Differential Method for Solving Terminal Control Problem

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Abstract

A terminal control problem with linear dynamics and boundary-value optimization problem is considered. To solve the problem, a new gradient non-iterative approach, based on the necessary and sufficient condition of extremality, is proposed. The approach is reduced to solving a functional variational inequality on the set of controls in a Hilbert space. This inequality is treated as a parametric family of finite-dimensional variational inequalities that depend on time as a parameter. At each fixed moment of time, we have a cross-section of the problem in time, i.e. finite-dimensional variational inequality. Each such inequality has its solution, which is found using the continuous gradient projection method. We prove the pointwise convergence of the entire family of methods to their solutions. Thus, a mapping is formed, when the solution of the variational inequality corresponds to each instant of time. This function is the desired optimal control of the original problem.

1 Formulation of the Problem

We consider a linear dynamical control system defined on a fixed time interval $[t_0, t_1]$ with a movable right-hand end. The dynamics of the controlled trajectory $x(\cdot)$ is described by a linear system of ordinary differential equations with fixed initial conditions and implicitly given terminal conditions on the right-hand end of the time interval. These conditions are defined as the solution of the optimization problem. There are constraints on controls $u(\cdot)$, which are given in the form of a geometric set. The problem has the form

$$\frac{d}{dt}x(t) = D(t)x(t) + B(t)u(t), \quad x(t_0) = x_0,\ x^*(t_1) = x_1^*, \tag{1}$$

$$x_1^* \in \mathrm{Argmin}\{\varphi(x_1) \mid x_1 \in X_1 \subset \mathbb{R}^n\}, \tag{2}$$

$$x(\cdot) \in \mathrm{AC}^n[t_0, t_1],\ \ u(\cdot) \in \mathrm{U} \text{ for almost all } t \in [t_0, t_1], \tag{3}$$

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where $D(t), B(t)$ are $n \times n$ and $n \times r$-continuous matrices ($r < n$). X_1 is the reachability set, i.e. the set of right-hand ends $x(t_1)$ of trajectories. Controls $u(\cdot) \in \mathrm{U}$, where $\mathrm{U} \subset \mathbb{R}^r$ is a convex closed set.

We understand any pair $(x(\cdot), u(\cdot)) \in \mathrm{AC}^n[t_0, t_1] \times \mathrm{U}$ satisfying the condition

$$x(t) = x(t_0) + \int_{t_0}^{t} (D(\tau)x(\tau) + B(\tau)u(\tau))d\tau, \quad t_0 \le t \le t_1. \tag{4}$$

as a solution to the differential system (1). Note that under these conditions the trajectory $x(\cdot)$ is an absolutely continuous function [Kolmogorov & Fomin, 2009]. The class of absolutely continuous functions is a linear variety that is everywhere dense in $\mathbb{L}_2^n[t_0, t_1]$. We denote this class as $\mathrm{AC}^n[t_0, t_1] \subset \mathbb{L}_2^n[t_0, t_1]$, where the closure $\overline{\mathrm{AC}}^n[t_0, t_1] \equiv \mathbb{L}_2^n[t_0, t_1]$. For any pair $(x(\cdot), u(\cdot)) \in \mathrm{AC}^n[t_0, t_1] \times \mathrm{U}$ both the Newton-Leibniz formula and, accordingly, the integration-by-parts formula are satisfied.[1] In [Vasiliev, 2011, Book 2, p. 443] it is shown that in the linear differential system (1) for any control $u(\cdot) \in \mathrm{U}$ there is a unique trajectory $x(\cdot)$, and this pair satisfies this identity (4).

In (1)-(3), it is necessary to find a control $u^*(\cdot) \in \mathrm{U}$ such that the corresponding trajectory $x^*(\cdot)$, being a solution of the differential system, connects the initial x_0 and the terminal $x^*(t_1)$ conditions.

