On the Resource Equivalences in Petri Nets with Invisible Transitions*

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Abstract. Two resources (submarkings) are called similar if in any marking any one of them can be replaced by another one without affecting the net's behaviour (modulo marking bisimulation). It is known that resource similarity is undecidable for general labelled Petri nets. In this paper we study the properties of resource similarity and resource bisimulation (a subset of complete similarity relation, closed under transition firing) in Petri nets with invisible transitions (where some transitions may be unlabelled and hence invisible for external observer). It is shown that for a proper subclass — p-saturated nets — the weak transfer property of resource bisimulation can be effectively checked.

1 Introduction

In this paper the behavior of Petri nets is investigated from the standpoint of bisimulation equivalence. The fundamental notion of bisimulation was introduced by R. Milner [9] and D. Park [10]. Two markings of a Petri net are called bisimilar if the choice of each of them as an initial marking gives the same visible behavior of the net. In [7] P. Jančar proved that bisimulation equivalence of markings is undecidable for a general Petri net.

In [1] C. Autant et al. introduced a notion of place bisimulation — a decidable bisimulation-induced equivalence on the finite set of places, that allows to find out some non-trivial behaviour-preserving net reductions. This relation and its applications were studied in [1, 2, 12].

The notion of resource similarity was introduced in [3]. In general a resource is a submarking. Two resources are similar if, having replaced one resource in any marking by another, we obtain the same observed behavior of the net. Resource bisimulation is a particular case of similarity that is closed under transition firing. Place bisimulation is a proper subset of resource bisimulation. Note that, unlike the place bisimulation [1], resource similarity and bisimulation are defined on the infinite set (of resources/submarkings).

Resource similarity and its modifications where studied in [3–5]. In particular it was proven that resource similarity is undecidable. However, it was shown that resource bisimulation can be effectively approximated and used as a basis of net reductions and adaptive control. For an overview, see [8].

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In this paper we consider an important generalization of labelled Petri nets, where some transitions may be unlabelled and hence invisible for external observer. Quite often when analyzing the system there is a need to abstract from the excessive information about its behavior. For example, it is convenient to hide all transitions, corresponding to the internal actions of the system. The information obtained in this case can be useful, in particular, to detect additional properties of the system in terms of its interaction with the environment.

Place bisimulations in Petri nets with invisible transitions were studied by C. Autant et al. in [2]. It was shown that unlabelled sequences of steps significantly complicate the calculations. However, there are specific nontrivial subclasses of Petri nets with invisible transitions, that have some nice properties w.r.t. place bisimulation.

In this paper we basically apply a similar approach to the resource equivalences. It is shown that, despite their non-trivial infinite structure, resource bisimulations can be effectively computed even in the case of nets with invisible transitions. In particular, it is shown that for a proper subclass — p-saturated nets — the weak transfer property of resource τp-bisimulation can be effectively checked. Moreover, we can underapproximate the largest τp-bisimulation by a parameterized algorithm.

The paper is organized as follows. Section 2 contains basic definitions. Specifically, in Subsection 2.1 we give some technical notions and lemmata on the properties of additively-transitively closed relations on multisets. Subsection 2.2 contains definitions of Petri nets and bisimulations. Subsections 2.3 and 2.4 give a short review on Petri net resources and resource equivalences (similarity and bisimulation). Section 3 deals with invisible transitions. In Subsections 3.1 and 3.2 we define the straightforward τ-generalizations of resource equivalences and study their properties. Subsections 3.3 and 3.4 describe the subclass of p-saturated nets and the corresponding notion of τp-bisimulation. In Subsection 3.5 we present an algorithm, computing the parameterized underapproximation of largest τp-bisimulation. Section 4 contains some conclusions.

2 Preliminaries

2.1 Relations on multisets

Let X be a finite set. A *multiset* m over a set X is a mapping $m : X \to Nat$, where Nat is the set of natural numbers (including zero), i.e. a multiset may contain several copies of the same element.

For two multisets m, m' we write $m \subseteq m'$ iff $\forall x \in X : m(x) \leq m'(x)$ (the inclusion relation). The sum and the union of two multisets m and m' are defined as usual: $\forall x \in X : m + m'(x) = m(x) + m'(x),\ m \cup m'(x) = max(m(x), m'(x))$. By $\mathcal{M}(X)$ we denote the set of all finite multisets over X.

Non-negative integer vectors are often used to encode multisets. Actually, the set of all multisets over finite X is a homomorphic image of $Nat^{|X|}$.

A binary relation $R \subseteq Nat^k \times Nat^k$ is a congruence if it is an equivalence relation and whenever $(v, w) \in R$ then $(v + u, w + u) \in R$ (here '+' denotes coordinate-wise addition). It was proved by L. Redei [11] that every congruence on Nat^k is generated by a finite set of pairs. Later P. Jančar [7] and J. Hirshfeld [6] presented a shorter proof and also showed that every congruence on Nat^k is a semilinear relation, i.e. it is a finite union of linear sets.

Let R^{AT} denote the additive-transitive closure (AT-closure) of the relation $R \subseteq \mathcal{M}(X) \times \mathcal{M}(X)$ (the minimal congruence, containing R).

Let $B \subseteq \mathcal{M}(X) \times \mathcal{M}(X)$ be a binary relation on multisets. A relation B' is called an *AT-basis* of B iff $(B')^{AT} = B^{AT}$. An AT-basis B' is called *minimal* iff there is no $B'' \subset B'$ such that $(B'')^{AT} = B^{AT}$.

