

A Local Variational Problem of Second Order for a Class of Optimal Control Problems with Nonsmooth Objective Function

Alexander P. Afanasiev

Institute for Information Transmission Problems,
Russian Academy of Sciences, Nakhimovsky Prospekt 36-1, Moscow 117218, Russia
National Research University Higher School of Economics,
Myasnitskaya str., 20, Moscow 101000, Russia
National University of Science and Technology MISiS,
Leninskiy Prospekt 4, Moscow 119049, Russia
Lomonosov Moscow State University,
GSP-1, Vorobievyy Gory, Moscow 119991, Russia
apa@iitp.ru

Elena V. Putilina

Institute for Information Transmission Problems,
Russian Academy of Sciences,
Nakhimovsky Prospekt 36-1, Moscow 117218, Russia
elena@iitp.ru

Abstract

An optimal control problem with a nonsmooth objective function defined on a polytope is studied. A scheme reducing the original problem to the problem of maximizing the sum of matrix elements lying above the main diagonal is constructed. Necessary conditions of optimality for this finite-dimensional optimization problem are established. A method of searching for the global maximum is suggested.

1 Introduction

Optimal control problems with nonsmooth objective functions of the form

$$J[u] = \min_i \{G_i x(\tau), i = \overline{1, N}\} \rightarrow \max_u, \quad (1)$$

under consideration in this work arise to study of the situation of switching during the contact of the optimum trajectory with phase and functional constraints. These situations are typical for optimal control problems with mixed constraints (see [Milyutin et al., 2004]). Moreover, the need to describe the situation of switching arises in the continuation of solutions on the parameter (see [Dikusar et al., 2001]) and when forming local variational problems (see [Afanasiev et al., 1990]). First-order degeneration in the objective function (1) is studied in (see [Afanasiev, 1998]). In this paper, we consider the second-degeneration for a local variational problem with the objective function of the form (1).

Copyright © by the paper's authors. Copying permitted for private and academic purposes.

In: Yu. G. Evtushenko, M. Yu. Khachay, O. V. Khamisov, Yu. A. Kochetov, V.U. Malkova, M.A. Posypkin (eds.): Proceedings of the OPTIMA-2017 Conference, Petrovac, Montenegro, 02-Oct-2017, published at <http://ceur-ws.org>

2 Formulation and Discussion of the Problem

We will study the problem

$$J[u] = \min_i \int_0^1 \langle P_i x(t), u(t) \rangle dt \rightarrow \max_u, i = \overline{1, R}, \quad (2)$$

$$\dot{x}(t) = u(t), \quad (3)$$

$$u(t) \in M, \quad (4)$$

$$x(\cdot): [0, 1] \rightarrow R^n, \quad u(\cdot): [0, 1] \rightarrow R^n, \quad (5)$$

M - is a polyhedron

$x(0) = 0, x(1) \in X, X \cap M \neq \emptyset$.

This problem arises in the study of the following important class of problems

$$J[u] = \min_i \{G_i(x(T)), i = \overline{1, p}\} \rightarrow \max_u, \quad (6)$$

$$\dot{x}(t) = u(t), \quad (7)$$

$$u(t) \in U(x(t)),$$

$$U(x(t)) = \{u: K(x(t)) = u(t) = L(x(t)), M(x(t)) \geq N(x(t))\}, \quad (8)$$

where $M(\cdot)$ - is an $m \times n$ matrix, $L(\cdot)$ - is an $m \times 1$ matrix, $K(\cdot)$ - is an $k \times n$ matrix, $N(\cdot)$ - is an $k \times 1$ matrix.

