An S4F-related monotonic modal logic

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Abstract. This paper introduces a novel monotonic modal logic, allowing us to capture the nonmonotonic variant of the modal logic **S4F**: we add a second new modal operator into the original language of **S4F**, and show that the resulting formalism is strong enough to characterise the *logical consequence* of (nonmonotonic) **S4F**, as well as its *minimal model* semantics. The latter underlies major forms of nonmonotonic logic, among which are (reflexive) autoepistemic logic, default logic, and nonmonotonic logic programming. The paper ends with a discussion of a general strategy, naturally embedding several nonmonotonic logics of a similar floor structure on which a (maximal) *cluster* sits.

Keywords: nonmonotonic **S4F**, minimal model semantics, monotonic modal logic

1 Introduction

The use of monotonic modal logics for describing nonmonotonic inference has a long tradition in Artificial Intelligence. There exists a considerable amount of research in the literature [1,2,3,4,5,6,7,8], logically capturing important forms of nonmonotonic reasoning. Theoretically, we obtain a clear and simple monotonic framework for studying further language extensions and possible generalisations. From a practical point of view, we can check nonmonotonic deduction with a validity proving procedure in a corresponding monotonic setting.

The modal logic **S4F** (aka, **S4.3.2**) is obtained from **S4** by adding the axiom schema

$$\mathrm{F}: (\varphi \wedge \mathsf{M}\mathsf{L}\psi) \rightarrow \mathsf{L}(\mathsf{M}\varphi \vee \psi)$$

[9] in which L is the epistemic modal operator, and M is its dual, defined by $\neg\mathsf{L}\neg$. A first and detailed investigation of this logic was given in [10]; yet in time, **S4F** has also found interesting theoretical applications in Knowledge Representation [11,12,13,14,15,16,17].

S4F is characterised by the class of Kripke models $(W, \mathcal{T}, V)$ in which $W = W_1 \cup W_2$ for some disjoint sets W_1 and W_2 such that W_2 is nonempty. Moreover, $x\mathcal{T}y$ if and only if $y \in W_2$ or $x \in W_1$. V is the *valuation* function such that $V(x)$ is a set of propositional

* I am grateful to Luis Fariñas del Cerro, Andreas Herzig, and Levan Uridia for motivation, and the anonymous reviewers for their useful comments. This paper has been supported by the institution "*Fundação para a Ciência e a Tecnologia (FCT)*", and the research unit "*Centro de Matemática, Aplicações Fundamentais e Investigação Operacional (CMAF-CIO)*" of the University of Lisbon, Portugal via the grant, identified by the number "UID/MAT/04561/2013".

variables for every $x \in W$. A *cluster* is simply a trivial **S5** model $(C, \mathcal{T}, V)$ such that $x\mathcal{T}y$ for every $x, y \in C$. In terms of Kripke semantics, **S5** is the modal logic, characterised by models in which the accessibility relation is an equivalence relation: it is reflexive, symmetric, and transitive. Now, we can alternatively identify an **S4F** model with the ordered triple (C_1, C_2, V) in which C_1 and C_2 are disjoint cluster structures, $C_2 \neq \emptyset$, and any world in C_2 can be accessed from every world in C_1.

This paper follows a similar approach to [18] and [8]: the former captures the reflexive autoepistemic reasoning [19,20] of nonmonotonic **SW5** [21,22,23]. The latter successfully embeds equilibrium logic [24,25], which is a logical foundation for answer set programming (**ASP**) [26,27,28], into a monotonic bimodal logic called **MEM**. All these works are, in essence, parts of a project that aims to reexamine the logical and mathematical foundations of nonmonotonic logics. The overall project will then culminate in a single monotonic modal framework, enabling us to obtain a unified perspective of various forms of nonmonotonic reasoning.

As a reference to the analogy between all such works, we here keep the same symbols $\mathcal{T}$ and $\mathcal{S}$[1] with [8,18] for the accessibility relations. Roughly speaking, [8,18] and this paper all propose Kripke models, composed of a union of 2-floor (disjoint) structures. In general, while the relation $\mathcal{T}$ helps access from '`bottom`' (*first floor*) to '`top`' (*second floor*), the relation $\mathcal{S}$ works in the opposite direction. However, the structures of `bottom` and `top` differ in all formalisms. In particular, the models here and in [18] are respectively the extensions of the Kripke models of **S4F** and **SW5** with the $\mathcal{S}$-relation; whereas **MEM** restricts `top` to a trivial cluster of a singleton, and forces all subsets of the `top` valuation to appear inside the `bottom` structure to check the minimality criterion of *equilibrium models* [24,25]. Similarly to [8,18], we also propose here a modal language $\mathcal{L}_{[T],[S]}$ with two (unary) modal operators, namely [T] and [S]. The former is a direct translation of L in the language of **S4F** ($\mathcal{L}_{\mathbf{S4F}}$) into $\mathcal{L}_{[T],[S]}$ via a mapping '$tr : \mathcal{L}_{\mathbf{S4F}} \longrightarrow \mathcal{L}_{[T],[S]}$'. The relations $\mathcal{T}$ and $\mathcal{S}$ respectively interpret the modal operators [T] and [S]. We call the resulting monotonic formalism **MLF**. We then add into **MLF** the *negatable axiom*, resulting in **MLF***: *modal logic of nonmonotonic S4F*. The negatable axiom ensures that the cluster C_1 (`bottom`) of **MLF** frames is nonempty, so it turns our frames into exactly 2-floor structures in **MLF***: both floors are maximal clusters w.r.t. the relation $\mathcal{T}$. Essentially, this axiom enables us to refute any nontautology of $\mathcal{L}_{[T],[S]}$ as it allows us to have all possible valuations in an **MLF*** model. Thus, we show that the formula $\langle T \rangle [T](\varphi \wedge [S]\neg\varphi)$ characterises maximal φ-clusters in **MLF***. This result paves the way to our final goal in which we capture nonmonotonic consequence (abbreviated '$\approx_{\mathbf{S4F}}$') of **S4F** in the monotonic modal logic **MLF***:

$$\varphi \approx_{\mathbf{S4F}} \psi \quad \text{if and only if} \quad [T](tr(\varphi) \wedge [S]\neg tr(\varphi)) \rightarrow [T]tr(\psi) \text{ is valid in } \mathbf{MLF}^*.$$

The rest of the paper is organised as follows. Section 2 introduces the monotonic modal logic **MLF**: we first define its bimodal language, and then propose two classes of frames, namely **K** and **F**. They are respectively based on standard Kripke frames, and the cluster-based component frames, which are in the form of a floor structure. We axiomatise the validities of our logic, and finally prove that **MLF** is sound and complete

[1] The symbols $\mathcal{T}$ and $\mathcal{S}$ of [8] respectively refer to '***Top***' and '***Subset***'. However, the relation $\mathcal{S}$ has a different character and meaning in this paper, which is similar to those of [18].

w.r.t. both semantics. In Section 3, we extend **MLF** with the negatable axiom, and call the resulting logic **MLF***. We introduce two kinds of model structures, **K*** and **F***, and end with the soundness and completeness results. Section 3.1 recalls minimal model semantics of nonmonotonic **S4F**: we define the preference relation, and then give the definition of a minimal model for **S4F**. Section 3.2 first captures minimal models of **S4F**, and then embeds the consequence relation of **S4F** into **MLF***. Section 4 discusses a general approach, allowing us to capture major nonmonotonic logics. Section 5 makes a brief overview of this paper, and mentions our future goals.

2 A monotonic modal logic related to nonmonotonic S4F

We here propose a new formalism called **MLF**, which is closely associated with **S4F**.

2.1 Language ($\mathcal{L}_{[T],[S]}$)

Throughout the paper we suppose $\mathbb{P}$ an infinite set of propositional variables, and $\mathbb{P}_\varphi$ its restriction to those of a formula φ. We also consider *Prop* as the set of all propositional formulas of our language. The language $\mathcal{L}_{[T],[S]}$ is formally defined by the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \to \varphi \mid [T]\varphi \mid [S]\varphi$$

where p ranges over $\mathbb{P}$. $\mathcal{L}_{[T],[S]}$ is therefore a bimodal language with the modalities [T] and [S]. As usual, $\top \overset{\text{def}}{=} \varphi \to \varphi$, $\bot \overset{\text{def}}{=} \neg(\varphi \to \varphi)$, $\varphi \vee \psi \overset{\text{def}}{=} \neg\varphi \to \psi$, $\varphi \wedge \psi \overset{\text{def}}{=} \neg(\varphi \to \neg\psi)$, and $\varphi \leftrightarrow \psi \overset{\text{def}}{=} (\varphi \to \psi) \wedge (\psi \to \varphi)$. Moreover, $\langle T\rangle\varphi$ and $\langle S\rangle\varphi$ respectively abbreviate $\neg[T]\neg\varphi$ and $\neg[S]\neg\varphi$.