Now we construct a special kind of minimal AT-basis for B. Define a partial order $\sqsubseteq$ on the set $B \subseteq \mathcal{M}(X) \times \mathcal{M}(X)$ of pairs of multisets as follows:

1. For loop (i.e. reflexive) pairs let

$$(r_1, r_1) \sqsubseteq (r_2, r_2) \overset{def}{\Leftrightarrow} r_1 \subseteq r_2;$$

2. For two non-loop pairs, the maximal loop constituents and the addend pairs of nonintersecting multisets are compared separately

$$(r_1 + o_1, r_1 + o_1') \sqsubseteq (r_2 + o_2, r_2 + o_2') \overset{def}{\Leftrightarrow}$$

$$\overset{def}{\Leftrightarrow} o_1 \cap o_1' = \emptyset \ \& \ o_2 \cap o_2' = \emptyset \ \& \ r_1 \subseteq r_2 \ \& \ o_1 \subseteq o_2 \ \& \ o_1' \subseteq o_2'.$$

3. A loop pair and a non-loop pair are always incomparable.

Let B_s denote the set of all minimal (with respect to $\sqsubseteq$) elements of B^{AT}.

Theorem 1. *[4] Let $B \subseteq \mathcal{M}(X) \times \mathcal{M}(X)$ be a symmetric and reflexive relation. Then B_s is an AT-basis of B and B_s is finite.*

We call B_s the *ground basis* of B. Obviously, it is finite.

There is also a useful

Lemma 1. *[4] Let $B \subseteq \mathcal{M}(X) \times \mathcal{M}(X)$ be a symmetric and reflexive relation, $(r, s) \in B^{AT}$. Then there exists a finite chain of pairs*

$$(r, a_1), (a_1, a_2), \ldots, (a_{k-1}, a_k), (a_k, s) \in (B_s)^A,$$

where $(B_s)^A$ is the additive closure of B_s.

2.2 Labelled Petri nets and bisimulations

Let P and T be disjoint sets of *places* and *transitions* and let $F : (P \times T) \cup (T \times P) \to Nat$. Then $N = (P, T, F)$ is a *Petri net.* A *marking* in a Petri net is a function $M : P \to Nat$, mapping each place to some natural number (possibly zero). Thus a marking may be considered as a multiset over the set of places.

Pictorially, P-elements are represented by circles, T-elements by boxes, and the flow relation F by directed arcs. Places may carry tokens represented by filled circles. A current marking M is designated by putting $M(p)$ tokens into each place $p \in P$. Tokens residing in a place are often interpreted as resources of some type consumed or produced by a transition firing. A marked Petri net (N, M_0) is a Petri net N together with an initial marking M_0.

For a transition $t \in T$ the *preset* ${}^\bullet t$ and the *postset* $t^\bullet$ are defined as the multisets over P such that ${}^\bullet t(p) = F(p,t)$ and $t^\bullet(p) = F(t,p)$ for each $p \in P$. A transition $t \in T$ is *enabled* in a marking M iff $\forall p \in P\ M(p) \geq F(p,t)$. An enabled transition t may *fire* yielding a new marking $M' =_{\text{def}} M - {}^\bullet t + t^\bullet$, i.e. $M'(p) = M(p) - F(p,t) + F(t,p)$ for each $p \in P$ (denoted $M \xrightarrow{t} M'$).

The transitions may *fire in parallel* (concurrently), if there are enough tokens for all of them. In particular, the transition may fire in parallel with itself. The concurrent firing of a multiset of transitions is called *a parallel step.* The precondition and postcondition for a multiset of transitions $\alpha \in \mathcal{M}(T)$ are:

$$ {}^\bullet\alpha =_{def} \sum_{t\in\alpha} {}^\bullet t, \qquad \alpha^\bullet =_{def} \sum_{t\in\alpha} t^\bullet. $$

Obviously, ${}^\bullet(\alpha + \beta) = {}^\bullet\alpha + {}^\bullet\beta, (\alpha + \beta)^\bullet = \alpha^\bullet + \beta^\bullet$.

To observe a net behavior transitions are labelled by special labels representing observable actions or events. Let Act be a set of action names. A *labelled Petri net* is a tuple $N = (P, T, F, l)$, where (P, T, F) is a Petri net and $l : T \to Act$ is a labelling function. It can be generalized to sequences:

for $\alpha \in T^*$ s.t. $\alpha = t\beta$ with $t \in T$ and $\beta \in T^*$ we have $l(\alpha) =_{def} l(t)l(\beta)$.

And also to multisets of transitions:

$$ \text{for } \alpha \in \mathcal{M}(T) \qquad l(\alpha) =_{def} \textstyle\sum_{t\in\alpha} l(t). $$

Here we use not a union but a sum of multisets.

Let $N = (P, T, F, l)$ be a labelled Petri net. We say that a relation $R \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the *transfer property* iff for all $(M_1, M_2) \in R$ and for every step $t \in T$, s.t. $M_1 \xrightarrow{t} M_1'$, there exists an imitating step $u \in T$, s.t. $l(t) = l(u)$, $M_2 \xrightarrow{u} M_2'$ and $(M_1', M_2') \in R$.

A relation R is called a *marking bisimulation*, if both R and R^{-1} conform to the transfer property.

