. 1. Software application *Model* for microeconomic System

The right side presents 6 kinds of graphs displayed by the application; their examples are set forth in the paper. Selected switch indicates that here the graph of trajectory $x(t)$ is selected. On the left side counters allow us to specify the parameters of the model and the initial values of the trajectory. After their setting automatically and immediately appear iterations of calculating the coordinates of a trajectory below the counters and their image in the center of the chart window, at that the number of iterations can be set on the scroll bar over graph. Software Module displays an animation of a selected path after pressing the button near with scroll bar. After pressing button *Modelview* we can see on the left and above information about the model, its equations and parameters. The model data and stored and we can go immediately to its window using the name of the model.

4 Investigation of Microeconomic System Using Desktop Application

4.1 From Stability to Chaos with Increasing of Firms' Number in the Market

One of the main assumptions of orthodox neoclassical microeconomics is the idea of automatic stabilization of a market as a result of increasing of independent firms' number under quantity competition. This is the realization of Adam Smith's 'invisible hand'. Let consider the behavior of our model with the growth of parameter n (total number of firms). Let $n = 34$; the number of reciprocator firms $k = 32$; $b = 200$; marginal cost $v = 2$; the share of profit in the objective function of reciprocator is $\alpha = 0.9$; probability of naive expectations is $q = 0.65$. The trajectory of dynamical system (3) with such initial parameters and initial output point $x_0 = 0.1$, $y_0 = 0.1$ is shown in the following fig.2.

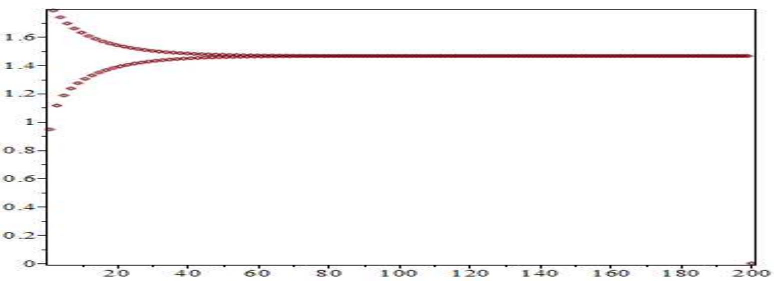

Fig. 2. Quantity trajectory of firm-reciprocators under initial conditions ($n = 34$, $k = 32$, $b = 200$, $v = 2$, $\alpha = 0.9$, $q = 0.605$, $x_0 = 0.1$, $y_0 = 0.1$)

Here along horizontal axis are given iteration of system (3) from $t = 1$ to $t = 200$, along ordinate axis are given corresponding quantities of reciprocator firm $x_i(t)$, $i = 1,...,k$. As you can see from the graph, the path quickly converges to the equilibrium quantity $x^* = x_i^* \approx 1.5$. The graph for quantity path of selfish firms $y_j(t)$, $i = 1,..., n-k$ looks like this one ($y* \approx 0.5$) under same conditions.

Let consider the graph of the trajectory for the same parameters except n . Now let $n = 36$ (fig. 3). In fig. 3 instead of equilibrium point there appeared bifurcation and a stable cycle where $x(t)$ approximates to point $x* \approx 1.9$ for even t and to point $x* \approx 0.7$ for odd t . After doubling the lag between iterations is either even or odd iterations. Thus either quantity $x* \approx 1.9$ or $x* \approx 0.7$ respectively will be equilibrium output.

Stable cycle has four points for $n = 44$ (fig. 4). There was a flip bifurcation.

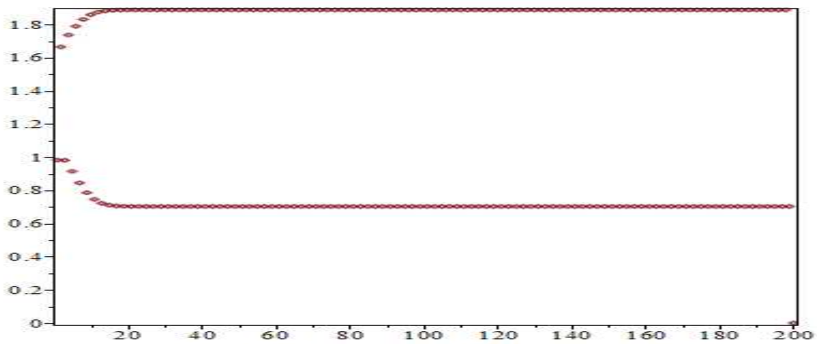

Fig. 3. Quantity trajectory of firm-reciprocators under initial conditions ($n = 36$)