2.2 Kripke semantics for MLF

We now describe the class **K** of Kripke frames for **MLF**. A **K**-*frame* is a triple $(W, \mathcal{T}, \mathcal{S})$:

- W is a non-empty set of possible worlds.
- $\mathcal{T}, \mathcal{S} \subseteq W \times W$ are binary relations such that for every $w, u, v \in W$,

$(w,w) \in \mathcal{T}$	refl($\mathcal{T}$)
$(w,u) \in \mathcal{T}$ and $(u,v) \in \mathcal{T} \Rightarrow (w,v) \in \mathcal{T}$	trans($\mathcal{T}$)
$(w,u) \in \mathcal{T}$, $(u,w) \notin \mathcal{T}$ and $(w,v) \in \mathcal{T} \Rightarrow (v,u) \in \mathcal{T}$	f($\mathcal{T}$)
$(w,u) \in \mathcal{S} \Rightarrow (u,u) \in \mathcal{S}$	refl$_2$($\mathcal{S}$)
$(w,u) \in \mathcal{S}$ and $(u,v) \in \mathcal{S} \Rightarrow u = v$	wtriv$_2$($\mathcal{S}$)
$(w,u) \in \mathcal{T} \Rightarrow (u,w) \in \mathcal{T}$ or $(u,w) \in \mathcal{S}$	msym($\mathcal{T},\mathcal{S}$)
$(w,u) \in \mathcal{S} \Rightarrow w = u$ or $(u,w) \in \mathcal{T}$	wmsym($\mathcal{S},\mathcal{T}$).

The first three properties above characterise the frames of the modal logic **S4F** [9]. Thus, a **K**-frame is an extension of an **S4F** frame by a second relation $\mathcal{S}$. Given a **K**-frame $\mathcal{F} = (W, \mathcal{T}, \mathcal{S})$, a **K**-model is a pair $\mathcal{M} = (\mathcal{F}, V)$ in which $V : W \to 2^{\mathbb{P}}$ is the map, assigning to each $w \in W$ a valuation $V(w)$. Then, given $w \in W$, a pointed **K**-model is a pair $\mathcal{M}_w = (\mathcal{M}, w)$, and similarly, a pointed **K**-frame is a pair $\mathcal{F}_w = (\mathcal{F}, w)$.

Truth conditions The truth conditions are standard: (for $p \in \mathbb{P}$)

$$\begin{array}{lll} \mathcal{M}, w \models_{\mathbf{MLF}} p & \text{if} & p \in V(w); \\ \mathcal{M}, w \models_{\mathbf{MLF}} \neg\varphi & \text{if} & \mathcal{M}, w \not\models_{\mathbf{MLF}} \varphi; \\ \mathcal{M}, w \models_{\mathbf{MLF}} \varphi \to \psi & \text{if} & \mathcal{M}, w \not\models_{\mathbf{MLF}} \varphi \text{ or } M, w \models_{\mathbf{MLF}} \psi; \\ \mathcal{M}, w \models_{\mathbf{MLF}} [\mathrm{T}]\varphi & \text{if} & \mathcal{M}, u \models_{\mathbf{MLF}} \varphi \text{ for every } u \text{ such that } w\mathcal{T}u; \\ \mathcal{M}, w \models_{\mathbf{MLF}} [\mathrm{S}]\varphi & \text{if} & \mathcal{M}, u \models_{\mathbf{MLF}} \varphi \text{ for every } u \text{ such that } w\mathcal{S}u. \end{array}$$

We say that φ is **MLF** *satisfiable* if $\mathcal{M}, w \models_{\mathbf{MLF}} \varphi$ for some **K**-model $\mathcal{M}$ and w in $\mathcal{M}$. Moreover, φ is **MLF** *valid* (for short, $\models_{\mathbf{MLF}} \varphi$) if $\mathcal{M}, w \models_{\mathbf{MLF}} \varphi$ for every w of every **K**-model $\mathcal{M}$. Then, φ is valid in $\mathcal{M}$ ($\mathcal{M} \models_{\mathbf{MLF}} \varphi$) when $\mathcal{M}, w \models_{\mathbf{MLF}} \varphi$ for every w in $\mathcal{M}$.

2.3 Cluster-based floor semantics for MLF

We here define the frames of a floor structure for **MLF**, and call their class **F**. The underlying idea is due to the property 'f($\mathcal{T}$)' of **K**-frames, and in fact, **F** is only a subclass of **K**. However, **F**-frames with some additional properties play an important role in the completeness proof. We now start with the definition of a $\mathcal{T}$-cluster[2] [22,29].

Definition 1. *Given a* **K**-*frame* $(W, \mathcal{T}, \mathcal{S})$*, let* C *be a subset of* W*. Then,*

- *C is a $\mathcal{T}$-cluster if $w\mathcal{T}u$ for every $w, u \in C$;*
- *C is maximal if no proper superset of C in W is a $\mathcal{T}$-cluster.*
- *C is a $\mathcal{T}$-cone if for every $w \in W$, and every $u \in C$, $u\mathcal{T}w$ implies $w \in C$;*
- *C is final if $w\mathcal{T}u$ for every $w \in W$ and every $u \in C$.*

It follows from Definition 1 that the restriction of $\mathcal{T}$ to a $\mathcal{T}$-cluster C (abbreviated $\mathcal{T}|_C$) is a *universal* relation, viz. $\mathcal{T}|_C = C \times C$. So, $(C, \mathcal{T})$ happens to be a trivial **S5** frame.

Given a **K**-frame $\mathcal{F} = (W, \mathcal{T}, \mathcal{S})$, the relation $\mathcal{T}$ partitions $\mathcal{F}$ into disjoint subframes $\mathcal{F}' = (W', \mathcal{T}, \mathcal{S})$ in which $W' = C_1 \cup C_2$ for some maximal clusters $C_1, C_2 \subseteq W' \subseteq W$ such that $C_1 \cap C_2 = \emptyset$, and $C_2 \neq \emptyset$ is a final cone in W'. Thus, $\mathcal{T}|_{W'} = (W' \times C_2) \cup (C_1 \times C_1)$. We now define the operators $\mathcal{T}(\cdot), \mathcal{S}(\cdot) : 2^W \longrightarrow 2^W$, respectively assigning to every $X \subseteq W$,

$$\mathcal{T}(X) = \{u \in W \ : \ w\mathcal{T}u \text{ for some } w \in X\};$$
$$\mathcal{S}(X) = \{u \in W \ : \ w\mathcal{S}u \ \text{ for some } w \in X\}.$$

When X=$\{w\}$, we simply write $\mathcal{T}(w)$ (resp. $\mathcal{S}(w)$), denoting the set of all worlds that w can access via $\mathcal{T}$ (resp. $\mathcal{S}$). Note that $\mathcal{T}(\cdot)$ and $\mathcal{S}(\cdot)$ are homomorphisms under union:

$$\mathcal{T}(X \cup Y) = \mathcal{T}(X) \cup \mathcal{T}(Y) \quad \text{and} \quad \mathcal{S}(X \cup Y) = \mathcal{S}(X) \cup \mathcal{S}(Y).$$

We now formally define the above-mentioned partitions of a **K**-frame w.r.t. $\mathcal{T}$.

Definition 2. *Given a* **K**-*frame* $\mathcal{F} = (W, \mathcal{T}, \mathcal{S})$*, let* $\mathbb{C} = (C_1, C_2)$ *be a pair of disjoint subsets of* W *such that* $C_2 \neq \emptyset$*. Then,* $\mathbb{C}$ *is called a* component *of* $\mathcal{F}$ *if:*

1. *C_1 and C_2 are maximal clusters;*

[2] Unless specified otherwise, any definition of this paper is given w.r.t. the relation $\mathcal{T}$.

2. $\mathcal{T} \cap (C_1 \times C_2) = C_1 \times C_2$.

