On the development of Steffensen's method and applications to Hammerstein equations

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Abstract

We construct an analog of Steffensen's method for solving nonlinear operator equations. Proposed method is a kind of two stage iterative process. We study the order of convergence of the method. It is shown that the proposed analog of Stefensen's method, which does not use derivatives, has higher order of convergency then Newton method, other generalizations of chord method or other known modifications of Steffensen's method. Results of numerical experiments with Hammerstein equation are presented.

1 Introduction

Aitken–Steffensen method (also known as Steffensen's method) is one of the effective methods for approximate solving of nonlinear equations [1, 2]. It is a kind of chord method, but in distinction of classical chord method to obtain next approximation it uses one current approximation and its simple iteration.

Sterffensen's method has some advantages in comparison with classical chord method, e.g. it does not require two initial approximations, it has quadratic convergence like Newton's method, moreover in particular cases it converges faster than Newton method. Steffensen's method is of high interest because it could be generalized for solving nonlinear equations in abstract spaces with preserving good characteristics mentioned above. Attempts to make such generalizations were previously made in [3, 4, 5, 6]. The review of existing modifications of the Aitken–Steffensen method are given in [7]: two-stage procedures, methods of high convergence for nonlinear scalar equations, generalizations for solving systems of nonlinear equations, predictor-corrector modifications. Present work continues investigations initiated in [8].

Let us consider an equation

$$F(x) = 0, \tag{1}$$

here F is a continuous mapping from a Banach space X to X. Equation (1) could also be written as a fixed point problem

$$x = \Phi(x),$$

where Φ is a mapping from X to X.

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We construct an analog of Steffensen's method for solving the operator equation (1) as follows

$$\begin{aligned} \tilde{x}^{(k+1)} &= x^{(k)} - \left[F\left(\tilde{x}^{(k)}, \Phi\left(\tilde{x}^{(k)}\right)\right)\right]^{-1} F\left(x^{(k)}\right), \\ x^{(k)} &= \tilde{x}^{(k)} - \left[F\left(\tilde{x}^{(k)}, \Phi\left(\tilde{x}^{(k)}\right)\right)\right]^{-1} F\left(\tilde{x}^{(k)}\right), \end{aligned} \tag{2}$$

where $F(x', x'') = E - \Phi(x', x'')$ is a first divided difference (see Definition 1), k = 0, 1, 2, ...

Definition 1 (see [9]). Let F be a continuous nonlinear mapping from a Banach space X to Y. Linear operator $F(x', x'')$ is called the first divided difference of operator F if the following requirements hold:

1. for each fixed $x', x'' \in X$ operator $F(x', x'')$ is such that $F(x', x'')(x' - x'') = F(x') - F(x'')$;
2. if the Frechet derivative $F'(x)$ exists then $F(x, x) = F'(x)$.

Note, that functional $F(x,' x'')$ is obviously nonlinear with respect to x', x''. Definition 1 does not defines a unique operator $F(x', x'')$, indeed the first divided difference could be specified in a numerous of ways. One of such specification is given in

Statement 1.[10] If there exists a continuous derivative $\Phi'(x)$ in the Gateau sense on the interval $[x', x'']$, then we can define the first divided difference $\Phi(x', x'')$ as a Riemann's abstract integral[1] [11]:

$$\Phi(x', x'') = \int_0^1 \Phi'\big(x'' + t(x' - x'')\big)\, dt.$$

Using this specification of the definition, we can obtain the estimate

$$\|\Phi(x', x'')\| = \left\| \int_0^1 \Phi'\big(x'' + t(x' - x'')\big)\, dt \right\| \leqslant \int_0^1 \left\|\Phi'\big(x'' + t(x' - x'')\big)\right\|\, dt \leqslant \int_0^1 M_1\, dt = M_1, \tag{3}$$

where M_1 is the maximum of the norm of the Gateau derivative Φ' on the segment $[x', x'']$.

Definition 2[9] For the operator $\Phi(x)$ and each fixed $x', x'', x''' \in X$ the second divided difference $\Phi(x', x'', x''')$ is a bilinear operator from the space X to the space $(X \to Y)$, that is $\Phi(x', x'', x''') \in X \to (X \to Y)$, such that

$$\Phi(x', x'', x''')(x' - x''') = \Phi(x', x'') - \Phi(x'', x'''), \tag{4}$$

where $\Phi(x', x'')$, $\Phi(x'', x''')$ are the operators of the first divided difference.