The model (1)-(3) contains two components: controlled dynamics and the optimization problem as a boundary-value problem. The boundary-value problem is a model of a controlled object: for example, the enterprize – in the economy, the disease history – in the medicine, the assembly line – in technology, the epidemic or the war – in society, and so on. All these objects act in some environment, in which perturbations constantly arise. From time to time, under the influence of perturbations, objects lose their equilibrium state and find themselves in a random position. The problem arises: by choosing control, to return the object from an arbitrary state $x_0 \in \mathbb{R}^n$ to the terminal state $x^*(t_1)$.

2 Classical Lagrangian and Original Problem

The problem presented above is a problem of terminal control, formulated in a Hilbert space. In the theory of convex programming in finite-dimensional spaces, for a primal problem there always exists a dual problem in the dual space. Carrying out the corresponding analogies with finite-dimensional spaces, we consider elements of the duality theory in a functional Hilbert space using the example of a convex terminal control problem (1)-(3). To this end, we introduce the linear convolution of the problem, known as the Lagrange function:

$$\mathcal{L}(x(t_1), x(\cdot), u(\cdot); \psi(\cdot)) = \varphi(x(t_1) + \int_{t_0}^{t_1} \langle \psi(t), D(t)x(t) + B(t)u(t) - \frac{d}{dt}x(t) \rangle dt \tag{5}$$

for all $(x(t_1), x(\cdot), u(\cdot)) \in X_1 \times \mathrm{AC}^n[t_0, t_1] \times \mathrm{U}$, $\psi(\cdot) \in \Psi_2^n[t_0, t_1]$. Here $\Psi_2^n[t_0, t_1]$ is a linear variety of absolutely continuous functions in a space that is conjugate to the space $\mathbb{L}_2^n[t_0, t_1]$ of primal variable $x(\cdot)$. Since the space $\mathbb{L}_2^n[t_0, t_1]$ is a Hilbert space, it is self-adjoint one.

In order to take advantage of the duality theory, we linearize the Lagrange function (5) at the solution point of the problem (1)-(3), i.e. at the point $(x^*(t_1), x^*(\cdot), u^*(\cdot))$. Since the Lagrange function has only one nonlinear term – the function $\varphi(x(t_1))$, then we replace this term by its linear approximation from the expansion of $\varphi(x(t_1))$ into a Taylor series. Then the linearized Lagrange function takes the form

$$\begin{gathered}\mathcal{L}(x(t_1), x(\cdot), u(\cdot); \psi(\cdot)) = \\ = \langle \nabla\varphi(x^*(t_1)), x(t_1) - x^*(t_1) \rangle + \int_{t_0}^{t_1} \langle \psi(t), D(t)x(t) + B(t)u(t) - \frac{d}{dt}x(t) \rangle dt\end{gathered} \tag{6}$$

for all $(x(t_1), x(\cdot), u(\cdot)) \in X_1 \times \mathrm{AC}^n[t_0, t_1] \times \mathrm{U}$, $\psi(\cdot) \in \Psi_2^n[t_0, t_1]$.

It is known [Ioffe & Tikhomirov, 1974] that the Lagrange functions for convex, regular (Slater's condition holds) problems in finite-dimensional and functional spaces have saddle points $(x_1^*, x^*(\cdot), u^*(\cdot); \psi^*(\cdot))$. The first

[1] The scalar products and norms in the introduced spaces are defined, respectively, as

$$\langle x(\cdot), y(\cdot) \rangle = \int_{t_0}^{t_1} \langle x(t), y(t) \rangle dt, \ \|x(\cdot)\|^2 = \int_{t_0}^{t_1} |x(t)|^2 dt,$$

$$\langle x(t), y(t) \rangle = \sum_1^n x_i(t) y_i(t), \ |x(t)|^2 = \sum_1^n x_i^2(t), \quad t_0 \le t \le t_1,$$

$$x(t) = (x_1(t), ..., x_n(t))^{\mathrm{T}}, \ y(t) = (y_1(t), ..., y_n(t))^{\mathrm{T}}.$$

three components are the solution to the problem (1)-(3), and the last component plays the role of the Lagrange multiplier and, accordingly, is the solution to the dual (conjugate) problem. Further, using the linearization of the Lagrangian function (6), we obtain a dual problem in an explicit form.