So, a component $\mathbb{C} = (C_1, C_2)$ has a 'two-layered' structure: C_1 is the *first floor* ('*F1-cluster*'), and C_2 is the *second floor* ('*F2-cluster*'). Clearly, C_2 is the final cone of the structure $\mathbb{C}$. Note that $\mathbb{C}$ can also be transformed into a special **K**-frame

$$\mathcal{F}^{\mathbb{C}} = \Big(C_1 \cup C_2,\ \ ((C_1 \cup C_2) \times C_2) \cup (C_1 \times C_1),\ \ (C_2 \times C_1) \cup \varDelta_{C_1}\Big) \tag{1}$$

where $\varDelta_{C_1}$ is the diagonal of $C_1 \times C_1$, i.e., $\varDelta_{C_1} = \{(w,w)\ :\ w \in C_1\}$. Given any two different components $\mathbb{C} = (C_1, C_2)$ and $\mathbb{C}' = (C_1', C_2')$ of a **K**-frame $\mathcal{F}$, $C_1 \cup C_2$ and $C_1' \cup C_2'$ are disjoint, and $\mathbb{C}$ and $\mathbb{C}'$ are disconnected in the sense that there is no $\mathcal{T}$-access (nor an $\mathcal{S}$-access) from one to the other. As a result, a **K**-frame $\mathcal{F}$ is composed of an arbitrary union of components; however, when $\mathcal{F}$ contains a component in which the F1-cluster is empty, and $\mathcal{S} \neq \emptyset$ (and so, $\mathcal{S}$ is arbitrary), (1) is not sufficient to recover $\mathcal{F}$. This ambiguity in the transformation will be solved in the following section as the proposed logic $\mathbf{MLF}^*$ does not accept components whose F1-cluster is empty.

Definition 3. *An* **F***-frame is a pair* $\mathbb{C} = (C_1, C_2)$*, having a component structure.*

We now define a function $\mu : \mathbf{F} \to \mathbf{K}$, assigning a **K**-frame $\mu(\mathbb{C}) = \mathcal{F}^{\mathbb{C}}$ (see (1)) to each **F**-frame $\mathbb{C}$. As two distinct components $\mathbb{C}$ and $\mathbb{C}'$ give rise to two distinct **K**-frames $\mathcal{F}^{\mathbb{C}}$ and $\mathcal{F}^{\mathbb{C}'}$, μ is 1-1, but not onto[3]. Thus, **F** is indeed a (proper) subclass of **K**.

Proposition 1. *Given a* **K***-frame* $\mathcal{F} = (W, \mathcal{T}, \mathcal{S})$*, let* $\mathbb{C} = (C_1, C_2)$ *be a component of* $\mathcal{F}$*, and* $w \in C_1 \cup C_2$*, then*

1. *if* $w \in C_1$*, then* $\mathcal{T}(w) = C_1 \cup C_2$*, and* $\mathcal{S}(w) = \{w\}$*;*
2. *if* $w \in C_2$ *, then* $\mathcal{T}(w) = C_2$*, and* $\mathcal{S}(w) = C_1$ *when* $C_1 \neq \emptyset$*; otherwise* $\mathcal{S}(w)$ *is arbitrary.*

The proof easily follows from the frame properties of **K**.

Corollary 1. *For a* **K***-frame* $\mathcal{F}$=$(W, \mathcal{T}, \mathcal{S})$*, and a component* $\mathbb{C}$=(C_1, C_2) *of* $\mathcal{F}$*, we have:*

1. $\mathcal{T}(C_1 \cup C_2) = C_1 \cup C_2$*;*
2. $\mathcal{S}(C_1 \cup C_2) \subseteq C_1 \cup C_2$.

Corollary 2. *Given a pointed* **K***-frame* $\mathcal{F}_w$*, let* $C = \mathcal{T}(w) \setminus C_1$ *if* w *is in an F1-cluster* C_1*; else if* w *is in an F2-cluster* C_2*, let* $C = \mathcal{T}(w)$*. Take* C'=$\mathcal{S}(C) \setminus C$*. Then,* $\mathbb{C}^{\mathcal{F}_w}$=$(C', C) \in \mathbf{F}$.

Note that the component generated by $w \in \mathcal{F}$ is exactly the one in which w is placed. So, any point from the same component forms itself. Using Corollary 2, we now define another function ν, assigning to each pointed **K**-frame $\mathcal{F}_w$ an **F**-frame $\mathbb{C}^{\mathcal{F}_w}$. Clearly, ν is not 1-1, but is onto. Finally, $\{\nu(\mathcal{F}_w) : w \in W\}$ identifies all the components in $\mathcal{F}$. The following proposition generalises this observation.

Proposition 2. *Given an* **F***-frame* $\mathbb{C} = (C_1, C_2)$ *and* $w \in C_1 \cup C_2$*, we have* $\nu(\mu(\mathbb{C}), w) = \mathbb{C}$.

[3] Note that there is no **F**-frame being mapped to (i) a **K**-frame containing more than one component structure in it, and (ii) a **K**-frame composed of only one component with a single (nonempty) cluster structure in which $\mathcal{S} \neq \emptyset$.

These transformations between frame structures of **MLF** enable us to define valuations also on the components $\mathbb{C} \in \mathbf{F}$, resulting in an alternative semantics for **MLF** via **F**-models. The new semantics can be viewed as a reformulation of the Kripke semantics: given a **K**-model $\mathcal{M} = (\mathcal{F}^{\mathbb{C}}, V)$ for some Kripke frame $\mathcal{F}^{\mathbb{C}} \in \mu(\mathbf{F})$ and a valuation V, one can transform $\mathcal{F}^{\mathbb{C}}$ to a component $\nu(\mathcal{F}^{\mathbb{C}}_w) = \mathbb{C} \in \mathbf{F}$ for some $w \in C_1 \cup C_2$ (see Proposition 2). This discussion allows us to define pairs $(\mathbb{C}, V)$ in which $\mathbb{C} \in \mathbf{F}$, and V is the valuation restricted to $\mathbb{C}$. Such valuated components are called '**F**-*models*', and they make it possible to transfer **K**-satisfaction to **F**-satisfaction.

Truth conditions (the modal cases) for an **F**-model $(\mathbb{C}, V) = (C_1, C_2, V)$ and $w \in C_1 \cup C_2$,
$(\mathbb{C}, V), w \models_{\mathbf{MLF}} [\mathrm{T}]\psi$ if and only if

- if $w \in C_1$ then $(\mathbb{C}, V), v \models_{\mathbf{MLF}} \psi$ for all $v \in C_1 \cup C_2$ (i.e., $(\mathbb{C}, V) \models_{\mathbf{MLF}} \psi$);
- if $w \in C_2$ then $(\mathbb{C}, V), v \models_{\mathbf{MLF}} \psi$ for all $v \in C_2$.

$(\mathbb{C}, V), w \models_{\mathbf{MLF}} [\mathrm{S}]\psi$ if and only if

- if $w \in C_1$ then $(\mathbb{C}, V), w \models_{\mathbf{MLF}} \psi$;
- if $w \in C_2$ then $(\mathbb{C}, V), v \models_{\mathbf{MLF}} \psi$ for all $v \in C_1$ if $C_1 \neq \emptyset$; else 'no strict conclusion'.

The next result reveals the relation between the Kripke and the floor sematics of **MLF**.

Proposition 3 (corollary of Proposition 2). *For an* **F**-*model* $(\mathbb{C}, V)$*,* $w \in \mathbb{C}$*, and* $\varphi \in \mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$*,* $(\mathbb{C}, V), w \models_{\mathbf{MLF}} \varphi$ *if and only if* $(\mathcal{F}^{\mathbb{C}}, V), w \models_{\mathbf{MLF}} \varphi$.

2.4 Axiomatisation of MLF

We here give an axiomatisation of **MLF**, and prove its completeness. Recall that K([T]), T([T]), 4([T]) and F([T]) characterise the modal logic **S4F** [30]. The monotonic logic defined by Table 1 is **MLF**. The schemas T_2([S]) and WTriv_2([S]) can be combined

K([T])	the modal logic **K** for [T]
K([S])	the modal logic **K** for [S]
T([T])	$[\mathrm{T}]\varphi \to \varphi$
4([T])	$[\mathrm{T}]\varphi \to [\mathrm{T}][\mathrm{T}]\varphi$
F([T])	$(\varphi \wedge \langle\mathrm{T}\rangle[\mathrm{T}]\psi) \to [\mathrm{T}](\langle\mathrm{T}\rangle\varphi \vee \psi)$
T_2([S])	$[\mathrm{S}]([\mathrm{S}]\varphi \to \varphi)$
WTriv_2([S])	$[\mathrm{S}](\varphi \to [\mathrm{S}]\varphi)$
MB([T],[S])	$\varphi \to [\mathrm{T}](\langle\mathrm{T}\rangle\varphi \vee \langle\mathrm{S}\rangle\varphi)$
WMB([S],[T])	$\varphi \to [\mathrm{S}](\varphi \vee \langle\mathrm{T}\rangle\varphi)$

Table 1. Axiomatisation of **MLF**

into the axiom Triv_2([S]), i.e., $[\mathrm{S}]([\mathrm{S}]\varphi \leftrightarrow \varphi)$, referring to the "triviality in the second $\mathcal{S}$-step". Finally, MB([T],[S]) and WMB([S],[T]) are familiar from tense logics.

2.5 Soundness and completeness of MLF

The axiom schemas given in Table 1 precisely characterise the class **K** of **MLF** frames. We only show that F([T]) describes the property f($\mathcal{T}$) of **K**-frames, but the rest is similar.