Statement 2.[10] According to [11] the second divided difference could be defined in terms of Riemann integrals:

$$\Phi(x', x'', x''') = \int_0^1 \int_0^{t_1} \Phi''\big(x''' + t_1(x'' - x''') + t_2(x' - x''')\big)\, dt_2\, dt_1. \tag{5}$$

Using (5), one can get the following estimate:

$$\|\Phi(x', x'', x''')\| = \left\| \int_0^1 \int_0^{t_1} \Phi''\big(x''' + t_1(x'' - x''') + t_2(x' - x''')\big)\, dt_2\, dt_1 \right\| \leqslant$$

$$\leqslant \int_0^1 \int_0^{t_1} \left\|\Phi''\big(x''' + t_1(x'' - x''') + t_2(x' - x'')\big)\right\| dt_2\, dt_1 \leqslant \int_0^1 \int_0^{t_1} M_2 dt_2\, dt_1 = \frac{1}{2} M_2, \tag{6}$$

here M_2 is the maximum of the norm of the second derivative $\max\limits_{\xi} \|\Phi''(\xi)\|$,
where $\xi = x''' + t_1(x'' - x''') + t_2(x' - x'')$, $t_1 \in [0, 1]$, $t_2 \in [0, t_1]$.

[1]The well-established terminology [12], [13] is used. The Riemann integral of a function of a real argument with values in an abstract Banach space.

2 Study of convergence

Let $F(x)$ be a continues, we also assume that the operator of the first divided difference is invertible, i.e. in the domain of our interest there exist $[F(x', x'')]^{-1} = [E - \Phi(x', x'')]^{-1}$.

Theorem 1 (proof could be found in [14]). Let the following conditions hold:

1. $\left\|F(\tilde{x}^{(0)})\right\| = \left\|\tilde{x}^{(0)} - \Phi\left(\tilde{x}^{(0)}\right)\right\| \leqslant \eta$;
2. there exists an open domain $\Omega \subseteq X$ such that for each $x', x'', x''' \in \Omega$ the following three estimations hold
   (a) $\left\|[F(x', x'')]^{-1}\right\| = \left\|[E - \Phi(x', x'')]^{-1}\right\| \leqslant B$;
   (b) $\|\Phi(x', x'')\| \leqslant M$;
   (c) $\|\Phi(x', x'') - \Phi(x'', x''')\| \leqslant K \|x' - x'''\|$,

   where B, M, K are constants;
3. $h = C_2 B^2 K M \eta < 1$, where C_2 is a constant which is not small;
4. the set Ω fully contains a closed ball
$$\left\|x - \tilde{x}^{(0)}\right\| \leqslant R, \tag{7}$$
   where $R = \dfrac{C_1 S_0}{C_2 B K M}$, C_1 is a constant, and $S_k = \sum\limits_{n=k}^{\infty} h^{2^n}$.

Then a) all elements of the sequence $\left(\tilde{x}^{(k)}\right)$, defined by the method (2) which starts from the certain $\tilde{x}^{(0)}$, lie in the ball (7), b) the sequence $\left(\tilde{x}^{(k)}\right)$ has a limit $\tilde{x}^*$ in the ball (7), and c) the estimation takes place

$$\left\|\tilde{x}^* - \tilde{x}^{(k)}\right\| \leqslant \frac{C_1}{C_2 B K M} S_k \quad (k = 0,\, 1,\, 2,\, \dots).$$

Theorem 2. Assume that

1. equation (1) has a solution in the ball
$$\left\|x - \tilde{x}^{(0)}\right\| \leqslant \rho; \tag{8}$$
2. for each x', x'', x''' from the ball $\left\|x - \tilde{x}^{(0)}\right\| \leqslant (1 + \alpha)\,\rho$ the following three estimations hold
   (a) $\left\|[F(x', x'')]^{-1}\right\| = \left\|[E - \Phi(x', x'')]^{-1}\right\| \leqslant B$;
   (b) $\|\Phi(x', x'')\| \leqslant M$;
   (c) $\|\Phi(x', x'') - \Phi(x'', x''')\| \leqslant K \|x' - x'''\|$,

   where B, M, K are constants, at that $\alpha = \max\{l^2\rho^2,\, M\}$, where $l = \sqrt{2C} B K M$, C is a constant;
3. $l\rho < 1$.