For every labelled Petri net there exists the largest marking bisimulation (denoted by $\sim$) and this bisimulation is an equivalence. It was proved by P. Jančar [7], that the marking bisimulation is undecidable for Petri nets.

2.3 Resource similarity

Informally, resources are parts of markings which may or may not provide this or that kind of net behavior.

Definition 1. *[4] Let $N = (P, T, W, l)$ be a labelled Petri net. A* resource $R \in \mathcal{M}(P)$ *in a Petri net N is a multiset over the set of places P.*

Resources r and s in N are called similar *(denoted $r \approx s$) iff for every marking $R \in \mathcal{M}(P)$, $r \subseteq R$ implies $R \sim R - r + s$.*

Thus if two resources are similar, then in every marking each of these resources can be replaced by another without changing the observable system's behavior. Some examples of similar resources are shown in Fig. 1.

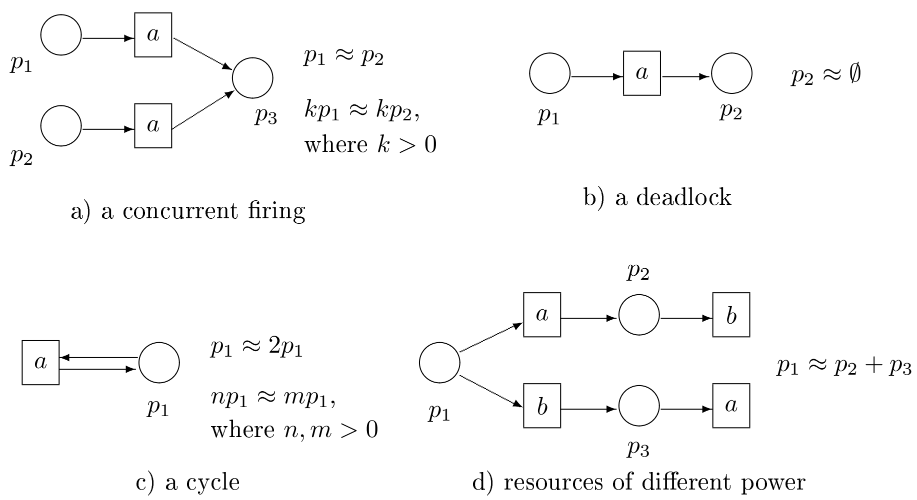


Fig. 1. Examples of similar resources.

Figure a) shows a Petri net containing two transitions labeled with the same label a and leading to the same marking p_3. Here the resources p_1 and p_2 are similar, as they lead to a completely identical observable behavior — action a producing a single token in p_3. Moreover, all the resources containing the same number of tokens in p_1 and p_2 are similar.

Figure b) shows a simple net consisting of a single transition. In this case the resource p_2 is similar to an empty resource, since it does not affect the behavior of the net (the place p_2 is redundant).

Figure c) depicts a cycle consisting of one transition and one place. Note that the set of markings of this net can be divided into two disjoint subsets — empty marking and all the others. With empty marking, the transition can not fire, for all others — it can fire any number of times. Note that for this net the largest marking bisimulation and the resource similarity coincide.

Figure d) shows a more complex network. We have $p_1 \approx p_2 + p_3$, that is, replacing one token in p_1 by two tokens (one in p_2 and one in p_3) does not affect the observable behavior of the net as a whole.

The similarity relation is an equivalence [4]. Moreover, it is monotonous:

Proposition 1. *[4] Let $N = (P, T, W, l)$ be a labelled Petri net, let r, s, u, v be resources of the net N. Then $r \approx s$ & $u \approx v \;\Rightarrow\; r + u \approx s + v$.*

Hence it has a finite ground basis. Unfortunately, from the undecidability of a stronger relation of place fusion [12] we get

Theorem 2. *[4] The resource similarity is undecidable for labelled Petri nets.*

2.4 Resource bisimulation

We defined a stronger equivalence relation, retaining the observable system's behavior:

Definition 2. *[4] An equivalence relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is called a resource bisimulation if B^{AT} is a marking bisimulation.*

Note that an AT-closure of a resource similarity is not necessarily a marking bisimulation. The next theorem states some important properties of resource bisimulations.

Theorem 3. *[4] Let N be a labelled Petri net. Then*

1. *if $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is a resource bisimulation and $(r_1, r_2) \in B$ then $r_1 \approx r_2$;*
2. *if B_1, B_2 are resource bisimulations for N then $B_1 \cup B_2$ is a resource bisimulation for N;*
3. *for any N there exists the largest resource bisimulation (denoted by $B(N)$), and it is an equivalence.*

Therefore $B(N)$ (as well as any other resource bisimulation) also has a finite ground basis.

The AT-closure of a resource bisimulation is a marking bisimulation, and hence, it conforms to the transfer property. Resource bisimulations satisfy a weak variant of the transfer property, when only 'adjacent' markings are considered for a transition t:

We say that a relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to *the weak transfer property* if for all $(r, s) \in B$, for each $t \in T$, such that ${}^\bullet t \cap r \neq \emptyset$, there exists an imitating transition $u \in T$, such that $l(t) = l(u)$ and, writing M_1 for ${}^\bullet t \cup r$ and M_2 for ${}^\bullet t - r + s$, we have $M_1 \xrightarrow{t} M_1{}'$ and $M_2 \xrightarrow{u} M_2{}'$ with $(M_1', M_2') \in B^{AT}$.