- Let $\mathcal{M}$=(W,$\mathcal{T}$,$\mathcal{S}$,V) be a **K**-model, satisfying f($\mathcal{T}$). We want to show that F([T]) is valid in $\mathcal{M}$. Let $w \in W$ be such that $\mathcal{M}, w \models_{\mathbf{MLF}} \varphi \wedge \langle \mathrm{T} \rangle [\mathrm{T}] \psi$ ($\star$). Then, it suffices to prove that $\mathcal{M}, w \models_{\mathbf{MLF}} [\mathrm{T}](\langle \mathrm{T} \rangle \varphi \vee \psi)$. For an arbitrary $u \in W$, assume that $(w,u) \in \mathcal{T}$.
  Case (1): let $(u,w) \in \mathcal{T}$. The assumption ($\star$) implies that $\mathcal{M}, w \models_{\mathbf{MLF}} \varphi$. Then, it also holds that $\mathcal{M}, u \models_{\mathbf{MLF}} \langle \mathrm{T} \rangle \varphi$; clearly, so does $\mathcal{M}, u \models_{\mathbf{MLF}} \langle \mathrm{T} \rangle \varphi \vee \psi$.
  Case (2): let $(u,w) \notin \mathcal{T}$. Then, by the assumption ($\star$), $\mathcal{M}, w \models_{\mathbf{MLF}} \langle \mathrm{T} \rangle [\mathrm{T}] \psi$. Thus, there is $v \in W$ such that $(w,v) \in \mathcal{T}$ and $\mathcal{M}, v \models_{\mathbf{MLF}} [\mathrm{T}] \psi$. As $\mathcal{M}$ satisfies f($\mathcal{T}$), we get $(v,u) \in \mathcal{T}$. As $\mathcal{M}, v \models_{\mathbf{MLF}} [\mathrm{T}] \psi$, we have $\mathcal{M}, u \models_{\mathbf{MLF}} \psi$; hence, $\mathcal{M}, u \models_{\mathbf{MLF}} \langle \mathrm{T} \rangle \varphi \vee \psi$.

- Let $\mathcal{F}$=(W,$\mathcal{T}$,$\mathcal{S}$) be a **K**-frame in which f($\mathcal{T}$) fails. So, there exists $w,u,v \in W$ with $(u,w) \notin \mathcal{T}$ while $(w,u) \in \mathcal{T}$ and $(w,v) \in \mathcal{T}$; yet $(v,u) \notin \mathcal{T}$. Thanks to the last 2 claims, we have $w \neq v$ (otherwise $(v,u) \notin \mathcal{T}$ would contradict $(w,u) \in \mathcal{T}$). Due to the first 2 claims, $w \neq u$ (otherwise, (w,u)=(u,w), and $(u,w) \in \mathcal{T}$). We now take a valuation V satisfying: $\mathcal{M}, w \models_{\mathbf{MLF}} \varphi$ ($\blacktriangle$), but $\mathcal{M}, x \not\models_{\mathbf{MLF}} \varphi$ for any $x \neq w$; similarly, $\mathcal{M}, u \not\models_{\mathbf{MLF}} \psi$ ($\blacktriangledown$), but $\mathcal{M}, y \models_{\mathbf{MLF}} \psi$ for every $y \neq u$. Since $(v,u) \notin \mathcal{T}$, and thanks to the choice of V, $\mathcal{M}, v \models_{\mathbf{MLF}} [\mathrm{T}] \psi$. As $(w,v) \in \mathcal{T}$, and also by using ($\blacktriangle$), we have $\mathcal{M}, w \models_{\mathbf{MLF}} \varphi \wedge \langle \mathrm{T} \rangle [\mathrm{T}] \psi$. On the other hand, $\mathcal{M}, u \models_{\mathbf{MLF}} [\mathrm{T}] \neg \varphi$ since $(u,w) \notin \mathcal{T}$ and w is the only point satisfying φ. Then, ($\blacktriangledown$) further implies that $\mathcal{M}, u \models_{\mathbf{MLF}} [\mathrm{T}] \neg \varphi \wedge \neg \psi$. Since $(w,u) \in \mathcal{T}$, we also get $\mathcal{M}, w \models_{\mathbf{MLF}} \langle \mathrm{T} \rangle ([\mathrm{T}] \neg \varphi \wedge \neg \psi)$. So, we are done.

Corollary 3. **MLF** *is sound w.r.t. the class* **K** *of frames.*

Here, we only need to show that the inference rules of **MLF** are validity-preserving.

Theorem 1. **MLF** *is complete w.r.t. the class of* **K***-frames.*

Proof. We use the method of canonical models (see [29]), so we first define the canonical model $\mathcal{M}^c = (W^c, \mathcal{T}^c, \mathcal{S}^c, V^c)$ in which

- W^c is the set of maximally consistent sets of **MLF**.
- $\mathcal{T}^c$ and $\mathcal{S}^c$ are the accessibility relations on W^c with:

$$\begin{array}{ll} \Gamma \mathcal{T}^c \Gamma' & \text{if and only if } \{\psi \,:\, [\mathrm{T}]\psi \in \Gamma\} \subseteq \Gamma'; \\ \Gamma \mathcal{S}^c \Gamma' & \text{if and only if } \{\psi \,:\, [\mathrm{S}]\psi \in \Gamma\} \subseteq \Gamma'. \end{array}$$

- V^c is the valuation s.t. $V^c(\Gamma) = \Gamma \cap \mathbb{P}$, for every $\Gamma \in W^c$.

By induction on φ, we prove a truth lemma saying: "$\Gamma \models_{\mathbf{MLF}} \varphi$ iff $\varphi \in \Gamma$" for every $\varphi \in \mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$. Then, it remains to show that $\mathcal{M}^c$ satisfies all constraints of **K**, and so is a legal **K**-model of **MLF**. We here give the proof only for wtriv$_2$($\mathcal{S}$) and wmsym($\mathcal{S}$,$\mathcal{T}$).
► The schema WTriv$_2$([S]) guarantees that $\mathcal{M}^c$ satisfies wtriv$_2$($\mathcal{S}$): let $\Gamma_1 \mathcal{S}^c \Gamma_2$ ($\star$) and $\Gamma_2 \mathcal{S}^c \Gamma_3$ ($\star\star$). Assume for a contradiction that $\Gamma_2 \neq \Gamma_3$. Thus, there exists $\varphi \in \Gamma_2$ with $\neg\varphi \in \Gamma_3$, implying that $\langle \mathrm{S} \rangle \neg \varphi \in \Gamma_2$ by the hypothesis ($\star\star$). Since Γ_2 is maximally consistent, $\varphi \wedge \langle \mathrm{S} \rangle \neg \varphi \in \Gamma_2$. So, using the hypothesis ($\star$), we get $\langle \mathrm{S} \rangle (\varphi \wedge \langle \mathrm{S} \rangle \neg \varphi) \in \Gamma_1$.

However, since Γ_1 is maximally consistent, any instance of $\mathrm{WTriv}_2([\mathrm{S}])$ is in Γ_1. Thus, $[\mathrm{S}](\varphi \to [\mathrm{S}]\varphi) \in \Gamma_1$, and it contradicts the consistency of Γ_1.
► The schema WMB([S],[T]) ensures that wmsym($\mathcal{S},\mathcal{T}$) holds in $\mathcal{M}^c$: suppose that $\Gamma \mathcal{S}^c \Gamma'$ ($\star$) for $\Gamma, \Gamma' \in W^c$. W.l.o.g., let $\Gamma \neq \Gamma'$. Then, there exists $\psi \in \Gamma'$ with $\neg\psi \in \Gamma$. We need to show that $\Gamma' \mathcal{T}^c \Gamma$. So, let φ be such that $[\mathrm{T}]\varphi \in \Gamma'$. As Γ' is maximally consistent, we have both $\varphi \vee \psi \in \Gamma'$ and $[\mathrm{T}]\varphi \vee [\mathrm{T}]\psi \in \Gamma'$. We know that $[\mathrm{T}]\varphi \vee [\mathrm{T}]\psi \to [\mathrm{T}](\varphi \vee \psi)$ is a theorem of **MLF**, so it is in Γ'. Then, by Modus Ponens (MP), we get $[\mathrm{T}](\varphi \vee \psi) \in \Gamma'$, further implying $(\varphi \vee \psi) \wedge [\mathrm{T}](\varphi \vee \psi) \in \Gamma'$ since we already have $(\varphi \vee \psi) \in \Gamma'$. The assumption ($\star$) gives us that $\langle \mathrm{S} \rangle \big((\varphi \vee \psi) \wedge [\mathrm{T}](\varphi \vee \psi) \big) \in \Gamma$. Since Γ is maximally consistent, any instance of WMB([S],[T]) is in Γ; in particular, so is $\langle \mathrm{S} \rangle ((\varphi \vee \psi) \wedge [\mathrm{T}](\varphi \vee \psi)) \to (\varphi \vee \psi)$. Finally, again by MP, we have $\varphi \vee \psi \in \Gamma$. Since $\neg\psi \in \Gamma$, it follows that $\varphi \in \Gamma$.