With the implementation of the procedure of continuation of the optimal trajectory (see[Afanasiev et al., 1990]), where T is a parameter, there arises a need in lifting the degeneracy

$$G_i(x(\tau)) = G_j(x(\tau)), \quad i \neq j \quad i, j \in p_0, \quad p_0 \subset p.$$

Without loss of generality, we may assume that $p_0 = p$. This case is studied in detail in (see [Afanasiev et al., 1990]). The functions $G_i(x(\tau))$ were presented in the form

$$\int_0^T \langle grad G_i(x(t)), \dot{x}(t) \rangle dt,$$

and the consideration was limited to the study of the linear programming problem of the form

$$u_{n+1} \rightarrow \max$$

$$u_{n+1} \geq \langle a_i, u \rangle, \quad i \in p,$$

$$\text{where } a_i = grad G_i(x_0),$$

$$u \in M, M - \text{polyhedron},$$

$$u \in R^n.$$

The second-order degeneration occurs when $grad G_i(x_0) = 0, i \in P$.

To remove degeneracy, let us put into consideration

$$\frac{\partial}{\partial x} grad G_i(\dot{x}) \Big|_{x=x_0} = \frac{\partial^2 G_i(x)}{\partial x^2} \Big|_{x=x_0} u,$$

and, under the assumption $\frac{\partial^2 G_i(x)}{\partial x^2} \Big|_{x=x_0} \neq \{0\}_{n \times n}, i \in P$, we introduce the notation $\frac{\partial^2 G_i(x)}{\partial x^2} \Big|_{x=x_0} = P_i$. And the objective function of problem (2) - (5) became

$$J[v] = \min_i \int_0^1 \langle P_i x(t), u(t) \rangle dt \rightarrow \max_u, \quad i \in P.$$

Further we assume that $P-$ is a skew-symmetric matrix, because a symmetric part of the matrix we may take out from under the integral sign.

3 Reduction of the Problem

Denote by $\{R_i = (r_1^i, r_2^i, \dots, r_n^i), \quad i = \overline{1, N}\}$ the set of vertices of a polyhedron M . Then, by the Caratheodory theorem,

$$U(t) = \sum_{i=1}^P R_i v_i(t), \quad v_i(t) \geq 0, \quad \sum_{i=1}^N v_i(t) = 1. \quad \text{Let } v_i(t) = \dot{y}_i(t).$$

The vectors R_i can be considered as columns of the matrix $R = \{R_i\}_{N \times n}$, and it is easy to see that $x(t) = R \int_0^t v(\tau) d\tau$, $y(0) = 0$, Integrants of the objective function take the form

$$\langle C_i y(t), v(t) \rangle, \quad \text{where } C_i = R^T A R.$$

Finally, the problem (2) - (5) takes the form:

$$J[v] = \min_i \int_0^1 \langle C_i y(t), v(t) \rangle dt \rightarrow \max_v, \quad (9)$$

$$\dot{y}(t) = v(t), \quad (10)$$

$$v(t) \in S, \text{ where } S - \text{simplex, } y(0) = 0, \quad Ry(1) = x(1) \in X, \quad (11)$$

4 Finite-dimensional Analogue of Problem (9) - (11)

Let $v(t)$ on the interval $[0, 1]$ takes finite number of values of L . Then

$$v(t) = \begin{cases} v_0 & \text{when } t \in [t_0, t_1] \\ v_1 & \text{when } t \in [t_1, t_2] \\ \dots & \\ v_i & \text{when } t \in [t_{i-1}, t_i] \\ \dots & \\ v_L & \text{when } t \in [t_{L-1}, t_L] \end{cases}$$

where $t_0 = 0$, $t_L = 1$.

It is clear that, when $t \in [t_{i-1}, t_i]$,

$$y(t) = \int_0^t v(\tau) d\tau = \sum_{\alpha=0}^{i-1} v_\alpha (t_\alpha - t_{\alpha-1}) + v_i (t - t_i).$$

Let us introduce the notation $v_\beta (t_{\beta+1} - t_\beta) = y_\beta$.

Substitute the obtained $y(t)$ and $y_\beta, \beta = \overline{0, L}$ into the objective function (9) into constraints (10) and (11)), and get

$$J[v] = \min_i \sum_{\alpha=1}^{L-1} \sum_{\beta=0}^{\alpha-1} \langle C_i y_\alpha, y_\beta \rangle \rightarrow \max_{\{y_i\}} \quad (12)$$

$$\sum_{\alpha=0}^{L-1} y_\alpha = y(1), \quad Ry(1) \in X \quad (13)$$

$$\langle y(1), 1_N \rangle = 1, \quad y(1) \geq 0, \quad 1_N = \underbrace{(1, 1, \dots, 1)}^T, \quad (14)$$

where $C_i = R^T P_i R$.