Theorem 4. *[4] A relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is a resource bisimulation iff B is an equivalence relation and it conforms to the weak transfer property.*

Due to this theorem to check whether a given finite relation B is a resource bisimulation, one needs to verify the weak transfer property for only a finite number of pairs of resources. In [4] we have shown that the largest resource bisimulation for resources with a bounded number of tokens can be effectively constructed (more precisely, it requires $\mathrm{O}(\max\{|P|\,\mathcal{R}^9, |T|^2|P|\,\mathcal{R}^7\})$ steps, where $\mathcal{R}$ is the number of resources in the consideration).

3 Petri nets with invisible transitions

In this section we investigate the possibilities of effectively constructing bisimulation-preserving relations for an extended class of systems — Petri nets with invisible transitions.

To distinguish visible and invisible transitions, a special τ symbol is added to the set of labels: $Act_\tau = Act \cup \{\tau\}$.

Definition 3. *A* labelled Petri net with invisible transitions *is a tuple* $N = (P, T, F, l)$*, where* (P, T, F) *is a Petri net and* $l : T \to Act_\tau$ *is an extended labelling function.*

Let $\sigma, \sigma' \in (Act_\tau)^*$ be sequences of action labels (with τ-s). Denote $\sigma =_\tau \sigma' \Leftrightarrow_{def} \sigma_{|Act} = \sigma'_{|Act}$ ("equal modulo τ"). For example, "$\tau\tau a\tau$" $=_\tau$ "a".

3.1 τ-bisimulation

Let $N = (P, T, F, l)$ be a labelled Petri net with invisible transitions. We say that a relation $R \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the *τ-transfer property* iff for all $(M_1, M_2) \in R$ and for every step $t \in T$, s.t. $M_1 \xrightarrow{t} M_1'$, there exists an imitating sequence of steps $\sigma \in T^*$ s.t. $l(t) =_\tau l(\sigma)$, $M_2 \xrightarrow{\sigma} M_2'$ and $(M_1', M_2') \in R$.

A relation R is called a *marking τ-bisimulation*, if both R and R^{-1} conform to the τ-transfer property. The largest τ-bisimulation is denoted by $\sim_\tau$.

Marking bisimulation is a special case of τ-bisimulation (for nets with no τ-s). It is a stronger relation. Consider as an example the net at Fig. 2. Markings p_1 and p_2 are not bisimilar, because at p_2 no transition with label a is active. But they are τ-bisimilar, because the invisible firing of t_2 changes the marking from p_2 to p_1.

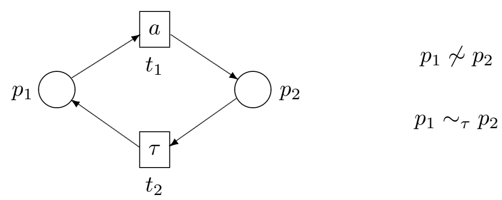


Fig. 2. τ-bisimulation is weaker than bisimulation

In particular, this implies the undecidability of τ-bisimulation in Petri nets with invisible transitions [7].

3.2 Resource similarity and bisimulation

The definition of resource similarity can be naturally generalized to the case of nets with invisible transitions:

Definition 4. *Let $N = (P, T, F, l)$ be a labelled Petri net with invisible transitions. Resources r and s are called* τ-similar *(denoted $r \approx_\tau s$) iff for every marking R, $r \subseteq R$ implies $R \sim_\tau R - r + s$.*

We can show that resource τ-similarity has all basic properties of resource similarity:

Proposition 2. *1. Resource τ-similarity is closed under addition and transitivity; hence it has finite AT-basis.*
2. Resource τ-similarity is undecidable.

Proof. 1) From the definitions.
2) From Th. 2 (note that τ-similarity is a generalization of basic resource similarity).

The definition of resource bisimulation also can be easily generalized:

Definition 5. *Let $N = (P, T, F, l)$ be a labelled Petri net with invisible transitions. An equivalence relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is called a* resource τ-bisimulation *if B^{AT} is a marking τ-bisimulation.*

Proposition 3. *Let $N = (P, T, F, l)$ be a labelled Petri net with invisible transitions. Then*

1. *if $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is a resource τ-bisimulation and $(r_1, r_2) \in B$ then $r_1 \approx_\tau r_2$;*
2. *if $B_1, B_2 \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ are resource τ-bisimulations then $B_1 \cup B_2$ is a resource τ-bisimulation;*
3. *for any N there exists the largest resource τ-bisimulation (denoted by $B_\tau(N)$), and it is an equivalence.*

Proof. The first statement follows directly from the definitions. Note, that there exists a resource τ-similarity which is not a τ-bisimulation.

The proof of the second statement is rather long and contains some technical details. It uses the decomposition of a given pair into a transitive chain of pairs, where pairs are constructed as sums of pairs from $(B_1)^{AT}$ and $(B_2)^{AT}$.

The third statement is an immediate corollary of the second. The largest resource τ-bisimulation is the union of all resource τ-bisimulations for N.