Then a) the solution $\tilde{x}^*$ of equation (1) is unique in the ball (8), b) sequence $\left(\tilde{x}^{(k)}\right)$, defined by the method (2), converges to $\tilde{x}^*$ with the third order, i.e. the following estimation of the convergence rate holds

$$\left\|\tilde{x}^* - \tilde{x}^{(k)}\right\| \leqslant \frac{1}{l}\,(l\rho)^{3^k} \quad (k = 0,\, 1,\, 2,\, \dots).$$

Theorem 2 shows that under natural conditions, the generalized method (2), that does not use the derivatives, has a third order of convergence, that is, greater than Newton's method or generalizations of the secant or Steffensen method. Theorems 1 and 2 impose less stringent conditions on the operator of the first divided difference than, e.g., in the papers [7, 15, 16] where some generalizations of the Steffensen's method were studied. Moreover, there are no theorems of convergence with order in [7, 15, 16] for the case of operator equations. The

number of function computations at each iteration step for modifications of the Steffensen method [7] is at least one more than for a generalized method (2) that requires three function computations at each iteration step.

The formal measure of quality is the efficiency index. Method (2) has higher efficiency index then classical Newton's or Steffensen's method, and the same as the two-step Steffensen's modification [15, 16].

Definition 3 The index of the efficiency of the iterative method is the quantity $p^{1/\theta}$, where p is the order of convergence of the method, θ is the number of calculations of the function at each iteration step.

3 Application to the nonlinear integral equations

We consider an integral equation of Hammerstein type

$$x(t) - g(t) - \int_a^b K(t,\, s) f(s,\, x(s)) ds = 0, \quad t \in [a,\, b]\,. \tag{9}$$

The equation (9) could also be considered as a special case of the equation (1) where

$$F(x) = x(t) - g(t) - \int_a^b K(t,\, s) f(s,\, x(s)) ds.$$

Let $X = C\,[a,\, b]$, the functions $g(t)$, $K(t,\, s)$, $f(s,\, x)$ are such that the conditions of theorem 2 are satisfied.

The operator of the first divided difference $F\,(x_1, x_2)$ is given by

$$F\,(x_1, x_2)\,h = h(t) - \int_a^b K(t,\, s) \frac{f\,(s, x_1(s)) - f\,(s, x_2(s))}{x_1(s) - x_2(s)} h(s) ds. \tag{10}$$

Then using (10) we can determine that iterative process (2) for equation (9) requires at each step solution of a linear integral equation with kernel $K(t,s)\psi_k(s)$ where

$$\psi_k(s) = \frac{f\left(s, \tilde{x}^{(k)}(s)\right) - f\left(s, \Phi\left(\tilde{x}^{(k)}(s)\right)\right)}{\tilde{x}^{(k)}(s) - \Phi\left(\tilde{x}^{(k)}(s)\right)}, \tag{11}$$

and calculations of

$$F(\tilde{x}^{(k)}(t)) = \tilde{x}^{(k)}(t) - g(t) - \int_a^b K(t,\, s) f(s,\, \tilde{x}^{(k)}(s)) ds, \tag{12}$$

and

$$F(x^{(k)}(t)) = x^{(k)}(t) - g(t) - \int_a^b K(t,\, s) f(s,\, x^{(k)}(s)) ds. \tag{13}$$

The numerical solution of the integral equation (9) raises a number of problems, for example, difficulties connected with an exact calculation of the integral in (12) and (13) and difficulties connected with solution of linear integral equation. Therefore, we approximate integrals in (12) and (13) by quadrature formulas and replace a kernel by degradate one. Let

$$K(t,s) = \sum_{i=0}^{n(k)} \alpha_i^{(k)}(t) \beta_i^{(k)}(s) \quad (t,s) \in [a,\, b] \times [a,\, b]\,, \tag{14}$$

be a degenerate kernel close to $K(t,s)$, and its approximation accuracy may vary depending on the number of iteration k. We take the operator $\hat{\Psi}_k$ as the operator of the form (10) in which $K(t,s)$ is replaced by a degenerate kernel (14). Further we consider two quadrature formulas:

$$\int_a^b G(s) ds \approx \sum_{l=0}^{N} A_l G(s_l), \quad \int_a^b G(s) ds \approx \sum_{l=0}^{N'} A'_l G(s'_l), \tag{15}$$

where the nodes $s_l \in [a,\, b]$, $s'_l \in \{s_l\}$, $N' \leqslant N$. We replace the integral in the discrepancies (12), (13) using the first of the formulas (15). Then the process (2) for the equation (9) will be written so