This problem is easily rewritten in the form of a problem of mathematical programming with a quadratic (bilinear) constraints. Using the skew-symmetric property of matrices C_i , we can derive the following equality,

$$\sum_{\alpha=1}^{L-1} \sum_{\beta=0}^{\alpha-1} \langle C_i y_\alpha, y_\beta \rangle = \sum_{\alpha=1}^{L-1} \sum_{\beta=\alpha+1}^L \langle C_i y_\alpha, y_\beta \rangle$$

and finally have

$$Z \rightarrow \max, \quad (15)$$

$$Z \leq \sum_{\alpha=1}^{L-1} \sum_{\beta=\alpha+1}^L \langle C_i y_\alpha, y_\beta \rangle, \quad (16)$$

$$\sum_{\alpha=0}^L y_\alpha = y(1), \quad (17)$$

$$\langle y(1), 1_N \rangle = 1, \quad y(1) \geq 0, \quad R y(1) \in Z. \quad (18)$$

To study properties of the solution of problem (15) - (18) we may to relax it by dropping constraints (18). Then, the following theorem is valid.

Theorem 1. *Among the solutions of problem (15) - (16) ($y(1)$ is f parameter) there is always such that $\langle y_\alpha, y_\beta \rangle = 0$ for any $\alpha \neq \beta$*

Proof. Let us replace problem (15) - (18) by the set of bilinear programming problems with linear constraints of the form

$$\sum_{\alpha=1}^{L-1} \sum_{\beta=\alpha+1}^L \langle C_i y_\alpha, y_\beta \rangle \rightarrow \max \quad (19)$$

$$\sum_{\alpha=0}^L y_\alpha = y(1), \quad i = \overline{1, p} \quad (20)$$

and let us solve each task separately, and among the solutions of problem (15) - (17) we will choose the one which has the maximum value of the objective function. (Obviously that this technique makes sense to use only in the proof).

Reasonable computational procedures will be discussed later. Thus, the task is came down to the problem of the form (19) - (20). This problem is investigated in detail in (see[Afanasyev, 1993]) where is set to among solutions to the problem of the form (19) - (20) there is always such that, $\langle y_\alpha, y_\beta \rangle = 0 \quad \forall \alpha \neq \beta$ \square

This theorem has important corollary.

Corollary 1. *Let the vector $y(1) \in R^N$ has a $N_0 \leq N$ non-zero components. Then, whatever L (the number of splits when constructing difference schemes), the solution of the problem (15) - (17) (and hence (15) - (18)) can be realized no more than N_0 of mutually orthogonal vectors J_α .*

5 The Theorem on the Bypass of the Vertices of the Polyhedron

Let us return to problem (2) - (5).

Theorem 2. *The solution of the problem (2) - (5) exists and can be always realized at the vertices of the polyhedron M . and the number of switchings does not exceed N , where N - is the number of vertices of polyhedron M .*

Proof. Consider the sequence $v^L(t)$ of simple (step) functions converging to a measurable function $v(t)$, which gives the optimal solution to the problem (9) - (11).

$$\lim_{L \rightarrow \infty} v^L(t) = v^*(t), \quad (21)$$

$$\lim_{L \rightarrow \infty} y^L(t) = \lim_{L \rightarrow \infty} \int_0^t v^L(\tau) d\tau \rightarrow y^*(t) = \int_0^t v^*(\tau) d\tau. \quad (22)$$

Let to each L there is a task (12) - (14), the solution of which, beginning with number N_0 due to a corollary of Theorem 1) does not depend on L . Therefore $y^*(t) = \{y_\alpha\}, \alpha = \overline{1, N_0}$. That is, the solution of problem (9) - (11) is completely determined by the solution of the problem (12) - (14) (finite-dimensional analogue). The transition from problem (9) - (11) to problem (2) - (5) is obvious. Theorem is proven. \square

6 Properties of the Solution of the Difference Problem (12) - (14)

By theorem 1, a vector y_α has only one nonzero component all nonzero components of vectors are the components y_α of the vector $y(1)$. The essence problem (12) - (14) is to find the suitable traversal of vertices of the polyhedron M .

