The equidistribution of some vincular patterns
on 132-avoiding permutations

Vincent Vajnovszki
LE2I, Université Bourgogne Franche-Comté
Dijon, France
vvajnov@u-bourgogne.fr

Abstract. A pattern in a permutation $\pi$ is a sub-permutation of $\pi$, and this paper deals mainly with length three patterns. In 2012 Bóna showed the rather surprising fact that the cumulative number of occurrences of the patterns 231 and 213 are the same on the set of permutations avoiding 132, even though the pattern based statistics 231 and 213 do not have the same distribution on this set. Here we show that if it is required for the symbols playing the role of 1 and 3 in the occurrences of 231 and 213 to be adjacent, then the obtained statistics are equidistributed on the set of 132-avoiding permutations. Actually, expressed in terms of vincular patterns, we prove bijectively the following more general results: the statistics based on the patterns 231, 213 and 213, together with other statistics, have the same joint distribution on $S_n(132)$ of length $n$ permutations avoiding 132, and so do the patterns 231 and 312; and up to trivial transformations, these statistics are the only based on length-three proper (not classical nor consecutive) vincular patterns which are equidistributed on a set of permutations avoiding a classical length-three pattern.

1 Introduction

In [2] Barnabei, Bonetti and Silimbani showed the equidistribution of some length-three consecutive patterns involvement statistics on the set of permutations avoiding the classical pattern 312 (or equivalently, 132), and in [4] Bóna showed the surprising fact that the total number of occurrences of the patterns 231 and 213 is the same on the set of 132-avoiding permutations, despite the pattern based statistics 231 and 213 having different distribution on this set. In [6], Homberger, generalizing Bóna’s result, gave the total number of occurrences of each classical length-three pattern on the set of 123-avoiding permutations, and showed that the total number of occurrences of the pattern 231 is the same in the set of 123- and 132-avoiding permutations, despite the pattern based statistic 231 having different distribution on these two sets.

Vincular patterns, introduced by Babson and Steingrímsson [1], are a generalization of the notion of patterns where, for example, some entries are required to occur consecutively, and in [8] Mansour considered permutations avoiding 132 and containing various length-three vincular patterns exactly 0 or 1 times.
Motivated by these, Burnstein and Elizalde gave in [5], in a much more general context, the total number of occurrences of any vincular pattern of length three on 231-avoiding (or equivalently, 132-avoiding) permutations, and more recently, Baxter [3] gave algorithmic methods to efficiently compute several statistics over some pattern-avoiding permutations.

In this paper we show that, on the set of 132-avoiding permutations, the vincular pattern based statistics $231$, $213$ and $213$ are equidistributed, and so are $231$ and $312$; and numerical evidence shows that, up to trivial transformations, these patterns are the only length-three proper (not classical nor consecutive) vincular patterns equidistributed on a set of permutations avoiding a classical length-three pattern.

It is worth to mention that, on the set of unrestricted permutations, the statistics $231$ and $312$ are trivially equidistributed, and so are $231$ and $213$ (which is all but obvious on 132-avoiding permutations), and this last distribution is different from that of $213$.

More precisely, in this paper we show bijectively the equidistribution on 132-avoiding permutations of the tuples of statistics

- $(231, 213, rlmin, rlmax)$ and $(213, 231, rlmax, rlmin)$,
- $(231, des)$ and $(213, des)$,
- $(213, des, 12)$ and $(213, des, 12)$,
- $(231, 312, des)$ and $(312, 231, des)$,

where $rlmax$, $rlmin$ and $des$ are respectively, the number of right-to-left maxima, right-to-left minima and descents. The corresponding bijections (the last of them being straightforward) are presented in Section 3.

2 Notations and definitions

A permutation of length $n$ is a bijection from the set $\{1, 2, \ldots, n\}$ to itself and we write permutations in one-line notation, that is, as words $\pi = \pi_1 \pi_2 \ldots \pi_n$, where $\pi_i$ is the image of $i$ under $\pi$. We let $S_n$ denote the set of permutations of length $n$.

2.1 Permutation patterns

Let $\sigma \in S_k$ and $\pi = \pi_1 \pi_2 \ldots \pi_n \in S_n$, $k \leq n$, be two permutations. One says that $\sigma$ occurs as a (classical) pattern in $\pi$ if there is a sequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$ is order-isomorphic with $\sigma$. For example, $231$ occurs as a pattern in $13452$, and the three occurrences of it are $342$, $352$ and $452$.

Vincular patterns were introduced in [1] and they were extensively studied since then (see Chapter 7 in [7] for a comprehensive description of results on these patterns). Vincular patterns generalize classical patterns and they are defined as follows:
Any pair of two adjacent letters may now be underlined, which means that
the corresponding letters in the permutation must be adjacent. (The original
notation for vincular patterns uses dashes: the absence of a dash between
two letters of a pattern means that these letters are adjacent in the permuta-
tion.) For example, the pattern 213 occurs in the permutation 425163 four
times, namely, as the subsequences 425, 416, 216 and 516. Note that, the
subsequences 426 and 213 are not occurrences of the pattern because their
last two letters are not adjacent in the permutation.

