The equidistribution of some vincular patterns on 132-avoiding permutations

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Abstract. A pattern in a permutation π is a sub-permutation of π, and this paper deals mainly with length three patterns. In 2012 Bóna showed the rather surprising fact that the cumulative number of occurrences of the patterns 231 and 213 are the same on the set of permutations avoiding 132, even though the pattern based statistics 231 and 213 do not have the same distribution on this set. Here we show that if it is required for the symbols playing the role of 1 and 3 in the occurrences of 231 and 213 to be adjacent, then the obtained statistics are equidistributed on the set of 132-avoiding permutations. Actually, expressed in terms of vincular patterns, we prove bijectively the following more general results: the statistics based on the patterns $2\underline{31}$, $2\underline{13}$ and $\underline{21}3$, together with other statistics, have the same joint distribution on $S_n(132)$ of length n permutations avoiding 132, and so do the patterns $\underline{23}1$ and $3\underline{12}$; and up to trivial transformations, these statistics are the only based on length-three proper (not classical nor consecutive) vincular patterns which are equidistributed on a set of permutations avoiding a classical length-three pattern.

1 Introduction

In [2] Barnabei, Bonetti and Silimbani showed the equidistribution of some length-three consecutive patterns involvement statistics on the set of permutations avoiding the classical pattern 312 (or equivalently, 132), and in [4] Bóna showed the surprising fact that the total number of occurrences of the patterns 231 and 213 is the same on the set of 132-avoiding permutations, despite the pattern based statistics 231 and 213 having different distribution on this set. In [6], Homberger, generalizing Bóna's result, gave the total number of occurrences of each classical length-three pattern on the set of 123-avoiding permutations, and showed that the total number of occurrences of the pattern 231 is the same in the set of 123- and 132-avoiding permutations, despite the pattern based statistic 231 having different distribution on these two sets.

Vincular patterns, introduced by Babson and Steingrímsson [1], are a generalization of the notion of patterns where, for example, some entries are required to occur consecutively, and in [8] Mansour considered permutations avoiding 132 and containing various length-three vincular patterns exactly 0 or 1 times.

Motivated by these, Burnstein and Elizalde gave in [5], in a much more general context, the total number of occurrences of any vincular pattern of length three on 231-avoiding (or equivalently, 132-avoiding) permutations, and more recently, Baxter [3] gave algorithmic methods to efficiently compute several statistics over some pattern-avoiding permutations.

In this paper we show that, on the set of 132-avoiding permutations, the vincular pattern based statistics $2\underline{31}$, $2\underline{13}$ and $\underline{21}3$ are equidistributed, and so are $\underline{23}1$ and $3\underline{12}$; and numerical evidence shows that, up to trivial transformations, these patterns are the only length-three proper (not classical nor consecutive) vincular patterns equidistributed on a set of permutations avoiding a classical length-three pattern.

It is worth to mention that, on the set of unrestricted permutations, the statistics $\underline{23}1$ and $3\underline{12}$ are trivially equidistributed, and so are $2\underline{31}$ and $2\underline{13}$ (which is all but obvious on 132-avoiding permutations), and this last distribution is different from that of $\underline{21}3$.

More precisely, in this paper we show bijectively the equidistribution on 132-avoiding permutations of the tuples of statistics

- $(2\underline{31}, 2\underline{13}, \mathsf{rlmin}, \mathsf{rlmax})$ and $(2\underline{13}, 2\underline{31}, \mathsf{rlmax}, \mathsf{rlmin})$,
- $(2\underline{31}, \mathsf{des})$ and $(2\underline{13}, \mathsf{des})$,
- $(\underline{21}3, \mathsf{des}, 1\underline{2}\rfloor)$ and $(2\underline{13}, \mathsf{des}, 1\underline{2}\rfloor)$,
- $(\underline{23}1, 3\underline{12}, \mathsf{des})$ and $(3\underline{12}, \underline{23}1, \mathsf{des})$,

where $\mathsf{rlmax}, \mathsf{rlmin}$ and des are respectively, the number of right-to-left maxima, right-to-left minima and descents. The corresponding bijections (the last of them being straightforward) are presented in Section 3.