***Soundness and completeness of* MLF *w.r.t.* F.** Since any component $\mathbb{C} \in \mathbf{F}$ can be converted to a **K**-frame $\mu(\mathbb{C})$, soundness follows from Corollary 3 and Proposition 2. As to completeness, for a non-theorem $\varphi \in \mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$, $\neg\varphi$ is consistent. Let $\Gamma_{\neg\varphi}$ be a maximally consistent set in the canonical model $\mathcal{M}^c$ such that $\neg\varphi \in \Gamma_{\neg\varphi}$. As the canonical frame $\mathcal{M}^c = (W^c, \mathcal{T}^c, \mathcal{S}^c)$ is a member of the class **K**, Proposition 1 and Proposition 2 allow us to define the component $\mathbb{C}^c = (C_1^c, C_2^c)$ with $\Gamma_{\neg\varphi} \in C_1^c \cup C_2^c$. Moreover, by Corollary 1, $C_1^c \cup C_2^c$ is closed under the operators $\mathcal{T}^c(\cdot)$ and $\mathcal{S}^c(\cdot)$. Therefore, modal satisfaction is preserved between $\mathcal{M}^c$ and $\mathbb{C}^c$. As a result, $\mathbb{C}^c, \Gamma_{\neg\varphi} \not\models_{\mathbf{MLF}} \varphi$ (i.e., $\mathbb{C}^c, \Gamma_{\neg\varphi} \models_{\mathbf{MLF}} \neg\varphi$).

3 Where we capture nonmonotonic S4F: Modal logic MLF*

We here propose an extension of **MLF** with a new axiom schema

$$\mathrm{Neg}([\mathrm{S}],[\mathrm{T}]):\quad \langle \mathrm{T} \rangle [\mathrm{T}]\varphi \to \langle \mathrm{T} \rangle \langle \mathrm{S} \rangle \neg\varphi$$

where $\varphi \in Prop$ is non-tautological. We call this schema '*negatable axiom*' and the resulting formalism **MLF***. **MLF***-models are of 2 kinds, namely **K*** and **F***. They are obtained respectively from the classes **K** and **F** by adding a 'model' constraint:

$$\mathrm{neg}(\mathcal{S},\mathcal{T})\text{: for every } P \subseteq \mathbb{P}, \text{ there exists a world } w \text{ such that } P = V(w).$$

In other words, **MLF***-models can falsify any nontheorem of our logic, i.e., for every such φ, there exists a world w such that $w \models_{\mathbf{MLF}^*} \neg\varphi$. Every **F***-model (C_1, C_2, V) now has an exactly 'two-floor' form: $C_1 \neq \emptyset$, and C_1 includes a world w, at which a propositional nontheorem φ, valid in C_2, is refuted. A **K***-model is indeed an arbitrary combination of **F***-models. Below we show that Neg([S],[T]) precisely corresponds to neg($\mathcal{S},\mathcal{T}$).

Proposition 4. *Given a* **K**-*model* $\mathcal{M} = (W, \mathcal{T}, \mathcal{S}, V)$ *in* **MLF**,

$$\mathrm{Neg}([\mathrm{S}],[\mathrm{T}]) \textit{ is valid in } \mathcal{M} \quad \textit{if and only if} \quad \mathcal{M} \textit{ is a } \mathbf{K}^*\textit{-model.}$$

Proof. Let $\mathcal{M} = (W, \mathcal{T}, \mathcal{S}, V)$ be a **K**-model of **MLF**.
($\Longrightarrow$): Assume that $\mathcal{M}$ is not a **K***-model. Then, there exists a nontautological propositional formula $\varphi \in Prop$ such that $\mathcal{M} \models_{\mathbf{MLF}} \varphi$. Clearly, $[\mathrm{T}]\varphi$, $[\mathrm{S}]\varphi$ and $[\mathrm{T}][\mathrm{S}]\varphi$ are all

valid in $\mathcal{M}$, but then so is $\langle\mathrm{T}\rangle[\mathrm{T}]\varphi$ (thanks to the reflexivity of $\mathcal{T}$). This implies that $\langle\mathrm{T}\rangle[\mathrm{T}]\varphi \wedge [\mathrm{T}][\mathrm{S}]\varphi$ is also valid in $\mathcal{M}$. Thus, Neg([S], [T]) is not valid in $\mathcal{M}$.
($\Longleftarrow$): Let $\mathcal{M}$ be a $\mathbf{K}^*$-model (•). Let $\varphi \in Prop$ be a nontheorem. Take $\beta = \langle\mathrm{T}\rangle[\mathrm{T}]\varphi \rightarrow \langle\mathrm{T}\rangle\langle\mathrm{S}\rangle\neg\varphi$. We need to show that $\mathcal{M} \models_{\mathbf{MLF}^*} \beta$. Let $w \in W$ be such that $\mathcal{M}, w \models_{\mathbf{MLF}^*} \langle\mathrm{T}\rangle[\mathrm{T}]\varphi$. We first consider the $\mathbf{F}$-model $\mathbb{C} = (C_1, C_2, V)$, generated by w as in Corollary 2. By construction, φ is valid in C_2, and (•) implies an existence of $u \in C_1$ such that u refutes φ. By the frame properties of $\mathbf{F}$, there exists $v \in C_2$ satisfying $v\mathcal{S}u$ and $\mathcal{M}, v \models_{\mathbf{MLF}^*} \langle\mathrm{S}\rangle\neg\varphi$. Regardless of the floor to which w belongs, $w\mathcal{T}v$, and $v \in C_2$. Thus, $\mathcal{M}, w \models_{\mathbf{MLF}^*} \langle\mathrm{T}\rangle\langle\mathrm{S}\rangle\neg\varphi$.

Proposition 5. *Given an* $\mathbf{F}$*-model* $(\mathbb{C}, V) = (C_1, C_2, V)$ *in* **MLF**,

$$\text{Neg}([\mathrm{S}],[\mathrm{T}]) \textit{ is valid in } (\mathbb{C}, V) \quad \textit{if and only if} \quad (\mathbb{C}, V) \textit{ is an } \mathbf{F}^*\textit{-model.}$$

Neg([S], [T]) has an elegant representation. However, as it makes the reasoning clear in the demanding proofs of this section, we find it handier to use the equivalent version

$$\text{Neg'}([\mathrm{S}],[\mathrm{T}]): \quad \langle\mathrm{T}\rangle[\mathrm{T}]\varphi \rightarrow \langle\mathrm{T}\rangle\langle\mathrm{S}\rangle(\neg\varphi \wedge Q)$$

of Neg([S], [T]) in which $\varphi \in Prop$ is a nontheorem, and Q is a conjunction of a finite set of literals (i.e., p or $\neg p$, for $p \in \mathbb{P}$) such that the set $\{\neg\varphi, Q\}$ is consistent.

Proposition 6. *For a* $\mathbf{K}^*$*-model* $\mathcal{M}=(W, \mathcal{T}, \mathcal{S}, V)$ *and* $w \in W$,

$$\mathcal{M}, w \models_{\mathbf{MLF}^*} \text{Neg}([\mathrm{S}],[\mathrm{T}]) \quad \textit{if and only if} \quad \mathcal{M}, w \models_{\mathbf{MLF}^*} \text{Neg'}([\mathrm{S}],[\mathrm{T}]).$$

Proof. The right-to-left direction is straightforward: take $Q = \emptyset$ and the result follows. For the opposite direction, we first assume that $\mathcal{M}, w \models_{\mathbf{MLF}^*} \text{Neg}([\mathrm{S}],[\mathrm{T}])$ (▲). Let $\varphi \in Prop$ be a nontheorem of $\mathbf{MLF}^*$ viz. $\mathcal{M}, w \models_{\mathbf{MLF}^*} \langle\mathrm{T}\rangle[\mathrm{T}]\varphi$ (▼). Let Q be a conjunction of finite literals such that $\neg\varphi \wedge Q$ is consistent. Then, $\varphi \vee \neg Q \in Prop$ is a nontheorem of $\mathbf{MLF}^*$. Moreover, from the assumption (▼), we also get $\mathcal{M}, w \models_{\mathbf{MLF}^*} \langle\mathrm{T}\rangle[\mathrm{T}](\varphi \vee \neg Q)$. Lastly, by the hypothesis (▲), we have $\mathcal{M}, w \models_{\mathbf{MLF}^*} \langle\mathrm{T}\rangle\langle\mathrm{S}\rangle(\neg\varphi \wedge Q)$.