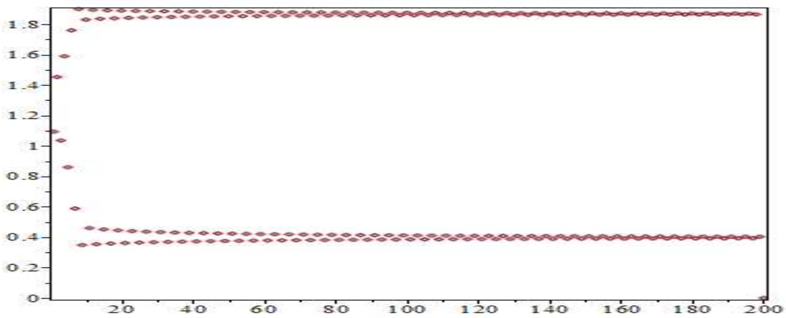

Fig. 4. Quantity trajectory of firm-reciprocators under initial conditions ($n = 43$)

The more firms' number the more series of doubling bifurcation cycle according to Shvarkovskii's scale. State of dynamic chaos already exists for $n = 100$ (fig. 5).

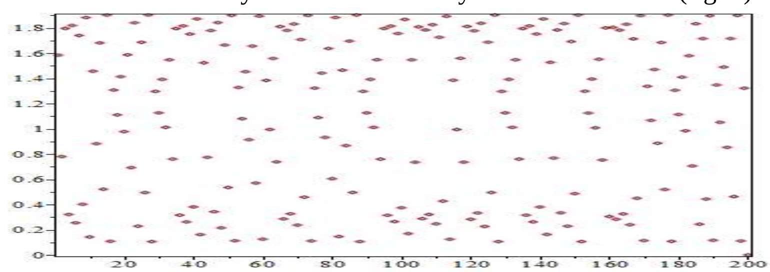

Fig. 5. The state of dynamic chaos for reciprocator firms' output ($n = 100$)

To understand chaos effect which contradicts to orthodox microeconomics during growth in the number of firms, let us consider how the number of reciprocator firms impacts on model dynamic for fixed n. Let $k = 1$, $n = 100$, all the other parameters are the same as above. The following figure shows graph $x(t)$ of corresponding trajectory (fig.6). This trajectory converges to Nash equilibrium.

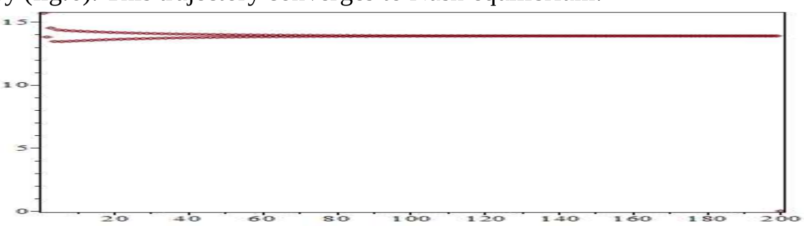

Fig. 6. Reciprocator firm's quantity trajectory for fixed n ($k = 1$)

Now let $k = 3$. There exists a flip bifurcation (fig.7).

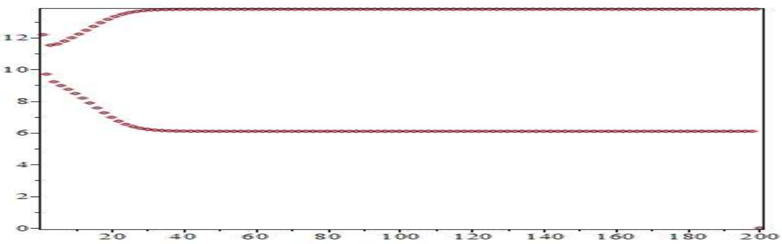

Fig. 7. Reciprocator firm's quantity trajectory for fixed n ($k = 3$)

Assume further that $k = 10$. We have a new flip bifurcation (fig.8).