2 Notations and definitions

A *permutation* of length n is a bijection from the set $\{1, 2, \dots, n\}$ to itself and we write permutations in *one-line notation*, that is, as words $\pi = \pi_1\pi_2\dots\pi_n$, where π_i is the image of i under π. We let S_n denote the set of permutations of length n.

2.1 Permutation patterns

Let $\sigma \in S_k$ and $\pi = \pi_1\pi_2\dots\pi_n \in S_n$, $k \leq n$, be two permutations. One says that σ occurs as a (classical) pattern in π if there is a sequence $1 \leq i_1 < i_2 < \dots < i_k \leq n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ is order-isomorphic with σ. For example, 231 occurs as a pattern in 13452, and the three occurrences of it are 342, 352 and 452.

Vincular patterns were introduced in [1] and they were extensively studied since then (see Chapter 7 in [7] for a comprehensive description of results on these patterns). Vincular patterns generalize classical patterns and they are defined as follows:

– Any pair of two adjacent letters may now be underlined, which means that the corresponding letters in the permutation must be adjacent. (The original notation for vincular patterns uses dashes: the absence of a dash between two letters of a pattern means that these letters are adjacent in the permutation.) For example, the pattern $2\underline{13}$ occurs in the permutation 425163 four times, namely, as the subsequences 425, 416, 216 and 516. Note that, the subsequences 426 and 213 are *not* occurrences of the pattern because their last two letters are not adjacent in the permutation.
– If a pattern begins (resp., ends) with a hook then its occurrence is required to begin (resp., end) with the leftmost (resp., rightmost) letter in the permutation. (In the original notation the role of hooks was played by square brackets.) For example, there are two occurrences of the pattern $\lfloor 2\underline{13}$ in the permutation 425163, which are the subsequences 425 and 416.

We denote by $S_n(\sigma)$ the set of permutations in S_n avoiding the pattern σ.

2.2 Statistics

A *statistic* on a set of permutations is simply a function from the set to $\mathbb{N}$. A classical example of statistic on S_n is the descent number

$$\mathsf{des}\,\pi = card\,\{i \;:\; 1 \leq i < n, \pi_i > \pi_{i+1}\},$$

for example $\mathsf{des}\,45312 = 2$.

In a permutation $\pi = \pi_1\pi_2 \ldots \pi_n$, π_i is a *right-to-left maximum* if $\pi_i > \pi_j$ for all $j > i$; and the number of right-to-left maxima of π is denoted by $\mathsf{rlmax}\,\pi$. Similarly, π_i is a *right-to-left minimum* if $\pi_i < \pi_j$ for all $j > i$; and the number of right-to-left minima of π is denoted by $\mathsf{rlmin}\,\pi$. Both, rlmax and rlmin are statistics on S_n.

For a set of permutations S, two statistics ST and ST' have the same distribution (or are equidistributed) on S if, for any k,

$$\text{card}\{\pi \in S : \mathsf{ST}\,\pi = k\} = \text{card}\{\pi \in S : \mathsf{ST}'\,\pi = k\},$$

and the tuples of statistics, or multistatistics, $(\mathsf{ST}_1, \mathsf{ST}_2, \ldots, \mathsf{ST}_p)$ and $(\mathsf{ST}'_1, \mathsf{ST}'_2, \ldots, \mathsf{ST}'_p)$ have the same distribution if, for any p-tuple $k = (k_1, k_2, \ldots, k_p)$,

$$\text{card}\{\pi \in S : (\mathsf{ST}_1, \mathsf{ST}_2, \ldots, \mathsf{ST}_p)\,\pi = k\} = \text{card}\{\pi \in S : (\mathsf{ST}'_1, \mathsf{ST}'_2, \ldots, \mathsf{ST}'_p)\,\pi = k\}.$$

For a permutation π and a (vincular) patterns σ we denote by $(\sigma)\,\pi$ the number of occurrences of this pattern in π, and (σ) becomes a permutation statistic. For example, $(\underline{21})\,\pi$ is $\mathsf{des}\,\pi$; $(21)\,\pi$ is the inverse number of π; and $(12\rfloor)\pi$ is the last value of π minus one. Similarly, for a set of (vincular) patterns $\{\sigma, \tau, \ldots\}$, we denote by $(\sigma{+}\tau{+}\cdots)\,\pi$ the number of occurrences of these patterns in π.

