The calculation of the spatial spectrum of multidimensional fractals using the fast Fourier transform

O.A. Mossouлина

1Samara National Research University, 34 Moskovskoe Shosse, 443086, Samara, Russia

Abstract

The Fast Fourier Transform was applied to spatial spectrum modeling of a one-dimensional fractal (Cantor set), a two-dimensional fractal (Sierpinski carpet), and a three-dimensional fractal (Menger sponge). A spectrum is developed for different levels. The spatial spectrum was also obtained and modeled for various filling parameters. The ParaView software package was used for 3D modeling.

Keywords: cantor set; Sierpinski carpet; fast Fourier transform; 3D modeling

1. Introduction

Many natural phenomena have distinctive features, which are often associated with fractal structures. Visually, fractals represent a geometric figure, replication of which is exactly the same at every scale [1]. This ability is called self-similarity. Fractals are interesting because of widespread presence in natural formations [1-3]. In this case, natural fractals are called "statistical", and artificial "exact". Statistical fractals can be observed in various polymers, biological structures, electrical circuits, galactic clusters and fluctuations in exchange prices [4]. Exact fractals are generated from mathematical approach [5]. Can these precise mathematical abstractions be found in physical reality? Yes, it is optical fractals [3]. This concept includes "diffractals" (diffraction pattern on fractal lattice) [6, 7], eigen modes of unstable resonators [8], distributions in nonlinear optics [3, 9].

Particularly interesting can be the coincidence of certain properties of "accurate" and "statistical" fractals [10], such as aerosols, smoke, moire [11-13], which is very important applied to optical signal transmission through a heterogeneous or random medium [14-17]. Examination of diffraction on fractal lattice [6, 7, 18-20] can solve other important problems - the formation of periodically self-reproducing fields [21-26], the creation of multi focus [27-30] or specified longitudinal distributions [31-33], and in achromatic depicting systems [34-37].

One of the most important characteristics of fractals is the spatial spectrum [38-41], which are also important in the analysis of crystal structures [42-44]. Taking into account possible multidimensionality of fractals, the calculation of the spatial spectrum can lead to problems associated with computational complexity, which depends on the technical capabilities of modern computers. The solution to the problem can be the usage of the fast calculation algorithm. Within this paper, the fast transformation is used to develop the spatial spectrum of multidimensional fractals with different characteristics.

2. The calculation of the spatial spectrum of multidimensional fractals

The first stage of the modeling is the implementation of a one-dimensional case. We take a unit segment $E_0 = [0,1]$. The next segment is formed according to the rule $E_i = [0,a_i] \cup [b_i,1]$, where $a$ and $b$ are the fractal parameters specified in the range of $(0,1)$, whereby $a < b$ and $a+b = 1$. We continue until reaching the desired order of the fractal. The intersection of all segments will be a simulated fractal.

$$E = \bigcap_{i=1}^{n} E_i, \quad \text{(1)}$$

where $n$ is the order of the fractal.

If the parameters are set to $a = \frac{1}{3}$ and $b = \frac{2}{3}$, then we get the Cantor set.

For programming is used a vector consisting of units, which is successively filled with zeros, according to input parameters and order.

To simulation for two-dimensional case, we used a similar implementation with some corrections. We took the unit square $E_0 = [0,1] \times [0,1]$ and the next one will take form of $E_i = (0,a_1] \cup [b_1,1] \times (0,a_2] \cup [b_2,1])$, where $a_1$, $a_2$, $b_1$ and $b_2$ are fractal parameters specified in the range of $(0,1)$, whereby $a_1 < b_1$, $a_2 < b_2$ and $a_1 + b_1 = 1$, $a_2 + b_2 = 1$. The simulated fractal can be found by the previously applied for the one-dimensional case formula (1). If we set the parameters $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{3}$, $b_1 = \frac{2}{3}$ and $b_2 = \frac{2}{3}$ we get a fractal called the Sierpinski carpet (Fig. 1).
The three-dimensional case is implemented reciprocally to the two-dimensional case. The unit cubes $E_1 = [0,1] \times [0,1] \times [0,1]$ and $E_2 = ([0,a_1] \cup [b_1,1]) \times ([0,a_2] \cup [b_2,1]) \times ([0,a_3] \cup [b_3,1])$ was taken, whereby $a_1$, $a_2$, $a_3$, $b_1$, $b_2$ and $b_3$ are fractal parameters specified in the range of $(0,1)$, whereby $a_1 < b_1$, $a_2 < b_2$, $a_3 < b_3$ and $a_1 + b_1 = 1$, $a_2 + b_2 = 1$, $a_3 + b_3 = 1$. If we set the parameters $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{3}$, $a_3 = \frac{1}{3}$, $b_1 = \frac{2}{3}$, $b_2 = \frac{2}{3}$ and $b_3 = \frac{2}{3}$ we get a three-dimensional fractal called Menger sponge (Fig. 2 a), the boundary section of which is a Sierpinsky carpet.

If we set the parameters $a_1 = \frac{1}{3}$, $a_2 = \frac{3}{8}$, $a_3 = \frac{1}{3}$, $b_1 = \frac{2}{3}$, $b_2 = \frac{5}{8}$ and $b_3 = \frac{2}{3}$ we get a scalable three-dimensional fractal (Fig. 3 a).

The Fast Fourier Transform was used to generate the spatial spectrum.

$$F(u) = \mathcal{F}[f(x)](u) = \int_{\mathbb{R}^3} f(x) \exp(-2\pi i u x) \, dx,$$

whereby $f(x)$ is the input function specified as a vector, which is a binary representation of the fractal, $F(u)$ is the output function, $\mathcal{F}[\cdot]$ is the Fourier transform operator.
The spatial spectrum was obtained from a two-dimensional fractal structure (Sierpinski carpet). The results for the different number of iterations and scale are presented in Table 1.

Table 1. Variability of the spectrum in relations to the number of iterations and scale.

<table>
<thead>
<tr>
<th>Number of iterations</th>
<th>Fractal Spectrum</th>
<th>Fractal Spectrum</th>
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<tbody>
<tr>
<td>2</td>
<td>![Image]</td>
<td>![Image]</td>
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<tr>
<td>3</td>
<td>![Image]</td>
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<tr>
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<td>5</td>
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As can be seen from the Table 1, with the number of iteration increasing, the spatial spectrum from the fractal structure becomes more complex and the energy at higher frequencies increases. However, the pattern of the spectrum maintains a regular structure, which is also characteristic of crystalline structures [42-44].

3. Conclusion

As a result of the work, the spatial spectrum was calculated and visualized from a two-dimensional (Sierpinski carpet) and a three-dimensional (Menger sponge) fractal structure using the Fast Fourier Transform algorithm.

Acknowledgements

The work was supported by the Ministry of Education and Science of the Russian Federation.

References