Definition 6. *We say that a relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the* weak τ-transfer property *if for all $(r, s) \in B$, $t \in T$ s.t. ${}^\bullet t \cap r \neq \emptyset$, there exists an imitating sequence of transitions $\sigma \in T^*$ s.t. $l(t) =_\tau l(\sigma)$ and, denoting $M_1 = {}^\bullet t \cup r$ and $M_2 = {}^\bullet t - r + s$, we have $M_1 \xrightarrow{t} M_1{}'$ and $M_2 \xrightarrow{\sigma} M_2{}'$ with $(M_1', M_2') \in B^{AT}$.*

Th. 4 in the case of Petri nets with invisible transitions works only in one direction:

Proposition 4. *If the relation conforms to the τ-transfer property then it conforms to the weak τ-transfer property; there exist relations, conforming to the weak τ-transfer property and not conforming to the τ-transfer property.*

Proof. ($\Rightarrow$) Since the weak τ-transfer property is the τ-transfer property for a bounded (finite) subset of pairs of resources.

($\not\Leftarrow$) Consider a net at Fig. 3 (this example is taken from [2]) and a relation

$$B = Id(P) \cup \{(p_1, p_2), (p_2, p_1), (p_3, p_4), (p_4, p_3)\}.$$

B conforms to the weak τ-transfer property. At the same time B is not a resource τ-bisimulation. Consider markings $M_1 = p_1 + p_3$ and $M_2 = p_2 + p_4$. The pair (M_1, M_2) belongs to the relation B^{AT}, but the markings are not bisimilar, because an action a is possible at M_2 (transition t_3) and is impossible at M_1.

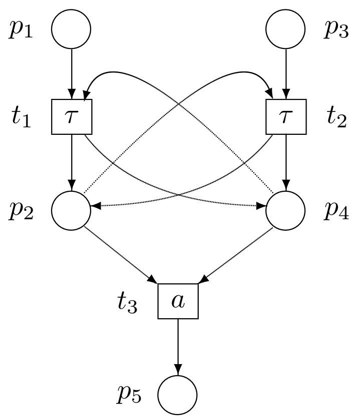


Fig. 3. Th. 4 does not hold for Petri nets with invisible transitions.

Hence the weak τ-transfer property can not be used to construct bisimulation. In the case of systems with invisible transitions it is even more important to strengthen the considered relations and/or to restrict the considered class of Petri nets.

3.3 Saturated nets

There exists a wide and important subclass of Petri nets with invisible transitions for which resource τ-bisimulation can be constructed using weak transfer property — so-called "p-saturated nets". In p-saturated nets [2] the firing of any sequence of transitions with at most one visible label can be simulated by a simultaneous (independent) firing of a certain set of transitions with the same label (called "parallel step").

Denote the set of transition sequences with at most one visible label:

$$T^{\times} =_{def} \{\sigma \in T^* \mid l(\sigma) \in Act_\tau\}.$$

Definition 7. *A labelled Petri net with invisible transitions $N = (P, T, F, l)$ is called* p-saturated *(or simply* saturated*), if for any sequence of transitions $\sigma \in T^{\times}$ there exists a parallel step $U \in \mathcal{M}(T)$ s.t. $l(U) =_\tau l(\sigma)$, ${}^\bullet U = {}^\bullet \sigma$ and $U^\bullet = \sigma^\bullet$.*

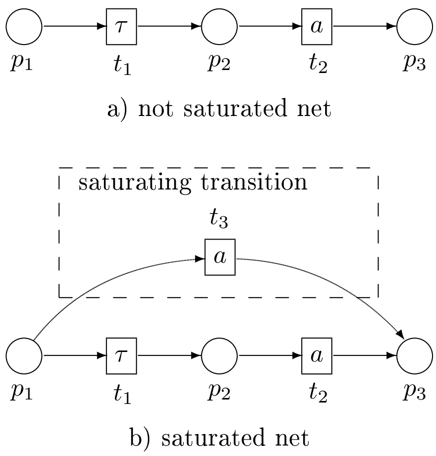


Fig. 4. An example of net saturation

In addition to saturated nets, there is an even broader class of *saturable* Petri nets. These are nets that can be transformed into saturated by adding a finite number of transitions while preserving the behavior of the net (in the sense of τ-bisimularity). On Fig. 4 a saturated net is shown, obtained by adding the transition t_3 to the unsaturated net.

It is known [2] that a net is p-saturated iff it is $2p$-saturated, i.e. all sequences of length 2 are saturated by parallel steps.

Not all nets are saturable [2]. An example is given at Fig. 5.

It is also easy to see that the net is saturable iff its "invisible subnet" is saturable (i.e. a net, obtained by removing all visible transitions).

3.4 τp-bisimulation

In [2] an equivalence stronger than τ-bisimulation was defined, called τp-bisimulation of markings. The transition in this case is modeled not by a sequence of transitions, but by a parallel step.