3 The main results

Our main results are stated in the following theorems.

Theorem 1 *There is a bijection* ϕ *from* $S_n(132)$ *to itself such that if* $\pi \in S_n(132)$*, then*

$$(2\underline{13}, 2\underline{31}, \mathsf{rlmin}, \mathsf{rlmax})\, \phi(\pi) = (2\underline{31}, 2\underline{13}, \mathsf{rlmax}, \mathsf{rlmin})\, \pi.$$

Theorem 2 *There is a bijection* ψ *from* $S_n(132)$ *to itself such that if* $\pi \in S_n(132)$*, then*

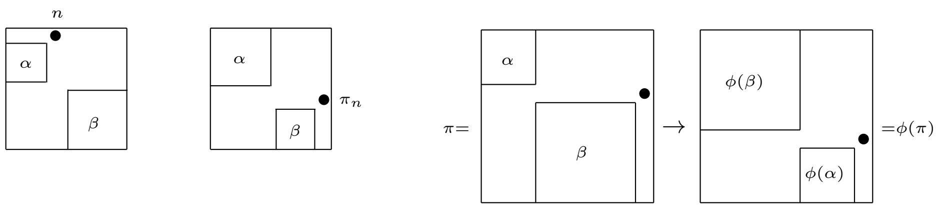
$$(2\underline{13}, \mathsf{des})\, \psi(\pi) = (2\underline{31}, \mathsf{des})\, \pi.$$

Theorem 3 *There is a bijection* μ *from* $S_n(132)$ *to itself such that if* $\pi \in S_n(132)$*, then*

$$(\underline{21}3, \mathsf{des}, 12\rfloor)\, \mu(\pi) = (2\underline{13}, \mathsf{des}, 12\rfloor)\, \pi.$$

Theorem 4 *If* $\pi \in S_n(132)$*, then* $(\underline{23}1, 3\underline{12}, \mathsf{des})\, \pi^{-1} = (3\underline{12}, \underline{23}1, \mathsf{des})\, \pi$.

All our bijections are constructive and based essentially on the recursive decomposition of 132-avding permutations.



(1) (2) (3)

Fig. 1. The decomposition: (1) $\pi = (\alpha \oplus 1) \ominus \beta$, and (2) $\pi = \alpha \ominus (\beta \oplus 1)$ of $\pi \in S_n(132)$, $n \geq 1$; and (3) the recursive definition of $\phi(\pi)$.

4 Conclusions

We showed bijectively the joint equidistribution on the set $S_n(132)$ of 132-avoiding permutations of some length-three vincular patterns together with other statistics. In particular, for the sets of vincular patterns $\{2\underline{31}, 2\underline{13}, \underline{21}3\}$ and $\{\underline{23}1, 3\underline{12}\}$, we showed that the patterns within each set are equidistributed on $S_n(132)$. By applying permutation symmetries, other similar results can be derived. For instance, from the equidistribution of $\underline{21}3$ and $2\underline{13}$ on $S_n(132)$ (belonging to the first set, see Subsection 3.3) it follows, by applying

- the reverse operation, the equidistribution of $3\underline{12}$ and $\underline{31}2$ on $S_n(231)$,
- the complement operation, the equidistribution of $\underline{23}1$ and $2\underline{31}$ on $S_n(312)$, and
- the complement and the reverse operations (in any order), the equidistribution of $1\underline{32}$ and $\underline{13}2$ on $S_n(213)$.

Moreover, computer experiments show that, up to these two symmetries, the patterns in $\{2\underline{31}, 2\underline{13}, \underline{21}3\}$ and those in $\{\underline{23}1, 3\underline{12}\}$ are the only length-three proper (not classical nor consecutive) vincular patterns which are equidistributed on a set of permutations avoiding a classical length-three pattern.

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