If a pattern begins (resp., ends) with a hook then its occurrence is required
to begin (resp., end) with the leftmost (resp., rightmost) letter in the per-
mutation. (In the original notation the role of hooks was played by square
brackets.) For example, there are two occurrences of the pattern 213
in the permutation 425163, which are the subsequences 425 and 416.

We denote by $S_n(\sigma)$ the set of permutations in $S_n$ avoiding the pattern $\sigma$.

2.2 Statistics

A statistic on a set of permutations is simply a function from the set to $\mathbb{N}$. A
classical example of statistic on $S_n$ is the descent number

$$\text{des } \pi = \text{card}\{i : 1 \leq i < n, \pi_i > \pi_{i+1}\},$$

for example $\text{des } 45312 = 2$.

In a permutation $\pi = \pi_1\pi_2\ldots\pi_n$, $\pi_i$ is a right-to-left maximum if $\pi_i > \pi_j$
for all $j > i$; and the number of right-to-left maxima of $\pi$ is denoted by $\text{rmax } \pi$.
Similarly, $\pi_i$ is a right-to-left minimum if $\pi_i < \pi_j$ for all $j > i$; and the number
of right-to-left minima of $\pi$ is denoted by $\text{rmin } \pi$. Both, $\text{rmax}$ and $\text{rmin}$ are
statistics on $S_n$.

For a set of permutations $S$, two statistics $ST$ and $ST'$ have the same distri-
bution (or are equidistributed) on $S$ if, for any $k$,

$$\text{card}\{\pi \in S : ST \pi = k\} = \text{card}\{\pi \in S : ST' \pi = k\},$$

and the tuples of statistics, or multistatistics, $(ST_1, ST_2, \ldots, ST_p)$ and $(ST'_1, ST'_2, \ldots, ST'_p)$
have the same distribution if, for any $p$-tuple $k = (k_1, k_2, \ldots, k_p)$,

$$\text{card}\{\pi \in S : (ST_1, ST_2, \ldots, ST_p) \pi = k\} = \text{card}\{\pi \in S : (ST'_1, ST'_2, \ldots, ST'_p) \pi = k\}.$$

For a permutation $\pi$ and a (vincular) patterns $\sigma$ we denote by $(\sigma) \pi$ the
number of occurrences of this pattern in $\pi$, and $(\sigma)$ becomes a permutation
statistic. For example, $(21) \pi$ is $\text{des } \pi$; $(21) \pi$ is the inverse number of $\pi$; and
$(12) \pi$ is the last value of $\pi$ minus one. Similarly, for a set of (vincular) patterns
$\{\sigma, \tau, \ldots\}$, we denote by $(\sigma + \tau + \cdots) \pi$ the number of occurrences of these patterns
in $\pi$. 
3 The main results

Our main results are stated in the following theorems.

**Theorem 1** There is a bijection \( \phi \) from \( S_n(132) \) to itself such that if \( \pi \in S_n(132) \), then

\[
(213, 231, \text{rlmin, rlmax}) \phi(\pi) = (231, 213, \text{rlmax}, \text{rlmin}) \pi.
\]

**Theorem 2** There is a bijection \( \psi \) from \( S_n(132) \) to itself such that if \( \pi \in S_n(132) \), then

\[
(213, \text{des}) \psi(\pi) = (231, \text{des}) \pi.
\]

**Theorem 3** There is a bijection \( \mu \) from \( S_n(132) \) to itself such that if \( \pi \in S_n(132) \), then

\[
(213, \text{des}, 12] \mu(\pi) = (213, \text{des}, 12] \pi.
\]

**Theorem 4** If \( \pi \in S_n(132) \), then \((231, 312, \text{des}) \pi^{-1} = (312, 231, \text{des}) \pi\).

All our bijections are constructive and based essentially on the recursive decomposition of 132-avoiding permutations.

![Decomposition Diagram](image)

**Fig. 1.** The decomposition: (1) \( \pi = (\alpha \oplus 1) \ominus \beta \), and (2) \( \pi = \alpha \ominus (\beta \oplus 1) \) of \( \pi \in S_n(132) \), \( n \geq 1 \); and (3) the recursive definition of \( \phi(\pi) \).

4 Conclusions

We showed bijectively the joint equidistribution on the set \( S_n(132) \) of 132-avoiding permutations of some length-three vincular patterns together with other statistics. In particular, for the sets of vincular patterns \{231, 213, 231\} and \{231, 312\}, we showed that the patterns within each set are equidistributed.
on $S_n(132)$. By applying permutation symmetries, other similar results can be derived. For instance, from the equidistribution of $213$ and $213$ on $S_n(132)$ (belonging to the first set, see Subsection 3.3) it follows, by applying

- the reverse operation, the equidistribution of $312$ and $312$ on $S_n(231)$,
- the complement operation, the equidistribution of $231$ and $231$ on $S_n(312)$, and
- the complement and the reverse operations (in any order), the equidistribution of $132$ and $132$ on $S_n(213)$.

Moreover, computer experiments show that, up to these two symmetries, the patterns in \{231, 213, 213\} and those in \{231, 312\} are the only length-three proper (not classical nor consecutive) vincular patterns which are equidistributed on a set of permutations avoiding a classical length-three pattern.

References