By definition, the saddle point $(x^*(t_1), x^*(\cdot), u^*(\cdot); \psi^*(\cdot))$ satisfies the saddle-point system of inequalities common both for the Lagrange function (5) and for its linearization (6). We write out these saddle points of the inequality for the linearization (6) [Antipin, 2014], [Antipin & Khoroshilova, 2015(1)], [Antipin & Khoroshilova, 2015(2)], [Antipin & Khoroshilova, 2015(3)], [Antipin & Khoroshilova, 2016(1)], [Antipin & Khoroshilova, 2016(2)]:

$$\langle \nabla\varphi(x^*(t_1), x^*(t_1) - x^*(t_1)\rangle + \int_{t_0}^{t_1} \langle \psi(t), D(t)x^*(t) + B(t)u^*(t) - \frac{d}{dt}x^*(t)\rangle dt$$

$$\leq \langle \nabla\varphi(x^*(t_1)), x^*(t_1) - x^*(t_1)\rangle + \int_{t_0}^{t_1} \langle \psi^*(t), D(t)x^*(t) + B(t)u^*(t) - \frac{d}{dt}x^*(t)\rangle dt$$

$$\leq \langle \nabla\varphi(x^*(t_1), x(t_1) - x^*(t_1)\rangle + \int_{t_0}^{t_1} \langle \psi^*(t), D(t)x(t) + B(t)u(t) - \frac{d}{dt}x(t)\rangle dt \tag{7}$$

for all $(x(t_1), x(\cdot), u(\cdot)) \in X_1 \times AC^n[t_0, t_1] \times \mathrm{U}$, $\psi(\cdot) \in \Psi_2^n[t_0, t_1]$. We show now that the primal and dual components of the saddle point for the linearized Lagrange function (6) are primal and dual solutions to the original problem (1)-(3).

The left-hand inequality of (7) is the problem of maximizing a linear function in the variable $\psi(\cdot)$ on the entire linear variety $\Psi_2^n[t_0, t_1]$:

$$\int_{t_0}^{t_1} \langle \psi(t), D(t)x^*(t) + B(t)u^*(t) - \frac{d}{dt}x^*(t)\rangle dt \leq \int_{t_0}^{t_1} \langle \psi^*(t), D(t)x^*(t) + B(t)u^*(t) - \frac{d}{dt}x^*(t)\rangle dt. \tag{8}$$

This inequality is true for all $\psi(\cdot) \in \Psi_2^n[t_0, t_1]$ only if

$$D(t)x^*(t) + B(t)u^*(t) - \frac{d}{dt}x^*(t) = 0, \quad x^*(t_0) = x_0. \tag{9}$$

The right-hand inequality of (7) is the problem of minimizing the Lagrange function in the variables $(x(t_1), x(\cdot), u(\cdot))$ if the function $\psi(t)$ is fixed: $\psi(t) = \psi^*(t)$. Let us show that the primal variables in $(x^*(t_1), x^*(\cdot), u^*(\cdot); \psi^*(\cdot))$ are the solution to (1)-(3).

In view of (9), from the right-hand inequality of (7), we have

$$\langle \nabla\varphi(x^*(t_1)), x^*(t_1)\rangle \leq \langle \nabla\varphi(x^*(t_1)), x(t_1)\rangle + \int_{t_0}^{t_1} \langle \psi^*(t), D(t)x(t) + B(t)u(t) - \frac{d}{dt}x(t)\rangle dt \tag{10}$$

for all $(x(t_1), x(\cdot), u(\cdot)) \in X_1 \times \mathrm{AC}^n[t_0, t_1] \times \mathrm{U}$.