We finally transform a valuated cluster (C, V) into an $\mathbf{F}^*$-model. We first construct a set

$$C_1 = \{x_\varphi \;:\; \text{for every } \varphi \in Prop \text{ such that } \neg\varphi \nvdash \bot,\; (C, V) \models_{\mathbf{MLF}} \varphi \;\text{ and }\; x_\varphi \notin C\}$$

into which we put a point $x_\varphi \notin C$ for every nontheorem φ that is valid in C. So, $C \cap C_1 = \emptyset$. Then, we extend the universal relation $\mathcal{T}$ on C to $\mathcal{T}' = \big((C_1 \cup C) \times C\big) \cup (C_1 \times C_1)$ on $C \cup C_1$. The valuation V defined over C is also extended to $V' : C_1 \cup C \longrightarrow \mathbb{P}$ satisfying: $V'|_C = V$, and $V'(x_\varphi)$ is designed to falsify φ. Hence, by definition, $(C_1, C, V') \in \mathbf{F}^*$.

Soundness and completeness of $\mathbf{MLF}^*$ We have seen that **MLF** is sound w.r.t. **F**, so Proposition 5 implies that $\mathbf{MLF}^*$ is sound w.r.t. $\mathbf{F}^*$. Since any $\mathbf{K}^*$-model is a combination of $\mathbf{F}^*$-models, we can generalise this result to $\mathbf{K}^*$. We here show that $\mathbf{MLF}^*$ is complete w.r.t. $\mathbf{F}^*$: first we take a canonical model $\mathcal{M}^c = (W^c, \mathcal{T}^c, \mathcal{S}^c, V^c)$ of $\mathbf{MLF}^*$ (see Theorem 1 for the details). Then, we define a valuated component $(\mathbb{C}^c, V^c) = (C_1^c, C_2^c, V^c)$ for $C_1^c, C_2^c \subseteq W^c$ as in Section 2.5. We want to show that $(\mathbb{C}^c, V^c)$ is an $\mathbf{F}^*$-model. So, it is enough to prove that Neg([S], [T]) ensures the property neg($\mathcal{S}, \mathcal{T}$).

First recall that every **F**-frame $\mathbb{C}$ corresponds to a **K**-frame $\mu(\mathbb{C}) = \mathcal{F}^{\mathbb{C}}$, and by Proposition 2, $\nu(\mu(\mathbb{C}), w) = \mathbb{C}$ for $w \in C_1 \cup C_2$. Thus, such $(\mu(\mathbb{C}^c), V^c)$ is a submodel of $\mathcal{M}^c$ since it is a $\mathbf{K}^*$-frame. For nontautological $\varphi \in Prop$, let us assume $\Gamma \models_{\mathbf{MLF}^*} \varphi$ (i.e., $\varphi \in \Gamma$) for every $\Gamma \in C_2^c$ (so, φ is consistent). This implies that $(\mathbb{C}^c, V^c), \Gamma \models_{\mathbf{MLF}^*} [\mathrm{T}]\varphi$ (i.e., $[\mathrm{T}]\varphi \in \Gamma$), for every $\Gamma \in C_2^c$. Using the fact that $\mu(\mathbb{C}^c)$ is part of the canonical model $\mathcal{M}^c$, we have $\mathcal{T}^c|_{C_1^c \cup C_2^c} \supset ((C_1^c \cup C_2^c) \times C_2^c)$. Thus, $(\mathbb{C}^c, V^c), \Gamma \models_{\mathbf{MLF}^*} \langle \mathrm{T} \rangle [\mathrm{T}]\varphi$ for every $\Gamma \in C_1^c \cup C_2^c$. As any instance of Neg([S], [T]) is valid in $(\mathbb{C}^c, V^c)$, $\langle \mathrm{T} \rangle \langle \mathrm{S} \rangle \neg\varphi \in \Gamma$ for every $\Gamma \in C_1^c \cup C_2^c$. In other words, $(\mathbb{C}^c, V^c) \models_{\mathbf{MLF}^*} \langle \mathrm{T} \rangle \langle \mathrm{S} \rangle \neg\varphi$. Thus, there exists $\Gamma' \in W^c$ such that $\Gamma \mathcal{T}^c \Gamma'$ and $\Gamma' \models_{\mathbf{MLF}^*} \langle \mathrm{S} \rangle \neg\varphi$ (i.e., $\langle \mathrm{S} \rangle \neg\varphi \in \Gamma'$). As $\mathcal{T}(C \cup A) = C \cup A$ in **F**, we have $\Gamma' \in C_1^c \cup C_2^c$. Moreover, there also exists $\Gamma'' \in W^c$ such that $\Gamma' \mathcal{S}^c \Gamma''$ and $\Gamma'' \models_{\mathbf{MLF}^*} \neg\varphi$. By Corollary 1, $\mathcal{S}(C_1^c \cup C_2^c) \subseteq C_1^c \cup C_2^c$, yet from our initial hypothesis, we obtain $\Gamma'' \in C_1^c$. To sum up, Γ'' is a maximally consistent set in $\mathbb{C}^c$ such that $(\mathbb{C}^c, V^c), \Gamma'' \not\models_{\mathbf{MLF}^*} \varphi$.

3.1 Minimal model semantics for nonmonotonic S4F

This section recalls the minimal model semantics for nonmonotonic **S4F** [22]. We first define a *preference* relation between **S4F** models, enabling us to check minimisation.

Definition 4. *An* **S4F** *model* $\mathcal{N} = (N, R, U)$ *is preferred over a valuated cluster* (C, V) *if*

- $N = C \cup C_1$ *for some (nonempty) set* C_1 *of possible worlds such that* $C \cap C_1 = \emptyset$*;*
- $R = (N \times C) \cup (C_1 \times C_1)$*;*
- *The valuations* V *and* U *agree on* C *(i.e.,* $V = U|_C$*);*
- *There exists* $\varphi \in Prop$ *such that* $C \models \varphi$ *and* $\mathcal{N} \not\models \varphi$*.*

We abbreviate it by $\mathcal{N} > (C, V)$. A valuated cluster (C, V) is then a *minimal model* of a theory (finite set of formulas) Γ in **S4F** if

- $(C, V), x \models \Gamma$ for every $x \in C$ (i.e., $(C, V) \models \Gamma$);
- $\mathcal{N} \not\models \Gamma$ for every $\mathcal{N}$ such that $\mathcal{N} > (C, V)$.

Finally, a formula φ is a *logical consequence* of a theory Γ in **S4F** (abbreviated $\Gamma \approx_{\mathbf{S4F}} \varphi$) if φ is valid in every minimal model of Γ. For example, $q \approx_{\mathbf{S4F}} \neg p \vee q$, yet $\neg p \vee q \not\approx_{\mathbf{S4F}} q$.

3.2 Embedding nonmonotonic S4F into MLF*

We here embed nonmonotonic **S4F** into $\mathbf{MLF}^*$. With this aim, we first translate the language of **S4F** ($\mathcal{L}_{\mathbf{S4F}}$) into $\mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$ via a mapping '*tr*': we simply and only replace $\mathsf{L} \in \mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$ by [T]. The following proposition proves that this translation is correct, and clarifies how to characterise minimal models of **S4F** in $\mathbf{MLF}^*$.

Proposition 7. *Given an* $\mathbf{F}^*$*-model* $(\mathbb{C}, V) = (C_1, C, V)$*, and* $\alpha \in \mathcal{L}_{\mathbf{S4F}}$*, we have:*

1. $(\mathbb{C}, V), w \models_{\mathbf{MLF}^*} tr(\alpha)$*, for every* $w \in C$ *if and only if* $(C, V|_C) \models \alpha$*.*
2. $(\mathbb{C}, V) \models_{\mathbf{MLF}^*} \langle \mathrm{T} \rangle [\mathrm{T}](tr(\alpha) \wedge [\mathrm{S}] \neg tr(\alpha))$ *if and only if* $(C, V|_C)$ *is a minimal model of* α*.*