$$\begin{aligned}
&\tilde{x}^{(k+1)}(t) = x^{(k)}(t) + \Delta_k(t),\\
&x^{(k)}(t) = \tilde{x}^{(k)}(t) + \Theta_k(t),\\
&\Theta_k(t) = -\hat{P}_k(t) + \sum_{i=0}^{n(k)} \alpha_i^{(k)}(t) c_i^{(k)},\\
&\Delta_k(t) = -\hat{Q}_k(t) + \sum_{i=0}^{n(k)} \alpha_i^{(k)}(t) d_i^{(k)},
\end{aligned} \tag{16}$$

where

$$\hat{Q}_k(t) = x^{(k)}(t) - g(t) - \sum_{l=0}^{N} A_l K(t,\, s_l) f(s_l,\, x^{(k)}(s_l)) ds, \tag{17}$$

$$\hat{P}_k(t) = \tilde{x}^{(k)}(t) - g(t) - \sum_{l=0}^{N} A_l K(t,\, s_l) f(s_l,\, \tilde{x}^{(k)}(s_l)) ds, \tag{18}$$

$k = 0,\, 1,\, 2,\, \ldots$, the constants $c_i^{(k)}$ are found from the linear system

$$c_i^{(k)} = -b_i^{(k)} + \sum_{j=0}^{n(k)} a_{ij}^{(k)} c_j^{(k)}, \quad i = 0,\, 1,\, \ldots, n(k), \tag{19}$$

in which

$$b_i^{(k)} = \int_a^b \beta_i^{(k)}(s) \psi_k(s) \hat{P}_k(s) ds \approx \sum_{l=0}^{N'} A'_l \beta_i^{(k)}(s'_l) \psi_k(s'_l) \hat{P}_k(s'_l) = \hat{b}_i^{(k)}, \tag{20}$$

$$a_{ij}^{(k)} = \int_a^b \beta_i^{(k)}(s) \psi_k(s) \alpha_j^{(k)}(s) ds \approx \sum_{l=0}^{N'} A'_l \beta_i^{(k)}(s'_l) \psi_k(s'_l) \alpha_j^{(k)}(s'_l) = \hat{a}_{ij}^{(k)}, \tag{21}$$

the constants $d_i^{(k)}$ are found from the linear system

$$d_i^{(k)} = -e_i^{(k)} + \sum_{j=0}^{n(k)} a_{ij}^{(k)} d_j^{(k)}, \quad i = 0,\, 1,\, \ldots,\, n(k), \tag{22}$$

in which $a_{ij}^{(k)}$ are calculated by the formula (21) and

$$e_i^{(k)} = \int_a^b \beta_i^{(k)}(s) \psi_k(s) \hat{Q}_k(s) ds \approx \sum_{l=0}^{N'} A'_l \beta_i^{(k)}(s'_l) \psi_k(s'_l) \hat{Q}_k(s'_l) = \hat{e}_i^{(k)}, \tag{23}$$

$\psi_k(s)$ looks similar to (11). The second quadrature formula (15) is used to calculate the integrals (20), (21) and (23).

The simple fact of the matter is, it is more convenient to implement the process (16) for grid functions[2] of approximations $\tilde{x}^{k+1}(t)$ by setting $t = t_i = s_i$, $i = 0,\, 1,\, \ldots,\, N$.

Example 1. Let us consider the integral equation in [18]:

$$x(t) = 1 - 0.4854t + t^2 + \int_0^1 ts \operatorname{arctg} x(s) ds, \qquad t \in [0,\, 1], \tag{24}$$

[2]A typical situation: instead of an exact solution — functions — we find a table of its approximate values (a skeleton), and then, if necessary, we make a replacement, for example, by spline interpolation. It is natural that the approximate solution obtained in this way belongs to a space that is narrower than the original solution space [17].

here $g(t) = 1 - 0.4854t + t^2$, $K(t,\, s) = ts$, $f(s,\, x(s)) = \operatorname{arctg} x(s)$. The exact solution of equation (24) is $x^*(t) = 1 + t^2$.