Considering the inequality (10) with an additional scalar constraint

$$\int_{t_0}^{t_1} \langle \psi^*(t), D(t)x(t) + B(t)u(t) - \frac{d}{dt}x(t)\rangle dt = 0, \tag{11}$$

we get that the linear function $\langle \nabla\varphi(x_1^*), x_1\rangle$ reaches its minimum on the set defined by the scalar constraint (11). But, according to (9), the solution $(x^*(t_1), x^*(\cdot), u^*(\cdot))$ belongs to a narrower set than (11). Therefore, this point remains a minimum on a subset of the solutions for the system (9), i.e.

$$\langle \nabla\varphi(x^*(t_1)), x^*(t_1)\rangle \leq \langle \nabla\varphi(x^*(t_1)), x(t_1)\rangle, \tag{12}$$

$$\frac{d}{dt}x(t) = D(t)x(t) + B(t)u(t) \tag{13}$$

for all $(x(t_1), x(\cdot), u(\cdot)) \in X_1 \times \mathrm{AC}^n[t_0, t_1] \times \mathrm{U}$. Thus, if the Lagrange function (6) has a saddle point, then its vector of primal components is a solution to (1)-(3).

3 Dual Lagrangian and Dual Problem

Let us show that the Lagrange function for linear dynamical problems allows to obtain the corresponding dual problems in conjugate spaces. Using the formulas for the transition to conjugate linear operators

$$\langle \psi(t), D(t)x(t)\rangle = \langle D^{\mathrm{T}}(t)\psi(t), x(t)\rangle,\ \langle \psi(t), B(t)u(t)\rangle = \langle B^{\mathrm{T}}(t)\psi(t), u(t)\rangle$$

and the integration-by-parts formula on the interval $[t_0, t_1]$

$$\langle \psi(t_1), x(t_1)\rangle - \langle \psi(t_0), x(t_0)\rangle = \int_{t_0}^{t_1} \langle \frac{d}{dt}\psi(t), x(t)\rangle dt + \int_{t_0}^{t_1} \langle \psi(t), \frac{d}{dt}x(t)\rangle dt,$$

we write out the Lagrangian conjugate with respect to (6):

$$\mathcal{L}^{\mathrm{T}}(\psi(\cdot); x(t_1), x(\cdot), u(\cdot)) =$$

$$= \langle \nabla\varphi(x^*(t_1) - \psi_1, x(t_1)\rangle + \int_{t_0}^{t_1} \langle D^{\mathrm{T}}(t)\psi(t) + \frac{d}{dt}\psi(t), x(t)\rangle dt + \int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi(t), u(t)\rangle dt + \langle \psi_0, x_0\rangle \tag{14}$$

for all $\psi(\cdot) \in \Psi_2^n[t_0, t_1]$, $(x(t_1), x(\cdot), u(\cdot)) \in \mathbb{R}^n \times \mathrm{AC}^n[t_0, t_1] \times \mathrm{U}$, where $\psi_1 = \psi(t_1)$.

The saddle point $(x^*(t_1), x^*(\cdot), u^*(\cdot); \psi_1^*, \psi^*(\cdot))$ satisfies the saddle-point system (7), which in the conjugate space has the form

$$\begin{aligned}
&\langle \nabla\varphi(x^*(t_1)) - \psi_1, x^*(t_1)\rangle + \langle \psi_0, x_0\rangle \\
&+ \int_{t_0}^{t_1} \langle D^{\mathrm{T}}(t)\psi(t) + \frac{d}{dt}\psi(t), x^*(t)\rangle dt + \int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi(t), u^*(t)\rangle dt + \langle \psi_0, x_0\rangle \\
&\le \langle \nabla\varphi(x^*(t_1)) - \psi^*(t_1), x^*(t_1)\rangle + \langle \psi_0^*, x_0\rangle \\
&+ \int_{t_0}^{t_1} \langle D^{\mathrm{T}}(t)\psi^*(t) + \frac{d}{dt}\psi^*(t), x^*(t)\rangle dt + \int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi^*(t), u^*(t)\rangle dt \\
&\le \nabla\varphi(x^*(t_1)) - \psi^*(t_1), x(t_1)\rangle + \langle \psi_0^*, x_0\rangle \\
&+ \int_{t_0}^{t_1} \langle D^{\mathrm{T}}(t)\psi^*(t) + \frac{d}{dt}\psi^*(t), x(t)\rangle dt + \int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi^*(t), u(t)\rangle dt
\end{aligned} \tag{15}$$

for all $(x(t_1), x(\cdot), u(\cdot)) \in X_1 \times \mathrm{AC}^n[t_0, t_1] \times \mathrm{U}$, $\psi_1 \in \mathbb{R}^n$, $\psi(\cdot) \in \Psi_2^n[t_0, t_1]$.