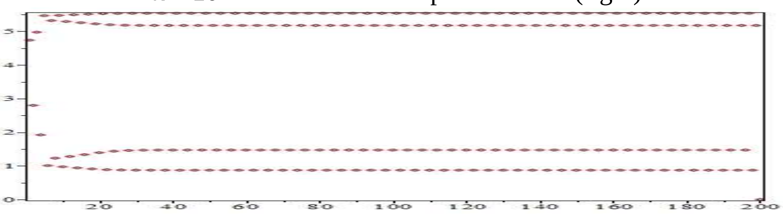

Fig. 8. Quantity trajectory of firm-reciprocators for fixed n ($k = 10$)

If $k = 32$ or $k = 100 - 32 = 68$ we will get the same chaos as in fig. 5. For $k = 100 - 10 = 90$, $k = 100 - 3 = 97$, $k = 100 - 1 = 99$ we obtain the same dynamics as in fig. 8, fig. 7 and fig. 6 correspondingly. Thus if different types of firms are uniformly presented in the market, quantity dynamics can be complex and transform to chaos after increasing firms' number. The destabilizing role of agents' number due to evolution is well-known for oligopoly games [15].

The reason of instability market share with increasing firms' number n is revealed in the following proposition 1. According to [14] there is a unique Nash equilibrium output in dynamic system (1):

$$\begin{cases} x^* = \dfrac{b}{vn}\left(1 - \dfrac{2\alpha - 1}{\alpha n}\right)\left(1 + \dfrac{1-\alpha}{\alpha}\dfrac{n-k}{k}\right), \\ y^* = \dfrac{b}{vn}\dfrac{2\alpha - 1}{\alpha}\left(1 - \dfrac{2\alpha - 1}{\alpha n}\right). \end{cases} \tag{4}$$

Here $x^* = x_1^* = \ldots = x_k^*$, $y^* = y_1^* = \ldots = y_{n-k}^*$. Then

Proposition 1. For any given b, $v > 0$ and α ($0 \le \alpha \le 1$) Jacobian $\boldsymbol{J}$ of system (1) for Nash equilibrium (4) is proportional to value n–1 for sufficiently large n. Its absolute value increases with growth of n, if $\left|\dfrac{k}{n}\right| > \varepsilon$ and $\left|\dfrac{k}{n} - \dfrac{3}{4}\right| > \varepsilon$ for any $\varepsilon > 0$.

Since the volume of the phase space under the influence of the dynamics of (1) at fixed point (4) is proportional to the absolute value of the Jacobian at this point, then proposition 1 means increased instability with the increase of n.

Proof. Here Jacobian J of system (1) at point (4) equals

$$\begin{pmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{pmatrix} = \begin{pmatrix} \frac{\partial x(t+1)}{\partial x(t)} & \frac{\partial x(t+1)}{\partial y(t)} \\ \frac{\partial y(t+1)}{\partial x(t)} & \frac{\partial y(t+1)}{\partial y(t)} \end{pmatrix}.$$

$$J_{xx} = \frac{\frac{b}{v}(k-1)}{2\sqrt{\frac{b}{v}((k-1)x_t + (n-k)y_t + d^2}} - (k-1), \qquad J_{xy} = \frac{\frac{b}{v}(n-k)}{2\sqrt{\frac{b}{v}((k-1)x_t + (n-k)y_t + d^2}} - (n-k),$$

$$J_{yx} = \frac{\frac{b}{v}k}{2\sqrt{\frac{b}{v}(kx_t + (n-k)y_t}} - k, \qquad J_{yy} = \frac{\frac{b}{v}(n-k-1)}{2\sqrt{\frac{b}{v}((k-1)x_t + (n-k-1)y_t}} - (n-k-1),$$

where $d = \frac{1}{2}\frac{1-\alpha}{\alpha}\frac{b}{vk}$. Then $\det J = J_{xx} \cdot J_{yy} - J_{xy} \cdot J_{yx} =$

$$= (1-n) \cdot (\frac{\frac{b}{v}}{2\sqrt{\frac{b}{v}((k-1)x_t + (n-k)y_t + d^2}} - 1) \cdot (\frac{\frac{b}{v}}{2\sqrt{\frac{b}{v}((k-1)x_t + (n-k-1)y_t}} - 1)$$

But for point (4) in the denominator $\frac{b}{v}((k-1)x^* + (n-k-1)y^*) =$

$$= \left(\frac{b}{v}\right)^2 \left[\frac{1-\alpha}{\alpha}(1-\frac{k}{n}) + \frac{2\alpha-1}{\alpha}(1-\frac{k}{n})\right] + o(\frac{1}{n}) = \left(\frac{b}{v}\right)^2 \cdot (1-\frac{k}{n}) + o(\frac{1}{n}),$$

where $o(\frac{1}{n}) \to 0$ for $n \to \infty$. Similarly we obtain following expression for second denominator:

$$\frac{b}{v} \cdot ((k-1) \cdot x^* + (n-k) \cdot y^*) + d^2 = \left(\frac{b}{v}\right)^2 \cdot \left(1 - \frac{k}{n}\right) + o\left(\frac{1}{n}\right).$$