Let us verify that the conditions of Theorem 2 are satisfied. The first divided difference for the operator on the right-hand side of the equation (24) has the form

$$\Phi\,(x_1; x_2)\,h = \int\limits_0^1 K(t,\, s)\frac{f\,(s, x_1(s)) - f\,(s, x_2(s))}{x_1(s) - x_2(s)}h(s)ds.$$

Since $K(t,\, s)f'_x\,(s,\, x(s)) = \dfrac{ts}{1 + x^2(s)}$ and the inequality

$$\max_{0\leqslant t\leqslant 1}\int\limits_0^1 |K(t,\, s)f'_x\,(s,\, x(s))|\,ds \leqslant 1/2$$

are satisfied in the whole space $C[0,\, 1]$, then according to the statement 1, $\|\Phi\,(x_1; x_2)\| \leqslant M = 1/2$, that is the condition 2b) is satisfied.

Since $K(t,\, s)f''_{xx}\,(s,\, x(s)) = -\dfrac{2ts\,x(s)}{(1 + x^2(s))^2}$, and inequality

$$\max_{0\leqslant t\leqslant 1}\int\limits_0^1 |K(t,\, s)f''_{xx}\,(s,\, x(s))|\,ds \leqslant \max_{0\leqslant t\leqslant 1}\int\limits_0^1 \frac{6\sqrt{3}\,ts}{16}ds \leqslant \frac{3\sqrt{3}}{16}$$

is satisfied in the whole space $C[0,\, 1]$, then according to the definition 2 and statement 2

$$\|\Phi\,(x_1; x_2) - \Phi\,(x_2; x_3)\| \leqslant K\,\|x_1 - x_3\| = \frac{3\sqrt{3}}{32}\|x_1 - x_3\|\,,$$

that is, the condition 2c) is satisfied.

Since $\|\Phi\,(x_1; x_2)\| \leqslant 1/2$ the following expansion and inequality hold

$$\|[E - \Phi(x', x'')]^{-1}\| = \sum_{n=0}^{\infty}\|\Phi(x', x'')\|^n \leqslant 2,$$

that is, the condition 2a) is satisfied.

We solved the equation (24) using the method (16). In the first series of numerical experiments, we investigated the influence of the initial approximation $\tilde{x}^{(0)}$ on the rate of convergence (here it is measured as a number of iterations required to achieve a given accuracy 10^{-5}).

Let us divide the segment $[0,\, 1]$ uniformly into $N = 10$ parts and form the time grid $t_i,\ i = 0, ..., N$. Let us put $N' = N$. We searched a approximate solution, ie, a grid function $\tilde{x}^k(t_i)$. We used the Simpson formulas with $N = N'$ nodes as a quadrature formulas for (15). The stop criterion was taken as $\|\tilde{x}^{(k+1)} - \tilde{x}^{(k)}\| < 10^{-5}$.

We considered the following initial conditions

1. $\tilde{x}^{(0)}(t) \equiv 1.5$,
2. $\tilde{x}^{(0)}(t) \equiv -1.5$,
3. $\tilde{x}^{(0)}(t) \equiv -10$,
4. $\tilde{x}^{(0)}(t) = -20 + 10|\sin(5\pi t)|$.

The results of the numerical experiments for (16) are presented in the table 1. In the first column, the nodes of the time grid t_k are listed, in the second column the values of the exact solution at the grid nodes $x^*(t_k)$ are listed. Columns three through six are similar: the differences between the exact $x^*(t_i)$ and the approximate solution $\tilde{x}^k(t_i)$ in the grid nodes are indicated, here the index k corresponds to the iteration number when the required accuracy has been achieved. In the lower part of the table, the norms of the difference between the exact $x^*(t_i)$ and the approximate solutions $\tilde{x}^k(t_i)$ are given; and the norm-maximum was used.

Table 1: The results of numerical experiments for the equation (24)