We will repeat the same transformations as in the previous section, where it was shown that the initial problem follows from the saddle-point system. But now we will get the dual problem.

From the right-hand inequality of (15), we have

$$\langle \nabla\varphi(x^*(t_1)) - \psi^*(t_1), x^*(t_1) - x(t_1)\rangle + \int_{t_0}^{t_1} \langle D^{\mathrm{T}}(t)\psi^*(t) + \frac{d}{dt}\psi^*(t), x^*(t) - x(t)\rangle dt +$$

$$+ \int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi^*(t), u^*(t) - u(t)\rangle dt \le 0$$

for all $(x(t_1), x(\cdot), u(\cdot)) \in X_1 \times \mathrm{AC}^n[t_0, t_1] \times \mathrm{U}$. By virtue of the independent change of each of the variables $(x(t_1), x(\cdot), u(\cdot))$ within its admissible subspaces (sets), the last inequality is decomposed into three independent inequalities:

$$\langle \nabla\varphi(x^*(t_1)) - \psi^*(t_1), x^*(t_1) - x(t_1)\rangle \le 0, \quad x(t_1) \in \mathbb{R}^n,$$

$$\int_{t_0}^{t_1} \langle D^{\mathrm{T}}(t)\psi^*(t) + \frac{d}{dt}\psi^*(t), x^*(t) - x(t)\rangle dt \le 0, \quad x(\cdot) \in \mathrm{AC}^n[t_0, t_1],$$

$$\int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi^*(t), u^*(t) - u(t)\rangle dt \le 0, \quad u(\cdot) \in \mathrm{U}.$$

Both first linear functionals reach a finite extremum on the entire subspace only in the case when their normals vanish, which leads to a system of problems

$$D^{\mathrm{T}}(t)\psi^*(t) + \frac{d}{dt}\psi^*(t) = 0,\ \nabla\varphi(x^*(t_1)) - \psi^*(t_1) = 0, \tag{16}$$

$$\int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi^*(t), u^*(t) - u(t)\rangle dt \le 0, \quad \forall u(\cdot) \in \mathrm{U}. \tag{17}$$

The left-hand inequality in (15), taking into account (16) and (17), can be rewritten as

$$\langle \nabla\varphi(x^*(t_1)) - \psi(t_1), x^*(t_1)\rangle + \langle \psi_0, x_0\rangle$$

$$+ \int_{t_0}^{t_1} \langle D^{\mathrm{T}}(t)\psi(t) + \frac{d}{dt}\psi(t), x^*(t)\rangle dt + \int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi(t), u^*(t)\rangle dt \le$$

$$\le \int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi^*(t), u^*(t)\rangle dt + \langle \psi_0^*, x_0\rangle.$$

We require additionally that the first two terms of this inequality satisfy conditions

$$\langle \nabla\varphi(x^*(t_1)) - \psi(t_1), x^*(t_1)\rangle \ge 0,$$

$$\int_{t_0}^{t_1} \langle D^{\mathrm{T}}(t)\psi(t) + \frac{d}{dt}\psi(t), x^*(t)\rangle dt \ge 0.$$

Then the initial inequality can be represented as an optimization problem of the form

$$(\psi^*(\cdot), \psi(t_1)) \in \mathrm{Argmax}\left\{ \int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi(t), u^*(t)\rangle dt \;\mid\; \langle \nabla\varphi(x^*(t_1)) - \psi(t_1), x^*(t_1)\rangle \ge 0, \right.$$

$$\left. \int_{t_0}^{t_1} \langle D^{\mathrm{T}}(t)\psi(t) + \frac{d}{dt}\psi(t), x^*(t)\rangle dt \ge 0. \right\}$$