But by the data $\left|\frac{k}{n} - \frac{3}{4}\right| > \varepsilon$ for $\varepsilon > 0$, which guarantees that the following expressions do not equal zero:

$$\frac{\frac{b}{v}}{2\sqrt{\frac{b}{v}((k-1)x_t + (n-k)y_t + d^2}} - 1 \neq 0 \text{ and } \frac{\frac{b}{v}}{2\sqrt{\frac{b}{v}((k-1)x_t + (n-k-1)y_t}} - 1 \neq 0 \text{ for all}$$

possible n, k, b, $v > 0$ and α ($0 \le \alpha \le 1$), Q.E.D.

4.2 The Crucial Factor Which Ensures Stable Equilibrium in the Market

How we can achieve stability of a competitive market with a large firms' number? We found that adaptive behavior is a way of achieving of market steady state [14].

Proposition 2. There is unique Nash equilibrium in a dynamic system with adaptive expectations (2) as follows

$$\begin{cases} x^* = \dfrac{b}{vk}\left(\dfrac{\alpha+(1-\alpha)\gamma}{2\alpha}\right)^2, \\ y^* = \dfrac{b}{v(n-k)}\dfrac{\alpha^2-((1-\alpha)\gamma)^2}{(2\alpha)^2}. \end{cases} \quad (5)$$

Here $x^* = x_1^* = \ldots = x_k^*$, $y^* = y_1^* = \ldots = y_{n-k}^*$. The trajectories of the system (2) converge to Nash equilibrium (5) for any acceptable initial values.

With the growth of adaptive expectations (i.e. with increase in p) stability enhances, predictability of the market becomes stronger; with the growth of naive expectations (i.e. with increasing q = 1 - q) the market loses stability, chaos increases. The process of loss of stability and transition to chaos of dynamic system (3) is the most visual in the bifurcation diagram (fig.9).

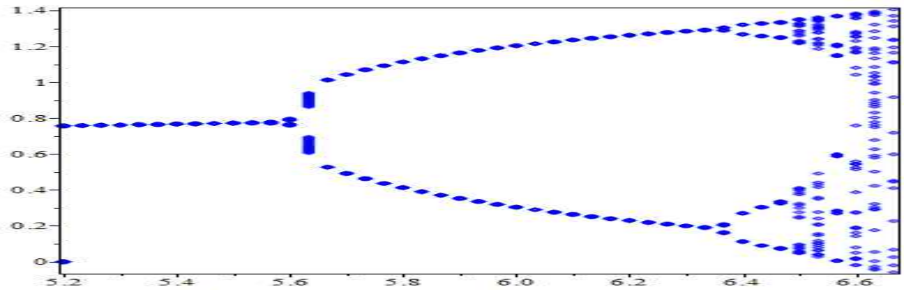

Fig. 9. The bifurcation diagram of dependence of quantity dynamics (3) on the probability of naive expectations (q)

As the above flip bifurcation can be interpreted as splitting of equilibrium state into several directions, one of which is selected by the market in the evolution of firms' strategies. If two-thirds of firms use naive expectation (q≈0.67), then there appears the state of dynamic chaos in the market. Facilities of *Model* application allow us to make sure that the above number is a universal constant which does not depend on model parameters and demand function.

4.3 Competition Between Different Types of Firms

If any type of firms increases their profit more quickly than their rivals then these firms will survive and expand their type of social responsibilities between all firms [8]. In our model, the profit ratio of reciprocator firm in time t $\pi_X(t) = (P - v)\cdot x(t)$ to profit of selfish firm $\pi_Y(t) = (P - v)\cdot y(t)$ in the same period will equal:

$$\lambda_{xy}(t) = \frac{\pi_x(t)}{\pi_y(t)} = \frac{(P(t)-v)x(t)}{(P(t)-v)y(t)} = \frac{x(t)}{y(t)}.$$

One more unexpected finding of our research during computational experiment is that in this model $\lambda_{xy}(t)$ is adiabatic invariant (constant) of a dynamical system, i.e. is almost independent for $t > 3$ for all acceptable values of parameters. As example, consider the phase curves for certain sets of parameter values used in section 4.1. The ration between the outputs of reciprocator and selfish firms remains unchanged both for steady state and dynamics chaos. For example profit ratio (phase curve) for dynamic chaos is presented in fig. 10. This phase curve corresponds to state of dynamic chaos in fig. 5 (fig. 10).