t_i	$x^*(t_i)$	$x^*(t_i) - \tilde{x}^{(1)}(t_i)$	$x^*(t_i) - \tilde{x}^{(1)}(t_i)$	$x^*(t_i) - \tilde{x}^{(1)}(t_i)$	$x^*(t_i) - \tilde{x}^{(1)}(t_i)$
0	1	0	0	0	0
0.1	1.01	$-2.816035 \cdot 10^{-6}$	$-2.816225 \cdot 10^{-6}$	$-2.816036 \cdot 10^{-6}$	$-2.816038 \cdot 10^{-6}$
0.2	1.04	$-5.632069 \cdot 10^{-6}$	$-5.632449 \cdot 10^{-6}$	$-5.632071 \cdot 10^{-6}$	$-5.632075 \cdot 10^{-6}$
0.3	1.09	$-8.448103 \cdot 10^{-6}$	$-8.448674 \cdot 10^{-6}$	$-8.448107 \cdot 10^{-6}$	$-8.448113 \cdot 10^{-6}$
0.4	1.16	$-1.126414 \cdot 10^{-5}$	$-1.126490 \cdot 10^{-5}$	$-1.126414 \cdot 10^{-5}$	$-1.126415 \cdot 10^{-5}$
0.5	1.25	$-1.408017 \cdot 10^{-5}$	$-1.408112 \cdot 10^{-5}$	$-1.408018 \cdot 10^{-5}$	$-1.408019 \cdot 10^{-5}$
0.6	1.36	$-1.689621 \cdot 10^{-5}$	$-1.689735 \cdot 10^{-5}$	$-1.689621 \cdot 10^{-5}$	$-1.689622 \cdot 10^{-5}$
0.7	1.49	$-1.971224 \cdot 10^{-5}$	$-1.971357 \cdot 10^{-5}$	$-1.971225 \cdot 10^{-5}$	$-1.971226 \cdot 10^{-5}$
0.8	1.64	$-2.252828 \cdot 10^{-5}$	$-2.252980 \cdot 10^{-5}$	$-2.252829 \cdot 10^{-5}$	$-2.252830 \cdot 10^{-5}$
0.9	1.81	$-2.534431 \cdot 10^{-5}$	$-2.534602 \cdot 10^{-5}$	$-2.534432 \cdot 10^{-5}$	$-2.534434 \cdot 10^{-5}$
1.0	2	$-2.816035 \cdot 10^{-5}$	$-2.816225 \cdot 10^{-5}$	$-2.816036 \cdot 10^{-5}$	$-2.816038 \cdot 10^{-5}$
		$2.816035 \cdot 10^{-5}$	$2.816225 \cdot 10^{-5}$	$2.816036 \cdot 10^{-5}$	$2.816038 \cdot 10^{-5}$

It is seen from the table that even for nonsmooth initial approximations which oscillates with high frequency and amplitude the method converges, and it is necessary only one iteration to achieve the required accuracy. For the initial approximation $\tilde{x}^{(0)}(t) = -20 + 10|\sin(5\pi t)|$ the experiment also shows good smoothing properties of the method.

The results obtained for the initial approximation $\tilde{x}^{(0)} = 1.5$ (stop criterion is $\|\tilde{x}^{(k+1)} - \tilde{x}^{(k)}\| < 10^{-6}$) exceed the results of applying the Steffensen's method [19] ($k = 2$) to this equation by the number of iterations, which in turn is better than Newton's method [20] (results of comparison of the Steffensen's method and the Newton's method for the equation (24) are given in [19]).

Example 2. Let us consider the integral equation, see [21]:

$$x(t) = g(t) - \int_{-1}^{1} e^{ts} \sin|x(s)|\, ds, \qquad t \in [-1,\, 1], \tag{25}$$

where $g(t) = \dfrac{2}{1+t^2}\left(1 + 0.5t + 0.5t^3 - \cos 1 \operatorname{ch} t + t \sin 1 \operatorname{sh} t\right)$.

The corresponding exact solution of the equation (25) is $x(t) = t$. We solved the equation (25) using the method (16), the initial approximation was taken as follow $\tilde{x}^{(0)} = g(t)$. The degenerate kernel (14), which approximates $K(t,\, s) = -e^{ts}$, was taken as the sum of $n(k)$ terms of the Taylor expansion. As the quadrature formulas (15) the Simpson's formulas were used with $N = N' = 81$. Stop criterion was taken as $\|\hat{Q}_k(t)\| < 10^{-6}$. The results of the numerical approximations (16) are given for three nodes in the table 2. From table 2 it is seen that the

Table 2: The results of numerical experiments for the equation (25)

k	$\tilde{x}^{(k+1)}(-1)$	$\tilde{x}^{(k+1)}(0)$	$\tilde{x}^{(k+1)}(1)$	$n(k)$
0	-0.8939872	0.01574576	0.97259075	8
1	-0.99999309	$9.99 \cdot 10^{-6}$	1.0000005	8
2	-0.99999999	$-1.57 \cdot 10^{-9}$	1.0000000	8

method (16) has converged in 3 iterations, while the analog of the Aitken-Steffensen method with a controlled step [21] converges in 9 iterations.

Note that application of the process (2) to the equation (25) requires at each iteration step to solve a linear system with N equations, while the iterative process (16) requires to solve a linear system with $n(k)$ equations.

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