Hence, taking into account (16), we have

$$(\psi^*(\cdot), \psi(t_1)) \in \mathrm{Argmax}\left\{ \int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi(t), u^*(t)\rangle dt \mid \right.$$

$$\left. D^{\mathrm{T}}(t)\psi(t) + \frac{d}{dt}\psi(t) = 0, \;\; \psi(t_1) = \nabla\varphi(x^*(t_1)) \right\}, \tag{18}$$

where $\psi(\cdot) \in \Psi_2^n[t_0, t_1]$.

Combining with (17),(18), we get the problem dual with respect to (1)-(3):

$$(\psi^*(\cdot), \psi(t_1)) \in \mathrm{Argmax}\left\{ \int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi(t), u^*(t)\rangle dt \mid \right. \tag{19}$$

$$\left. D^{\mathrm{T}}(t)\psi(t) + \frac{d}{dt}\psi(t) = 0, \;\; \psi(t_1) = \nabla\varphi(x^*(t_1)) \right\}, \tag{20}$$

$$\int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi^*(t), u^*(t) - u(t)\rangle dt \le 0, \quad u(\cdot) \in \mathrm{U}. \tag{21}$$

4 Boundary-Value Differential System

We consider together the left-hand inequality of the saddle-point system (7) for the classical Lagrangian and the right-hand inequality of the dual saddle-point system (15) for the conjugate Lagrangian. Of these systems, partial subsystems (9) and (16),(17) were obtained as a consequence. We write them out and arrive at the following boundary-value problem:

$$\frac{d}{dt}x^*(t) = D(t)x^*(t) + B(t)u^*(t), \quad x^*(t_0) = x_0, \tag{22}$$

$$\frac{d}{dt}\psi^*(t) + D^{\mathrm{T}}(t)\psi^*(t) = 0, \quad \psi^*(t_1) = \nabla\varphi(x^*(t_1)), \tag{23}$$

$$\int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi^*(t), u^*(t) - u(t)\rangle dt \le 0, \quad u(\cdot) \in \mathrm{U}. \tag{24}$$

The variational inequality of this system can be rewritten in the equivalent form of the operator equation with the projection operator on the corresponding convex closed set U [Vasiliev, 2011]. Then we obtain a system of differential and operator equations:

$$\frac{d}{dt}x^*(t) = D(t)x^*(t) + B(t)u^*(t), \quad x^*(t_0) = x_0, \tag{25}$$

$$\frac{d}{dt}\psi^*(t) + D^{\mathrm{T}}(t)\psi^*(t) = 0, \quad \psi^*(t_1) = \nabla\varphi(x^*(t_1)), \tag{26}$$

$$u^*(t) = \pi_{\mathrm{U}}(u^*(t) - \alpha B^{\mathrm{T}}(t)\psi^*(t)), \ t_0 \le t \le t_1, \tag{27}$$

where $\pi_{\mathrm{U}}(\cdot)$ is the projection operator on the set of controls U, $\alpha > 0$.

5 Continuous Method for Solving the Boundary-Value Differential System

In order to solve the system (22)-(24), which is a sufficient extremality condition for the problem (1)-(3), we can formulate the gradient projection method. In our situation, it looks like this

$$\frac{d}{dt}x(t) = D(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \tag{28}$$

$$\frac{d}{dt}\psi(t) + D^{\mathrm{T}}(t)\psi(t) = 0, \quad \psi(t_1) = \nabla\varphi(x(t_1)), \tag{29}$$

$$\frac{d}{dt}u(t) + u(t) = \pi_{\mathrm{U}}(u(t) - \alpha B^{\mathrm{T}}(t)\psi(t)). \tag{30}$$

The process (28)-(30) is a family of finite-dimensional continuous gradient projection methods, each of which converges to the solution of the problem for a given $t \in [t_0, t_1]$ [Antipin, 1994], [Liao, 2005]. In general, the whole family converges point-by-point.