Proof. The proof of the first item is by induction on α. As to the second item, for the proof of the '*only if*' part, we first assume $(\mathbb{C}, V) \models_{\mathbf{MLF}^*} \langle \mathrm{T} \rangle [\mathrm{T}](tr(\alpha) \wedge [\mathrm{S}]\neg tr(\alpha))$ (♦).
(1) From (♦), we obtain that $(\mathbb{C}, V), u \models_{\mathbf{MLF}^*} tr(\alpha)$ (▲), and $(\mathbb{C}, V), u \models_{\mathbf{MLF}^*} [\mathrm{S}]\neg tr(\alpha)$ (▼) for every $u \in C$ (consider: for $w \in C_1$, (♦) implies that there is $u \in C_1 \cup C$ such that $w\mathcal{T}u$ and $(\mathbb{C}, V), u \models_{\mathbf{MLF}^*} [\mathrm{T}](tr(\alpha) \wedge [\mathrm{S}]\neg tr(\alpha))$. So, $u \in C$; otherwise it yields a contradiction). Then, using Proposition 7.1 and (▲), we get $(C, V|_C) \models \alpha$. So, the first condition holds.
(2) By definition, it remains to show that $\mathcal{N} \not\models \alpha$ for every **S4F** model $\mathcal{N}$ such that $\mathcal{N} \succ (C, V|_C)$. Let $\mathcal{N} = (N, R, U)$ be a preferred model over the valuated cluster $(C, V|_C)$ satisfying: $N = C \cup C'$ for some (cluster) C' such that $C \cap C' = \emptyset$, $R = (N \times C) \cup (C' \times C')$, and $U|_C = V|_C$. By Definition 4, we also know that there exists $\psi \in Prop$ such that $(C, V|_C) \models \psi$ (•), but $\mathcal{N} \not\models \psi$. Therefore, there exists $r \in C'$ viz. $\mathcal{N}, r \not\models \psi$ (i.e., $\mathcal{N}, r \models \neg\psi$).
(3) As $(\mathbb{C}, V)$ is an $\mathbf{F}^*$-model, Neg([S], [T]) is valid in it; due to Proposition 6, so is Neg'([S], [T]). Hence, $(\mathbb{C}, V) \models \langle \mathrm{T} \rangle [\mathrm{T}]\varphi \rightarrow \langle \mathrm{T} \rangle \langle \mathrm{S} \rangle (\neg\varphi \wedge Q)$ for a non-theorem $\varphi \in Prop$ of $\mathcal{L}_{\mathbf{MLF}^*}$, and a conjunction of a finite set of literals Q such that $\{\neg\varphi, Q\}$ is consistent.
(4) By (•) in the item (2) and also using Lemma 7.1, we get $(\mathbb{C}, V), u \models_{\mathbf{MLF}^*} tr(\psi)$ for every $u \in C$. Since $(\mathbb{C}, V)$ is an $\mathbf{F}^*$-model, we also have $(\mathbb{C}, V), u \models_{\mathbf{MLF}^*} [\mathrm{T}]tr(\psi)$; even $(\mathbb{C}, V), u \models_{\mathbf{MLF}^*} \langle \mathrm{T} \rangle [\mathrm{T}]tr(\psi)$ for every $u \in C$ (♣). Moreover, we know that $tr(\psi)$ is not a tautology; otherwise $\mathcal{N}, r \models \psi$. Let $Q' = \left(\bigwedge_{p \in \left(\mathbb{P}_\alpha \cap U(r)\right)} p\right) \wedge \left(\bigwedge_{q \in \left(\mathbb{P}_\alpha \setminus U(r)\right)} \neg q\right)$. It is clear that $\mathcal{N}, r \models Q'$, but we also know that $\mathcal{N}, r \models \neg\psi$, so we have $\mathcal{N}, r \models \neg\psi \wedge Q'$. We so conclude that $\{\neg\psi, Q'\}$ is consistent; then so is $\{\neg tr(\psi), Q'\}$. As $(\mathbb{C}, V)$ is an $\mathbf{F}^*$-model, an instance of the negatable axiom, namely $\langle \mathrm{T} \rangle [\mathrm{T}]tr(\psi) \rightarrow \langle \mathrm{T} \rangle \langle \mathrm{S} \rangle (\neg tr(\psi) \wedge Q')$, is valid in $(\mathbb{C}, V)$. So, (♣) implies that $(\mathbb{C}, V), u \models_{\mathbf{MLF}^*} \langle \mathrm{T} \rangle \langle \mathrm{S} \rangle (\neg tr(\psi) \wedge Q')$ for every $u \in C$. This means that there exists a point $x_\psi \in C_1$ such that $(\mathbb{C}, V), x_\psi \models_{\mathbf{MLF}^*} \neg tr(\psi) \wedge Q'$, i.e., $(\mathbb{C}, V), x_\psi \models_{\mathbf{MLF}^*} \neg tr(\psi)$ and $(\mathbb{C}, V), x_\psi \models_{\mathbf{MLF}^*} Q'$. As a result, $V(x_\psi) \cap \mathbb{P}_{tr(\alpha)} = U(r) \cap \mathbb{P}_\alpha$.
(5) Note that r and x_ψ agree on $\mathbb{P}_\alpha$. By (▼), we also have $(\mathbb{C}, V), x \models_{\mathbf{MLF}^*} \neg tr(\alpha)$ for every $x \in C_1$; in particular, $(\mathbb{C}, V), x_\psi \models_{\mathbf{MLF}^*} \neg tr(\alpha)$. To summarise the observation above:

1. The pointed model $\left((\{x_\psi\}, C, V|_{(C \cup \{x_\psi\})}), x_\psi\right)$ in $\mathbf{MLF}^*$, and the pointed model $(\mathcal{N}, r)$ in **S4F** have the similar structure: both contain the same maximal valuated cluster $(C, V|_C)$ and one additional reflexive point that can have access to all points of C;
2. $\mathbb{P}_\alpha = \mathbb{P}_{tr(\alpha)}$ and $V(x_\psi) \cap \mathbb{P}_{tr(\alpha)} = U(r) \cap \mathbb{P}_\alpha$;
3. Both α and $tr(\alpha)$ are the exact copies of each other, except that one contains L wherever the other contains [T] (note that $tr(\alpha)$ contains neither [S] nor $\langle \mathrm{S} \rangle$).

Then, it follows that $\mathcal{N}, r \not\models \alpha$, which further implies that $\mathcal{N} \not\models \alpha$. By definition, $(C, V|_C)$ is a minimal model for α. The other part of the proof is similar.

We are now ready to show how we capture the logical consequence of **S4F** in $\mathbf{MLF}^*$.

Theorem 2. *For* $\alpha, \beta \in \mathcal{L}_{\mathbf{S4F}}$, $\alpha \approx_{\mathbf{S4F}} \beta$ *iff* $\models_{\mathbf{MLF}^*} [\mathrm{T}](tr(\alpha) \wedge [\mathrm{S}]\neg tr(\alpha)) \rightarrow [\mathrm{T}]tr(\beta)$.

Proof. We first take $\zeta = [\mathrm{T}](tr(\alpha) \wedge [\mathrm{S}]\neg tr(\alpha)) \rightarrow [\mathrm{T}]tr(\beta)$.
($\Longrightarrow$): Assume that $\alpha \approx_{\mathbf{S4F}} \beta$ in **S4F** (▲). Let $(\mathbb{C}, V) = (C_1, C_2, V)$ be an $\mathbf{F}^*$-model. Then $(C_2, V|_{C_2})$ is a valuated cluster over C_2. We need to show that $(\mathbb{C}, V) \models_{\mathbf{MLF}^*} \zeta$. "For every $w \in C_1$, $(\mathbb{C}, V), w \models_{\mathbf{MLF}^*} \zeta$" trivially holds: by the frame constraints w.r.t. $\mathcal{T}$ in $\mathbf{MLF}^*$, $(\mathbb{C}, V), w \not\models_{\mathbf{MLF}^*} [\mathrm{T}](tr(\alpha) \wedge [\mathrm{S}]\neg tr(\alpha))$ for any $w \in C_1$ (otherwise, $(\mathbb{C}, V) \models_{\mathbf{MLF}^*} tr(\alpha)$, but also $(\mathbb{C}, V) \models_{\mathbf{MLF}^*} \neg tr(\alpha)$, yielding a contradiction). Let $x \in C_2$ be such that $(\mathbb{C}, V), x \models_{\mathbf{MLF}^*}$