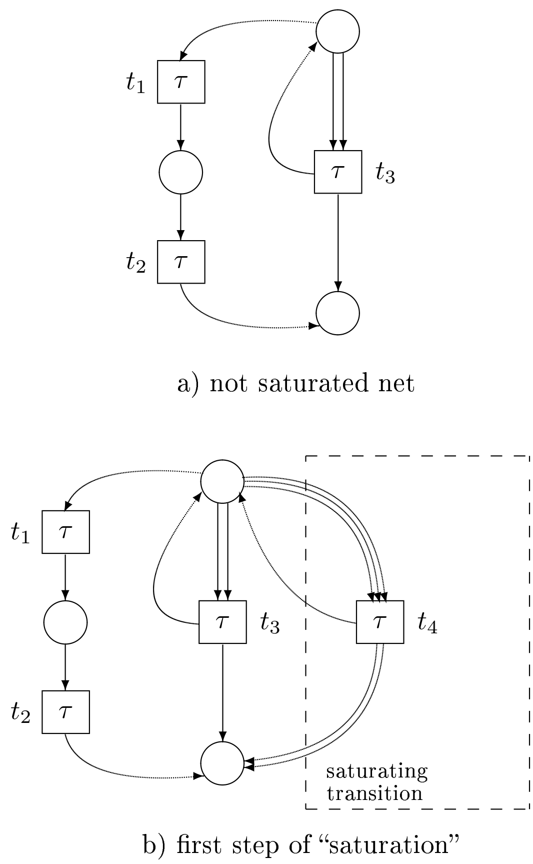


Fig. 5. Not saturable net

Definition 8. *[2] Let $N = (P, T, F, l)$ be a labelled Petri net with invisible transitions. We say that a relation $R \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the* τp-transfer property *if for all $(M_1, M_2) \in R$ and for each $t \in T$ s.t. $M_1 \xrightarrow{t} M_1'$, there exists an imitating parallel step $U \in \mathcal{M}(T)$ s.t. $l(t) =_\tau l(U)$, $M_2 \xrightarrow{U} M_2'$ and $(M_1', M_2') \in R$.*

Definition 9. *[2] A relation R is called a* marking τp-bisimulation*, if both R and R^{-1} conform to the τp-transfer property.*

It is known [2] that for any net there exists the largest τp-bisimulation (denoted by $\sim_{\tau p}$).

In saturated Petri nets τp-bisimulation coincides with τ-bisimulation [2]:

$$M_1 \sim_{\tau p} M_2 \quad \Leftrightarrow \quad M_1 \sim_\tau M_2.$$

Now we are ready to define a resource τp-similarity:

Definition 10. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions. Resources r and s are called* τp-similar *(denoted $r \approx_{\tau p} s$) iff for every marking R, $r \subseteq R$ implies $R \sim_{\tau p} R - r + s$.*

From the equality of $\sim_{\tau p}$ and $\sim_\tau$ in saturated nets we immediately have:

Corollary 1. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions, $r, s \in \mathcal{M}(P)$. Then*

$$r \approx_{\tau p} s \quad \Leftrightarrow \quad r \approx_\tau s.$$

So, in saturated nets it is sufficient to look for τp-similarities.

Definition 11. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions. An equivalence relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is called a* resource τp-bisimulation *if B^{AT} is a marking τp-bisimulation.*

In the case of τp-relations all basic properties also hold:

Proposition 5.
1. *Resource τp-similarity is closed under addition and transitivity; hence it has finite AT-basis.*
2. *Resource τp-similarity is undecidable.*
3. *If $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is a resource τp-bisimulation and $(r_1, r_2) \in B$ then $r_1 \approx_{\tau p} r_2$.*
4. *If $B_1, B_2 \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ are resource τp-bisimulations then $B_1 \cup B_2$ is a resource τp-bisimulation;*
5. *For any N there exists the largest resource τp-bisimulation (denoted by $B_{\tau p}(N)$), and it is an equivalence.*

Proof.

1) Immediately from the definition of resource τp-similarity.

2) From Cor. 1 and the undecidability of $(\approx_\tau)$.

3) Immediately from the definitions.

4) The proof is almost the same as in Prop. 3: the only difference is that we consider not an imitating transition but an imitating parallel step.

5) Note that we can take a union of all resource τp-bisimulations.

Definition 12. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions. We say that a relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the* weak τp-transfer property *if for all $(r, s) \in B$, $t \in T$ s.t. ${}^\bullet t \cap r \neq \emptyset$, there exists an imitating parallel step $U \in \mathcal{M}(T)$ s.t. $l(t) =_\tau l(U)$ and, denoting $M_1 = {}^\bullet t \cup r$ and $M_2 = {}^\bullet t - r + s$, we have $M_1 \xrightarrow{t} M_1{}'$ and $M_2 \xrightarrow{U} M_2{}'$ with $(M_1', M_2') \in B^{AT}$.*

In saturated nets the weak τp-transfer property is a necessary and sufficient condition for its extended version, which guarantees the imitation of a parallel step rather than a single transition:

Definition 13. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions. We say that a relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the* extended weak τp-transfer property *if for all $(r, s) \in B$ and any parallel step $V \in \mathcal{M}(T)$ s.t. ${}^\bullet V \cap r \neq \emptyset$, there exists an imitating parallel step $U \in \mathcal{M}(T)$ s.t. $l(V) =_\tau l(U)$ and, denoting $M_1 = {}^\bullet V \cup r$ and $M_2 = {}^\bullet V - r + s$, we have $M_1 \xrightarrow{V} M_1{}'$ and $M_2 \xrightarrow{U} M_2{}'$ with $(M_1', M_2') \in B^{AT}$.*

Lemma 2. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions. The relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the weak τp-transfer property iff it conforms to the extended weak τp-transfer property.*

Proof. $(\Leftarrow)$ Since the weak transfer property is a special case of the extended weak transfer property.

$(\Rightarrow)$ Assume the converse: the extended property does not hold, so there exists $(M_1, M_2) \in B^{AT}$, $V = \{t_1, \ldots, t_k\} \in \mathcal{M}(T)$ with $M_1 \xrightarrow{V} M_1'$, s.t. there exists no imitating parallel step $U \in \mathcal{M}(T)$ with the same visible label $l(V) =_\tau l(U)$ and $M_2 \xrightarrow{U} M_2'$ and $(M_1', M_2') \in B^{AT}$.