Theorem. *If the set of solutions $(x^*(t_1), x^*(\cdot), u^*(\cdot); \psi^*(\cdot))$ to the problem (28)-(30) is not empty and belongs to the space $X_1 \times AC^n[t_0, t_1] \times \mathrm{U} \times \Psi_2^n[t_0, t_1]$, the terminal function is convex, U is strictly convex closed set, then the family of continuous gradient projection methods converges pointwise to the control $u^*(\cdot) \in \mathrm{U}$ as $t \to \infty$.*

Proof. We represent (29) and (26) in the form of variational inequalities

$$\langle \nabla\varphi(x(t_1)) - \psi_1, x^*(t_1) - x(t_1)\rangle + \int_{t_0}^{t_1} \langle D^T(t)\psi(t) + \frac{d}{dt}\psi(t), x^*(t) - x(t)\rangle dt \ge 0,$$

$$\langle \nabla\varphi(x^*(t_1)) - \psi_1^*, x^*(t_1) - x(t_1)\rangle + \int_{t_0}^{t_1} \langle D^T(t)\psi^*(t) + \frac{d}{dt}\psi^*(t), x^*(t) - x(t)\rangle dt \ge 0, \tag{31}$$

add together both inequalities, then

$$\langle \nabla\varphi(x(t_1)) - \nabla\varphi(x^*(t_1)), x^*(t_1) - x(t_1)\rangle - \langle \psi_1 - \psi_1^*, x^*(t_1) - x(t_1)\rangle$$
$$+ \int_{t_0}^{t_1} \langle D^T(t)(\psi(t) - \psi^*(t)) + \frac{d}{dt}(\psi(t) - \psi^*(t)), x^*(t) - x(t)\rangle dt \ge 0.$$

Using the integration by parts formula from Section 3, we transform (28) and obtain

$$\int_{t_0}^{t_1} \langle \psi(t) - \psi^*(t), D(t)(x^*(t) - x(t)) - \frac{d}{dt}(x^*(t) - x(t))\rangle dt \ge 0. \tag{32}$$

We represent the equation (30) in the form of a variational inequality

$$\langle \frac{d}{dt}u(t) + \alpha B^T(t)\psi(t), z(t) - u(t) - \frac{d}{dt}u(t)\rangle \ge 0 \tag{33}$$

for all $z(\cdot) \in \mathrm{U}$. We put $z(t) = u^*(t)$ in (33), then we get

$$\langle \frac{d}{dt}u(t) + \alpha B^T(t)\psi(t), u^*(t) - u(t) - \frac{d}{dt}u(t)\rangle \geq 0. \tag{34}$$

Taking into account that $\frac{d}{dt}u(t) = \frac{d}{dt}(u(t) - u^*(t))$, from (33) we have

$$-\langle \frac{1}{2}\frac{d}{dt}(u(t) - u^*(t)), u(t) - u^*(t)\rangle - \left|\frac{d}{dt}u(t)\right|^2 + \alpha\langle B^T(t)\psi(t), u^* - u(t)\rangle - \alpha\langle B^T(t)\psi(t), \frac{d}{dt}u(t)\rangle \geq 0.$$

Hence, we obtain

$$\frac{1}{2}\frac{d}{dt}|u(t) - u^*(t)|^2 + \left|\frac{d}{dt}u(t)\right|^2 + \alpha\langle B^T(t)\psi(t), u(t) - u^*(t)\rangle + \alpha\langle B^T(t)\psi(t), \frac{d}{dt}u(t)\rangle \leq 0. \tag{35}$$

We set $u(t) := u(t) + \frac{d}{dt}u(t)$ in (24), then

$$\int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi^*(t), u^*(t) - u(t)\rangle dt - \int_{t_0}^{t_1} \langle B^{\mathrm{T}}(t)\psi^*(t), \frac{d}{dt}u(t)\rangle dt \leq 0. \tag{36}$$