[T]$(tr(\alpha) \wedge [\text{S}]\neg tr(\alpha))$. We know that $\mathcal{T}|_{C_2}$ is a universal relation, so "for all $x \in C_2$, $(\mathbb{C}, V), x \models_{\mathbf{MLF}^*} \langle\text{T}\rangle[\text{T}](tr(\alpha) \wedge [\text{S}]\neg tr(\alpha))$" trivially follows. Then, by Proposition 7.2, we conclude that $(C_2, V|_{C_2})$ is a minimal model for α. Then, as $\alpha \approx_{\mathbf{S4F}} \beta$ by the hypothesis (▲), β is valid in $(C_2, V|_{C_2})$, i.e., $(C_2, V|_{C_2}) \models \beta$. Thus, Proposition 7.1 gives us that $(\mathbb{C}, V), z \models_{\mathbf{MLF}^*} tr(\beta)$ for every $z \in C_2$. Since C_2 is a cluster which is a final cone, we also have $(\mathbb{C}, V), z \models_{\mathbf{MLF}^*} [\text{T}]tr(\beta)$ for every $z \in C$; in particular, $(\mathbb{C}, V), x \models_{\mathbf{MLF}^*} [\text{T}]tr(\beta)$.
($\Longleftarrow$): Assume that ζ is valid in $\mathbf{MLF}^*$ (▼). We need to prove that $\alpha \approx_{\mathbf{S4F}} \beta$. Let (C, V) be a minimal model of α. Then, we take an **S4F** model $\mathcal{N} = ((C \cup C'), R, U)$ preferred over (C, V). viz. $\mathcal{N} > (C, V)$. Thus, $\mathcal{N} \not\models \alpha$ (♦). Now, let us construct $(\mathbb{C}, \overline{V}) = (C_1, C_2, \overline{V})$ as follows: take C_2 as the maximal α-cluster C (i.e., exactly the same cluster C as in (C, V)), and $C_1 = \{r \ : \ \mathcal{N}, r \not\models \alpha\}$. Simply, restrict R and U to $C_1 \cup C_2$, respectively resulting in $\mathcal{T}$ and $\overline{V}$. Finally, arrange $\mathcal{S}$ in a way that would satisfy all the frame constraints of **MLF**. Thus, $(\mathbb{C}, \overline{V})$ is clearly an $\mathbf{F}^*$-model. By the minimal model definition, $(C, V) \models \alpha$. Then, Proposition 7.1 and (♦) imply that $(\mathbb{C}, \overline{V}), x \models_{\mathbf{MLF}^*} tr(\alpha)$ for every $x \in C_2$, and for every $y \in C_1$, $(\mathbb{C}, \overline{V}), y \models_{\mathbf{MLF}^*} \neg tr(\alpha)$. As $(\mathbb{C}, \overline{V})$ is an $\mathbf{F}^*$-model, we have $(\mathbb{C}, \overline{V}), x \models_{\mathbf{MLF}^*} [\text{S}]\neg tr(\alpha)$ for every $x \in C_2$. As a result, $(\mathbb{C}, \overline{V}), x \models_{\mathbf{MLF}^*} tr(\alpha) \wedge [\text{S}]\neg tr(\alpha)$ for every $x \in C_2$. Since C_2 is a cluster which is a final cone, we further have $(\mathbb{C}, \overline{V}), x \models_{\mathbf{MLF}^*} [\text{T}](tr(\alpha) \wedge [\text{S}]\neg tr(\alpha))$ for each $x \in C_2$. From (▼), it also follows that $(\mathbb{C}, \overline{V}), x \models_{\mathbf{MLF}^*} [\text{T}]tr(\beta)$ for every $x \in C_2$. Clearly, $tr(\beta)$ is also valid in C_2. Finally, Proposition 7.1 implies that $(C, V) \models \beta$ in **S4F**.

Corollary 4. *For $\alpha \in \mathcal{L}_{\mathbf{S4F}}$, α has a minimal model if and only if* $[\text{T}](tr(\alpha) \wedge [\text{S}]\neg tr(\alpha))$ *is satisfiable in* $\mathbf{MLF}^*$. (hint: take $\beta = \bot$ in Theorem 2.)

4 Relation to other nonmonotonic formalisms

In this section, we briefly discuss a general strategy, unifying some major nonmonotonic reasonings among which are autoepistemic logic (**AEL**) [31], reflexive autoepistemic logic (*R***AEL**) [23], equilibrium logic (and so **ASP**), and nonmonotonic **S4F**. The emphasis is on the 2-floor semantics; the second floor charaterises the minimal model of a formula, and the first floor checks the minimality criterion. This approach can be generalised to other formalisms such as default logic [32] and **MBNF** [33] as there exists a good amount of research in the literature, studying such relations [34,35,36,15]. In particular, nonmonotonic **S4F** and default logic has a strong connection as it is explained and analysed in [14,15]. So, the $\mathbf{MLF}^*$ encoding of nonmonotonic **S4F** leads the potential encoding of default logic.

AEL and *R***AEL** [21,23] are the nonmonotonic variants [22] of respectively the modal logics **KD45** and **SW5** [9,29]. We have recently proposed two new monotonic modal logics called $\mathbf{MAE}^*$ and $\mathbf{M}R\mathbf{AE}^*$, respectively capturing **AEL** and *R***AEL**. They are obtained from $\mathbf{MLF}^*$ by replacing only the axioms characterising **S4F** (i.e., S, 4, F) by ones, characterising respectively **KD45** and **SW5** (i.e., groups of axioms D, 4, 5 and T, 4, W5). The models of $\mathbf{MAE}^*$, $\mathbf{M}R\mathbf{AE}^*$, and $\mathbf{MLF}^*$ are all composed of a union of 2-floor structures: in each, the second floor is a maximal cluster which is a final cone of the 2-floor part of the model; where they differ is the structure of the first floor. While a first floor in $\mathbf{MLF}^*$ is a maximal cluster, that of $\mathbf{MAE}^*$ contains irreflexive and isolated worlds w.r.t. the $\mathcal{T}$-relation (in a sense that, any two different worlds of the first floor are not related to each other by the accessibility relation $\mathcal{T}$). Moreover, the **M***R***AE*** models are nothing, but the reflexive closures of the **MAE*** models w.r.t. the relation $\mathcal{T}$. Interestingly, the same mechanism applied to **S4F** performs successfully for **KD45** and **SW5** as well when everything else remains the same: the implication $[\mathrm{T}](tr(\alpha) \wedge [\mathrm{S}]\neg tr(\alpha)) \rightarrow [\mathrm{T}]tr(\beta)$, capturing nonmonotonic consequence of **S4F**, and the formula $\langle \mathrm{T}\rangle[\mathrm{T}](tr(\alpha) \wedge [\mathrm{S}]\neg tr(\alpha))$ characterising minimal model semantics in **S4F** perfectly work for the nonmonotonic variants of **KD45** and **SW5** as well.

Our research has also a large overlap with [8], embedding equilibrium logic (and so, **ASP**) into a monotonic bimodal logic called **MEM**. The models of **MEM** are roughly described in the introduction. The main result of this paper is also given via a similar implication: the validity of $tr(\alpha) \wedge [\mathrm{S}]\neg tr(\alpha)) \rightarrow tr(\beta)$ in **MEM** captures the nonmonotonic consequence, $\alpha \mathrel{|\!\!\approx} \beta$, of equilibrium logic. However, it is easy to check that the formula $[\mathrm{T}](tr(\alpha) \wedge [\mathrm{S}]\neg tr(\alpha)) \rightarrow [\mathrm{T}]tr(\beta)$ of this paper also gives the same result. This analogy between all these works enables us to classify **MEM** under the same approach. Still, we need to provide a stronger result that would help reinforce the relations between **MEM** and **MLF***. For instance, [14] proves that the well-known Gödel's translation into the modal logic **S4** is still valid for translating the logic of here-and-there (a 3-valued monotonic logic on which equilibrium logic is built) [37,25] into the modal logic **S4F**. A natural question that may arise is whether a similar translation can be used to encode **MEM** into **MLF***, which is the subject of a future work.

5 Conclusion and further research

In this paper, we design a novel monotonic modal logic, namely **MLF***, that captures nonmonotonic **S4F**. We demonstrate this embedding by translating the language of **S4F** into that of **MLF***. This way, we see that **MLF*** is able to characterise the existence of a minimal model as well as logical consequence in nonmonotonic **S4F**.

Our work provides an alternative to Levesque's monotonic bimodal logic of only knowing [38,4,5,6], by which he captures four kinds of nonmonotonic logic, including autoepistemic logic: his language has two modal operators, namely B and N. B is similar to [T]. N is characterised by the complement of the relation, interpreting B. Levesque's frame constraints on the accessibility relation differ from ours, and he identifies the nonmonotonic consequence '$\alpha \mathrel{|\!\!\approx} \beta$' with the implication

$$(\mathsf{B}\; tr(\alpha) \wedge \mathsf{N}\; \neg tr(\alpha)) \rightarrow \mathsf{B}\; tr(\beta).$$

Levesque attacked the same problem with an emphasis on the only knowing notion. However, his reasoning does not attempt to unify, and does not provide a general mechanism either. In particular, he applied his approach to neither **SW5** nor **S4F**.

As a future work, we will implement this general methodology to capture minimal model reasoning, underlying many other nonmonotonic formalisms. This paper, together with other works on **KD45**, **SW5**, and **ASP** [8] stand a very strong initiative by their possible straightforward implementations to different kinds of nonmonotonic formalisms of similar floor-based semantics. Such research will then enable us to compare various forms of nonmonotonic formalisms in a single monotonic modal setting.

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