Let us we go back to the maximization of the expression

$$\sum_{\alpha=1}^{L-1} \sum_{\beta=\alpha+1}^L \langle C_i y_\alpha, y_\beta \rangle \quad \text{where} \quad \sum_{\alpha=1}^N y_\alpha = y(1)$$

in view of the above. it is clear how the original problem is reduced to the following: consider the matrix $\{y_\alpha C_i^{\alpha\beta} y_\beta\}_{N \times N}$ - and put the task of finding such a permutation rows and columns with the same names to maximize the sum of above-diagonal elements in the resulting matrix. Thus we can formulate the following optimization problem.

Suppose that there is a set of skew-symmetric matrices $A_i = \{a_{\alpha\beta}^i\}_{n \times n}, i = \overline{1, k}$. Let $S(A_i)$ denote the sum of the above-diagonal elements. It is necessary to find a set of permutations of rows and columns with the same names such that

$$\min_i S(A_i) \rightarrow \max. \quad (23)$$

In (see[Afanasiev et al., 1990]) it is shown that, when $i = 1$, then nonnegativity of all partial sums of above-diagonal elements along each strings is the necessary and sufficient condition for the maximum, i.e.

$$\sum_{\beta=\alpha+1}^{\alpha=\gamma} a_{\alpha\beta}^i \geq 0, \quad \alpha = \overline{1, n}, \quad \beta = \overline{1, n}, \quad \gamma = \overline{1, n}. \quad (24)$$

Generally speaking, they are local maximum conditions. The fact that the problem (22) is a global optimization problem shows the following example Matrix

$$\begin{pmatrix} 0 & 6 & 2 & -7 \\ -6 & 0 & 3 & -2 \\ -2 & -3 & 0 & 10 \\ 7 & 2 & -10 & 0 \end{pmatrix},$$

with the sum of the above-diagonal elements $6 + 2 - 7 + 3 - 2 + 10 = 12$ satisfying the condition (23), by using the permutation (3, 4, 1, 2) can be reduced to the matrix

$$\begin{pmatrix} 0 & 10 & -2 & -3 \\ -10 & 0 & 7 & 2 \\ 2 & -7 & 0 & 6 \\ 3 & -2 & -6 & 0 \end{pmatrix}$$

with the sum of the above-diagonal elements $10 - 2 - 3 + 7 + 2 + 6 = 20$ which also satisfy condition (23).

Thus, even in the case of $i = 1$, to solve problem (22) should be involved methods of search for global extremum. However, the use of conditions(23) in problem (22) when $i > 1$ allows one to apply the method of branches and boundaries.

Acknowledgements

This work was supported by Russian Scientific Foundation, Project 16-11-10352.

References

- [Milyutin et al., 2004] Milyutin A. A., Dmitruk A. V., Osmolovskii N. P. (2004). Maximum principle in optimal control *Moscow State University, Faculty of Mechanics and Mathematics*. Moscow, 168 p.
- [Dikusar et al., 2001] Dikusar V. V., Koshka M., Figura A. (2001). A Method of continuation on a parameter in solving boundary value problems in optimal control. *Differents. equations, 37:4*, 453–45.

- [Afanasiev et al., 1990] Afanasiev A. P., Dikumar V. V., Milyutin A. A., Chukanov S. V. (1990) *A necessary condition in optimal control*, Moscow: Nauka.
- [Afanasiev, 1998] Afanasiev A. P. (1998) *On local variational problems in the presence of functional constraints*. *Differents. equations, volume 34:11*, 1459–1463.
- [Afanasiev, 1993] Afanasiev A. P. (1993) *Generalized isoperimetric problem on a polyhedron* *Differents. Equations, Vol. 29:11*, 1856–1867.