Adding the last two inequalities and combining the result with the inequality (32), we obtain

$$\frac{1}{2}\frac{d}{dt}|u(t) - u^*(t)|^2 + \left|\frac{d}{dt}u(t)\right|^2$$

$$+\alpha\langle B^T(t)(\psi(t) - \psi^*(t)), u(t) - u^*\rangle + \alpha\langle B^T(t)(\psi(t) - \psi^*(t)), \frac{d}{dt}u(t)\rangle \leq 0, \tag{37}$$

$$\int_{t_0}^{t_1} \langle \psi(t) - \psi^*(t), D(t)(x^*(t) - x(t)) - \frac{d}{dt}(x^*(t) - x(t))\rangle dt \geq 0. \tag{38}$$

Hence, we find

$$\frac{1}{2}\frac{d}{dt}|u(t) - u^*(t)|^2 + \left|\frac{d}{dt}u(t)\right|^2 + \alpha\langle B^T(t)(\psi(t) - \psi^*(t)), \frac{d}{dt}u(t)\rangle \leq 0,$$

$$\alpha\int_{t_0}^{t_1} \langle \psi(t) - \psi^*(t), -\frac{d}{dt}(x^*(t) - x(t)) + D(t)(x^*(t) - x(t)) + B^T(t)(u(t) - u^*(t))\rangle dt \geq 0. \tag{39}$$

The last term on the right-hand side of the inequality is zero by virtue of (22) and (28). The second term can be transformed to the form of the total derivative in the direction, i.e. $\frac{d}{dt}F(x(t)) = \frac{d}{dx}F(x) \cdot \frac{d}{dt}x(t)$. In view of the foregoing, the inequality (39) can be represented in the form

$$\frac{1}{2}\frac{d}{dt}|u(t) - u^*(t)|^2 + \left|\frac{d}{dt}u(t)\right|^2 + \alpha\frac{d}{dt}F(x) \leq 0.$$

From here

$$\frac{1}{2}|u(t_1) - u^*(t_1)|^2 + \int_{t_0}^{t_1} \left|\frac{d}{dt}u(t)\right|^2 dt + \alpha F(x(t_1)) \leq \frac{1}{2}|u(t_0) - u^*(t_0)|^2 + \alpha F(x(t_0)).$$

It follows from the estimate obtained that the control is bounded ($\frac{1}{2}|u(t_1) - u^*(t_1)|^2 \leq C$) and the integral converges as $t \to \infty$ ($\int\limits_{t_0}^{t_1} \left|\frac{d}{dt}u(t)\right|^2 dt < \infty$). Assuming the existence of $\varepsilon > 0$, such that $\left|\frac{d}{dt}u(t)\right| > \varepsilon$ for all $t > t_0$, we arrive at a contradiction with the convergence of the integral. Consequently, there exists a subsequence of time moments $t_i \to \infty$ such that $\left|\frac{d}{dt}u(t)\right|^2 \to \infty$. Since $u(t)$ is bounded, we again choose a subsequence of times, which we also denote by t_i such that $u(t_i) \to u^{'}$ and $\left|\frac{d}{dt}u(t)\right|^2 \to 0$. In the system (28)-(30), passing to this limit in this subsequence as $t_i \to \infty$, we obtain

$$\frac{d}{dt}x^{'}(t) = D(t)x^{'}(t) + B(t)u^{'}(t), \quad x^{'}(t_0) = x_0, \tag{40}$$

$$\frac{d}{dt}\psi^{'}(t) + D^{\mathrm{T}}(t)\psi^{'}(t) = 0, \quad \psi^{'}(t_1) = \nabla\varphi(x^{'}(t_1)), \tag{41}$$

$$u^{'}(t) = \pi_U(u^{'}(t) - \alpha B^{\mathrm{T}}(t)\psi^{'}(t)). \tag{42}$$

Comparing this system with (22), one can see that the solution obtained is the primal and dual solutions to the original problem: $(x^{'}(t_1), x^{'}(\cdot), u^{'}(\cdot); \psi^{'}(\cdot)) = (x^*(t_1), x^*(\cdot), u^*(\cdot); \psi^*(\cdot))$. The theorem is proved.

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