Consider the transition firing $M_1 \xrightarrow{t_1} M_1^1$. From the weak τp-transfer property it follows that this transition has an imitating parallel step $M_2 \xrightarrow{W_1} M_2^1$ such that $(M_1^1, M_2^1) \in B^{AT}$.

Note that $V = \{t_1, \ldots, t_k\}$ is a parallel step at marking M_1, hence after the firing of one of these transitions all other are still enabled. Therefore we can repeat the previous reasoning for the new pair of markings $(M_1^1, M_2^1) \in B^{AT}$ and transition t_2. And continue this until t_k:

$$
\begin{array}{rcl}
M_1 & B^{AT} & M_2 \\
t_1 \downarrow & & \downarrow W_1 \\
M_1^1 & B^{AT} & M_2^1 \\
t_2 \downarrow & & \downarrow W_2 \\
M_1^2 & B^{AT} & M_2^2 \\
t_3 \downarrow & & \downarrow W_3 \\
\dots & & \dots \\
t_k \downarrow & & \downarrow W_k \\
M_1' = M_1^k & B^{AT} & M_2^k = M_2'
\end{array}
$$

At the end we got a sequence of parallel steps

$$M_2 \xrightarrow{W_1} M_2^1 \xrightarrow{W_2} M_2^2 \xrightarrow{W_3} \dots \xrightarrow{W_k} M_2^k = M_2',$$

imitating the firing of parallel step $M_1 \xrightarrow{V} M_1'$. The net is saturated so for any sequence of transitions (note that a parallel step also can be considered as a sequence of transitions) there exists an imitating parallel step with the same label, precondition and postcondition ($M_2 \xrightarrow{U} M_2'$) – q.e.d.

Note that, unlike the weak transfer property, the extended weak transfer property can not be effectively checked by the search of resource pairs, since the set of parallel steps is infinite.

Theorem 5. *Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions. An equivalence relation $B \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ conforms to the weak τp-transfer property iff B is a resource τp-bisimulation.*

Proof. ($\Leftarrow$) Since the weak τp-transfer property is the τp-transfer property for a bounded (finite) subset of pairs of resources.

($\Rightarrow$) The proof is similar to the proof of Th. 4, with the additional use of Lm. 2.

Assume the converse: let B^{AT} does not conform to the τp-transfer property, i.e. there exist $(M_1, M_2) \in B^{AT}$, $t \in T$ with $M_1 \xrightarrow{t} M_1'$, s.t. there are no imitating parallel step $U \in \mathcal{M}(T)$ with $l(t) = l(U)$, $M_2 \xrightarrow{U} M_2'$ and $(M_1', M_2') \in B^{AT}$.

Consider a pair of markings $(M_1, M_2) \in B^{AT}$. From Lm. 1 this pair can be obtained by a transitive closure of several pairs from B^A (additive closure of B):

$$(H_1, H_2), (H_2, H_3), \dots, (H_{k-1}, H_k) \in B^A, \text{ where } H_1 = M_1, H_k = M_2.$$

Consider the pair (H_1, H_2).

$$(H_1, H_2) = (r_1 + r_2 + \dots + r_l, s_1 + s_2 + \dots + s_l), \text{ where } (r_i, s_i) \in B$$

$H_1 = {}^\bullet t \cup r_1 + F_1$. Due to the weak transfer property for the pair (r_1, s_1) there exists an imitating parallel step $V \in \mathcal{M}(T)$ s.t. $l(t) = l(V)$, ${}^\bullet t \cup r_1 \xrightarrow{t} G_1$ and ${}^\bullet t - r_1 + s_1 \xrightarrow{V} G_2$, where $(G_1, G_2) \in B^{AT}$.

Since ${}^\bullet t \cup r_1 \subseteq H_1$, we can add the resource $F = H_1 - {}^\bullet t \cup r_1$ to preconditions and postconditions:

$$
\begin{gathered}
{}^\bullet t \cup r_1 + F \xrightarrow{t} G_1 + F \\
{}^\bullet t - r_1 + s_1 + F \xrightarrow{V} G_2 + F
\end{gathered}
$$

From the reflexivity of B and the additive closure of B^{AT} the new pair of markings is also decomposable by $B : (G_1 + F, G_2 + F) \in B^{AT}$.

We obtained a new marking $H_1' = {}^\bullet t - r_1 + s_1 + F = H_1 - r_1 + s_1$. Note that it still contains $r_2 + \ldots + r_l$. Therefore, we can apply the same reasoning one more time, replacing resource r_2 by the bisimilar resource s_2, now using Lm. 2 and constructing an imitating parallel step not for a transition but for a parallel step V.

Apply this $l - 1$ times. Using transitive closure of B^{AT}, at the end we obtain a parallel step W that can imitate t at marking H_2.

Now proceed to the next pair (H_2, H_3) and repeat the procedure for the parallel step W. And so on, until the last pair (H_{k-1}, H_k). Finally we obtain a parallel step U that can imitate t at marking $H_k = M_2$.

Thus, in saturated nets the weak τp-transfer property can be used in the construction of resource τp-bisimulation.