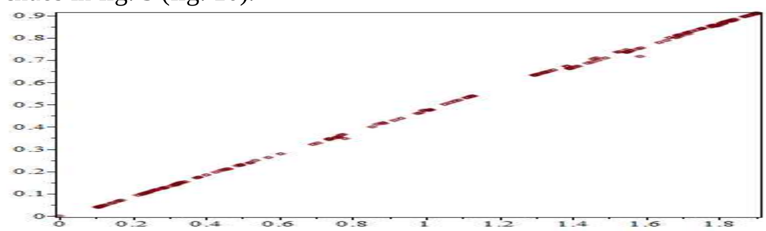

Fig. 10. Adiabatic invariant of a dynamical system for quantity ratio ($n = 100$)

The more chaotic dynamics, the more densely populated points of phase curve which coincide with line segment, whose slope is equal to $\frac{y(t)}{x(t)} = \frac{1}{\lambda_{xy}(t)}$. It means that ratio between firms' output with different responsibilities remains almost unchanged. Every conceivable examples and parameters set can be easily viewed through the application *Model* and gives the same result.

4.4 Generalization of Microeconomic System Model Properties

Finally, consider one more property of model with adaptive expectations (2).

Proposition 3. The total quantity of reciprocator firms exceeds the total quantity of selfish firms in model (2) for sufficiently large t ($t > 3$) for all values of parameters.

Proof. In accordance with proposition 2 the trajectories of system (2) converge to a Nash equilibrium (5) for any acceptable initial values. Since value $\lambda_{xy} = \frac{x(t)}{y(t)}$ is constant for $t > 3$, then it is sufficient to check proposition only at Nash equilibrium (5). But at (5):

$$k \cdot x^* - (n-k) \cdot y^* = \frac{b}{v(2\alpha)^2}\{(\alpha + (1-\alpha)\gamma)^2 - (\alpha^2 - ((1-\alpha)\gamma)^2)\} =$$

$$= \frac{b}{v(2\alpha)^2}(\alpha + (1-\alpha)\gamma)\{(\alpha + (1-\alpha)\gamma) - (\alpha - (1-\alpha)\gamma)\} =$$

$$= \frac{b}{v(2\alpha)^2}(\alpha + (1-\alpha)\gamma)2(1-\alpha)\gamma > 0, Q.E.D.$$

Does this fact is model's general property which does not depend on the choice of demand function? No. We show this through considering a similar result for a model using linear demand $P = b - c \cdot Q$ function instead of non-linear one $P = \frac{b}{Q}$.

Proposition 4. There is unique Nash equilibrium for dynamic microeconomic system which consists of selfish and reciprocity firms with adaptive expectation and linear demand function:

$$\begin{cases} x^* = \dfrac{\left[1-\alpha\cdot(n-k+3)\right]\cdot M}{k\cdot(1-\alpha)-\alpha\cdot(n+1)}, \\ y^* = \dfrac{\alpha\cdot(k-1)\cdot M}{k\cdot(1-\alpha)-\alpha\cdot(n+1)}. \end{cases} \tag{6}$$

where $M = \dfrac{b-v}{c}$, $x^* = x_1^* = \ldots = x_k^*$, $y^* = y_1^* = \ldots = y_{n-k}^*$. The trajectories of this system converge to a fixed point (6) for any acceptable initial values.

Proof. Without loss of generality we assume that $\alpha_1 = \ldots = \alpha_k = \alpha$ we simplify the system:

$$\begin{cases} x+(k-1)\cdot\dfrac{2\alpha-1}{3\alpha-1}\cdot x+(n-k)\cdot\dfrac{2\alpha-1}{3\alpha-1}\cdot y = M, \\ \dfrac{1}{2}k\cdot x+y+\dfrac{1}{2}(n-k-1)\cdot y = \dfrac{M}{2}. \end{cases} \tag{7}$$

where $x_1 = \ldots = x_k$ are quantities of reciprocity firms; $y_1 = \ldots = y_{n-k}$ - quantities of selfish firms. The solutions of system (7) are the equilibrium quantities in proposition 4, Q.E.D.