3.5 Approximation

As in ordinary Petri nets, in the case of saturated (saturable) nets with invisible transitions there is a way of constructing an approximation of the maximal resource τp-bisimulation. If we consider not an infinite set of network resources, but only it's finite subset, then it will be possible to check the weak τp-transfer property.

Let $N = (P, T, F, l)$ be a saturated labelled Petri net with invisible transitions, $q \in Nat$ — some parameter. By $\mathcal{M}_q(P)$ we denote the set of all resources, containing not more then q tokens in the net: $\mathcal{M}_q(P) = \{r \in \mathcal{M}(P) \ : \ |r| < q\}$.

The largest resource τp-bisimulation on $\mathcal{M}_q(P)$ is defined as the union of all resource τp-bisimulations on $\mathcal{M}_q(P)$. We denote it by $B_{\tau p}(N, q)$. Since $\mathcal{M}_q(P)$ is finite, we can use the weak transfer property to compute $B_{\tau p}(N, q)$.

Definition 14. *(Underapproximation of largest resource τp-bisimulation)*

Input: *A saturated labelled Petri net with invisible transitions $N = (P, T, F, l)$, parameter $q \in Nat$.*

Output: *Relation $B_{\tau p}(N, q)$.*

Step 1: *Let $C = \{(\emptyset, \emptyset)\}$ — an empty set of pairs (considered as a relation over $\mathcal{M}_q(P)$).*

Step 2: *Compute* $B = (\mathcal{M}_q(P) \times \mathcal{M}_q(P)) \setminus C$*. Since* $\mathcal{M}_q(P)$ *is finite the set of pairs* B *is also finite.*

Step 3: *Compute* B_s *— the ground basis of* B*.*

Step 4: *Check, whether* B_s *conforms to the weak* τp*-transfer property: it is sufficient to test all non-reflexive elements of* B_s*.*

- *If all pairs conforms to the weak* τp*-transfer property then stop and return* B *— the bisimulation.*
- *Otherwise there are* $(r, s) \in B_s^{nr}$ *and* $t \in T$ *with* ${}^\bullet t \cap r \neq \emptyset$*, s.t. the firing* $M_1 \xrightarrow{t} M_1{}'$ *with* $M_1 = {}^\bullet t \cup r$ *can not be imitated by a parallel step with the same label and with precondition* $M_2 = {}^\bullet t - r + s$*. In this case add* (r, s) *and* (s, r) *to* C *and go back to Step 2.*

For any marking the set of active parallel steps is finite. Also note that the set $\mathcal{M}_q(P) \times \mathcal{M}_q(P)$ is finite. Hence the algorithm always stops.

Denote by $\mathcal{R} = |\mathcal{M}_q(P)|$ the size of the set of considered resources.

At the Step 2 we search through the set of all parallel steps with at most one visible label, that can fire at marking M_2. Each invisible transition can participate in the parallel step at most $|M_2|$ times, since it uses at least one input token.[1] There is also at most one visible transition. Hence we have to check at most $|T||M_2|^{|T|}$ multisets of transitions.

The size of marking $M_2 = {}^\bullet t - r + s$ can be evaluated as $\mathrm{O}(|s|) = \mathrm{O}(q)$.

Using our previous estimations of complexity for ground basis calculation (polynomial w.r.t. $\mathcal{R}$) and the complexity of other steps of algorithm (polynomial w.r.t. the size of the net), we obtain the overall complexity of

$$\mathrm{O}(\max\{|P|\ \mathcal{R}^9, |T|^2 q^{|T|} |P|\ \mathcal{R}^7\}).$$

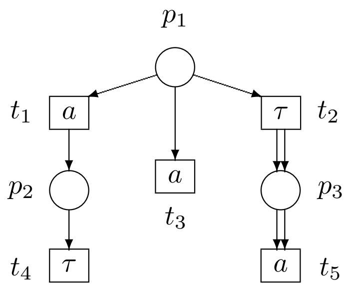
So in the case of nets with invisible transitions the complexity of the algorithm increased significantly (the linear dependence on $|T|$ was replaced by an exponential one). Such a jump is explained by the transition from sets of transitions to multisets.

Consider an example of calculations (Fig. 6). With $q = 1$ we found that resource p_2 is τp-similar to an empty resource (i.e. the place p_2 is redundant). Increasing the parameter ($q = 2$), we obtained one more pair of similar resources $p_1 \approx_{\tau p} 2p_3$.

4 Conclusion

The proposed method for finding pairs of similar resources is of particular interest for certain applications. In addition, the use of resource bisimulation allows one to reduce a Petri net with conservation of its behavior. This reduction is

[1] Without loss of generality we can assume that a net contains no invisible transitions with empty preconditions, since such transitions are redundant and can always be removed from the net along with places, included in their postconditions.



$q = 1: \quad p_2 \approx_{\tau p} \emptyset$

$q = 2: \quad p_1 \approx_{\tau p} 2p_3$

Fig. 6. An example of approximation: resource τp-bisimulation of a saturated Petri net with invisible transitions

important when analyzing properties of the Petri net, since the computational complexity of the majority of algorithms used in analysis depends exponentially on the size of the net.

Important open questions concern decidability and complexity of related algorithmic problems. For example, we have already shown that all types of resource similarity (ordinary, τ-, τp-) are undecidable. On the other hand, the problem of $B(N)$ (and $B_\tau(N)$, and $B_{\tau p}(N)$) computability is still open. We have introduced only the underapproximations.

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