Proposition 5. Reciprocator firm (for $k \ge 2$):

(a) produces more product in the market than selfish one $x^* > y^*$ if and only if the share of his private interest is within the interval: $\alpha \in (0;\alpha_1)\cup(\alpha_2;1)$;

(b) produces less product in the market than selfish firm $x^* < y^*$ if the share of his private interest is within the interval: $\alpha \in (\alpha_1;\alpha_2)$, where $\alpha_1 = \dfrac{1}{n+2}$, $\alpha_2 = \dfrac{k}{n+k+1}$.

Proof. According to (6) inequality $x^* > y^*$ is equivalent to inequality:

$$(n+2)\cdot(n+k+1)\cdot\alpha^2 - (n\cdot(k+1)+3k+1)\cdot\alpha + k > 0.$$

The solutions of corresponding equation are $\alpha_1 = \dfrac{1}{n+2}$, $\alpha_2 = \dfrac{k}{n+k+1}$, Q.E.D.

So in compliance with proposition 3 $\lambda_{xy} = \dfrac{x(t)}{y(t)} > \dfrac{n-k}{k} = 1$ if $k = n-k$ for all α.

However according to proposition 5 $\lambda_{xy} = \dfrac{x*}{y*} < 1$ if $\alpha \in (\alpha_1;\alpha_2)$ where $\alpha_1 < \alpha_2$ for $k = n-k$, $k > 1$. Thus the result of proposition 3 is not generalized for linear demand functions.

The following fig.11 shows the graphs of dependance of Nash equilibrium point (6) coordinates' x* and y* on firms' number n according to proposition 4 at fixed parameters $k = 4$, $\alpha = 0.04$ and $M = 50$.

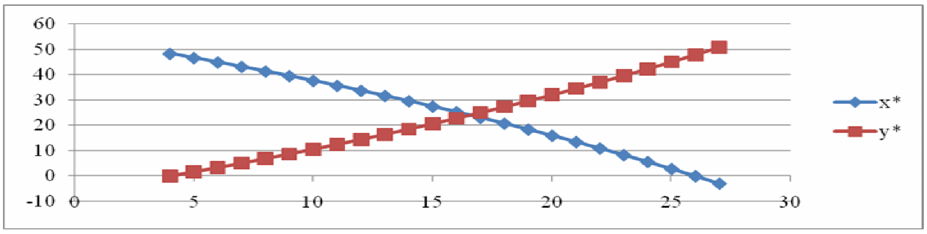

Fig. 11. Dependence of Nash equilibrium (6) on firms' number ($n = 100$)

These graphs are set at $5 \le n \le 26$. Out of this interval linear demand model is not defined, and coordinates x^* or $y^* = 0$ have invalid negative values. For $n = 5$, $y^* = 0$ there are no selfish firms in the market; for $n = 26$, $x^* = 0$ reciprocator firms have been pushed out. As we see the ratio of profit λ_{xy} can vary from zero to infinity, depending on market conditions, in particular on firms' number.

5 Conclusion

Thus we have synthesized the simplest possible mathematical model of microeconomics in accordance with agent based computational economics paradigm. This is the model of competition between reciprocator and selfish firms which plan their output using adaptive approach with probability p and naïve one with a probability $1 - p$.

Desktop C# application *Model* has been created specially for our research for the computational experiments. As a result of simulation experiment we have found that flip bifurcations occur with an increase firms' number in the market. Such bifurcations can be interpreted as separation of equilibrium state into several ways, one of which is selected by the market due to the evolution of firms' strategies. A market moves from stability to chaos with an increase in parameter n and finally has reached dynamic.

The crucial factor which ensures sustainable equilibrium in the market is the adaptive approach. In the market with only adaptive expectations there is unique Nash equilibrium which is stable for all possible values of the parameters. If no less than two-thirds of firms use naive expectations, then it will appear the state of dynamic chaos in the market. During numerical investigations we found that the ration between the outputs of reciprocator and selfish firms remains unchanged both for steady state and dynamics chaos.

The total quantity of reciprocator firms exceeds the total quantity of selfish firms for nonlinear demand function. This property is not generalized for linear demand functions. Reciprocity firms will have more market share than selfish ones if their private interest is either sufficiently high or very low. Selfish firms will have more output than reciprocity ones if their reciprocity share is average.

On this basis we plan to study complex real systems, which, in our opinion, involve the construction of a neural network which simulates the real market based on a very simple model.

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